King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics
MATH 101 – Calculus I
EXAM I
2009-2010 (091)

Monday, November 2, 2009 Allowed Time: 2 Hours

Name: KEY

ID Number: ______ Serial Number: ______

Section Number: ______ Instructor's Name: ______

Instrunctions:

1. Write neatly and legibly. You may lose points for messy work.
2. **Show all your work.** No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 10 different problems (6 pages + cover page).

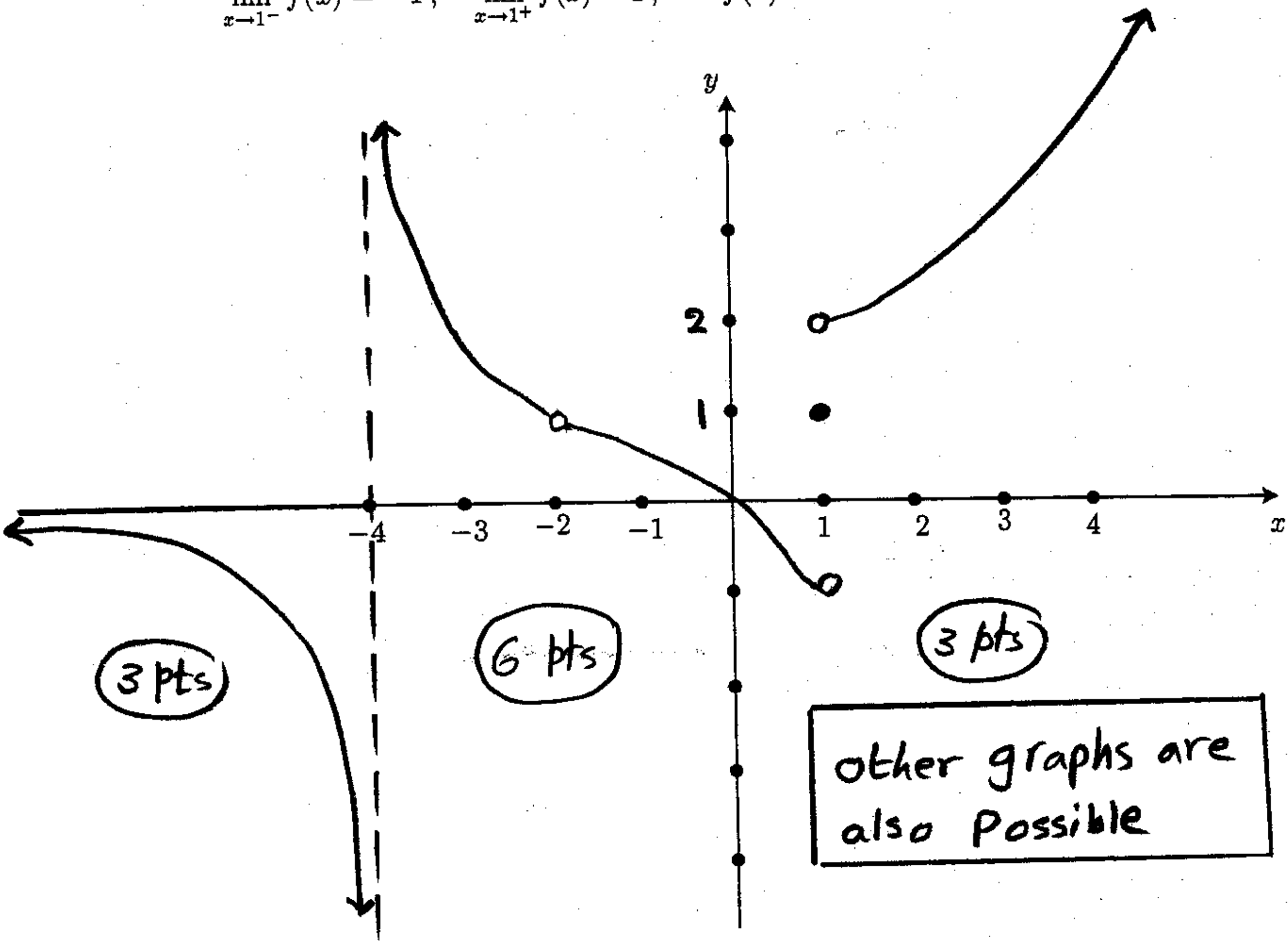
Problem No.	Points	Maxiumum Points
1		12
2		7
3(a,b,c)		18
4		12
5		6
6		10
7		7
8		12
9		12
10		4
Total:		100

1. (12-points) Sketch the graph of a function f that satisfies all of the following conditions:

$\lim_{x \to -4^-} f(x) = -\infty$; $\lim_{x \to -4^+} f(x) = \infty$; $\lim_{x \to -\infty} f(x) = 0$;

$\lim_{x \to -2} f(x) = 1$; f is undefined at -2;

$\lim_{x \to 1^-} f(x) = -1$; $\lim_{x \to 1^+} f(x) = 2$; $f(1) = 1$



2. (7-points) If $x^3 - x + 4 \le x + f(x) \le 3x^2 + 1$ for all real numbers x, then find $\lim_{x \to 1} f(x)$. (Give reasons to your steps).

Add $-x$ to all sides of the given inequality

$\Rightarrow \quad x^3 - 2x + 4 \le f(x) \le 3x^2 - x + 1$ (1 pt)

But $\lim_{x \to 1} (x^3 - 2x + 4) = 1 - 2 + 4 = 3$ (2 pts)

and $\lim_{x \to 1} (3x^2 - x + 1) = 3 - 1 + 1 = 3$ (2 pts)

$\Rightarrow \lim_{x \to 1} f(x) = 3$ by the Squeeze Theorem (2 pts)

3. Evaluate the limit, if it exists:

(a) (6-points) $\lim_{x\to 1/2}\left(\frac{2}{2x-1}-\frac{3}{2x^2+x-1}\right)$.

We cannot substitute $x=\frac{1}{2}$ because it makes each denominator in the given expression zero

$$\lim_{x\to\frac{1}{2}}\left(\frac{2}{2x-1}-\frac{3}{2x^2+x-1}\right)=\lim_{x\to\frac{1}{2}}\left(\frac{2}{2x-1}-\frac{3}{(2x-1)(x+1)}\right) \quad 2\text{ pts}$$

$$=\lim_{x\to\frac{1}{2}}\frac{2x+2-3}{(2x-1)(x+1)}=\lim_{x\to\frac{1}{2}}\frac{2x-1}{(2x-1)(x+1)} \quad 2\text{ pts}$$

$$=\lim_{x\to\frac{1}{2}}\frac{1}{x+1}=\frac{1}{\frac{3}{2}}=\frac{2}{3} \quad 2\text{ pts}$$

(b) (6-points) Let $f(x)=\left[\frac{1}{2}x+1\right]$ be the greatest integer less than or equal to $\frac{1}{2}x+1$. Find each of the following limits:

(i) $\lim_{x\to-2^-}f(x)=\lim_{x\to-2^-}\left[\frac{1}{2}x+1\right]=-1$ 2 pts

(ii) $\lim_{x\to-2^+}f(x)=\lim_{x\to-2^+}\left[\frac{1}{2}x+1\right]=0$ 2 pts

(iii) $\lim_{x\to-2}f(x)$ Does Not Exist. 2 pts

(c) (6-points) $\lim_{x\to 3^-}\frac{|x^2-9|}{x-3}$. $=\lim_{x\to 3^-}\frac{-(x^2-9)}{x-3}$ 2 pts

$$=\lim_{x\to 3^-}\frac{-(x-3)(x+3)}{(x-3)} \quad 2\text{ pts}$$

$$=\lim_{x\to 3^-}-(x+3)=-6 \quad 2\text{ pts}$$

4. (12-points) Find the horizontal asymptotes of the graph of the function

$$f(x) = \arctan \frac{\sqrt{9x^2+2}}{3x+7}.$$

Since arctan is a continuous function, then (1 pt)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \arctan \frac{\sqrt{9x^2+2}}{3x+7} = \arctan\left(\lim_{x\to\infty} \frac{\sqrt{9x^2+2}}{3x+7}\right) \quad \text{(2 pts)}$$

$$= \arctan\left(\lim_{x\to\infty} \frac{|x|\sqrt{9+\frac{2}{x^2}}}{x(3+\frac{7}{x})}\right) = \arctan\left(\lim_{x\to\infty} \frac{x\sqrt{9+\frac{2}{x^2}}}{x(3+\frac{7}{x})}\right) \quad \text{(2 pts)}$$

$$= \arctan\left(\lim_{x\to\infty} \frac{\sqrt{9+\frac{2}{x^2}}}{3+\frac{7}{x}}\right) = \arctan \frac{\sqrt{9}}{3} = \arctan 1 = \frac{\pi}{4} \quad \text{(2 pts)}$$

And
$$\lim_{x\to-\infty} f(x) = \arctan\left(\lim_{x\to-\infty} \frac{|x|\sqrt{9+\frac{2}{x^2}}}{x(3+\frac{7}{x})}\right) = \arctan\left(\lim_{x\to-\infty} \frac{-x\sqrt{9+\frac{2}{x^2}}}{x(3+\frac{7}{x})}\right) \quad \text{(1 pt)}$$

$$= \arctan\left(\lim_{x\to-\infty} \frac{-\sqrt{9+\frac{2}{x^2}}}{3+\frac{7}{x}}\right) = \arctan\left(-\frac{\sqrt{9}}{3}\right) = \arctan -1 = -\frac{\pi}{4}$$

(2 pts)

$\Rightarrow$ The horizontal asymptotes are $y = \frac{\pi}{4}$ and $y = -\frac{\pi}{4}$

(2 pt)

5. (6-points) Let $f(x) = \dfrac{4-x^2}{2-x-x^2}$. Find the following limits (write the answer as a real number, ∞, or $-\infty$).

(a) $\lim_{x\to 1^-} f(x)$. $= \lim_{x\to 1^-} \dfrac{(2-x)(2+x)}{(1-x)(2+x)} = \lim_{x\to 1^-} \dfrac{2-x}{1-x}$ (1 pt)

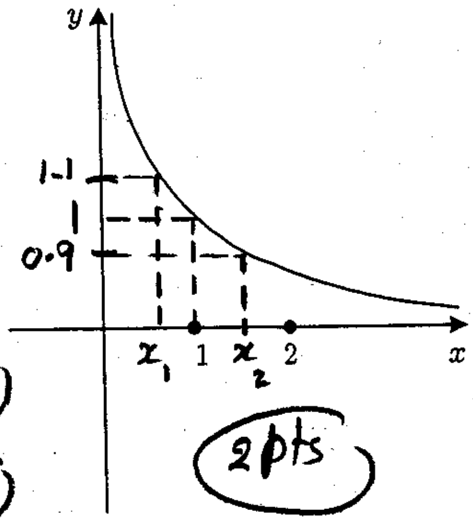
$= \infty$ (2 pts)

(b) $\lim_{x\to 1^+} f(x)$. $= \lim_{x\to 1^+} \dfrac{2-x}{1-x}$ (1 pt)

$= -\infty$ (2pts)

6. (10-points) Use the graph of $f(x) = \dfrac{1}{x}$ to find the largest number δ such that if $|x-1| < \delta$, then $|f(x) - 1| < 0.1$. (Show your work and write your answer in simplest rational form $\dfrac{p}{q}$).

Let x_1 and x_2 as shown in the figure



2pts

$\Rightarrow x_1 = \dfrac{1}{1.1} = \dfrac{10}{11}$ 1pt

and $x_2 = \dfrac{1}{0.9} = \dfrac{10}{9}$ 1pt

Let $\delta_1 = 1 - x_1 = 1 - \dfrac{10}{11} = \dfrac{1}{11}$ 2pts

and $\delta_2 = x_2 - 1 = \dfrac{10}{9} - 1 = \dfrac{1}{9}$ 2pts

$\Rightarrow$ The required δ = Smaller of (δ_1, δ_2)

$= \dfrac{1}{11}$ 2pts

7. (7-points) Use the Intermediate Value Theorem to show that there is a root of the equation $e^{-x^2} = x$ between 0 and 1.

Let $f(x) = e^{-x^2} - x$

since $f_1(x) = e^{-x^2}$ and $f_2(x) = x$ are continuous functions on $[0,1]$, then f is also continuous on $[0,1]$ 2pts

We have $f(0) = 1 - 0 = 1 > 0$ and $f(1) = \frac{1}{e} - 1 < 0$ 1pt

$\Rightarrow$ The Intermediate Value Theorem says there is a number c with $0 < c < 1$ such that $f(c) = 0$ 2pts

$\Rightarrow e^{-x^2} - x = 0$ has at least one root c in the interval $(0, 1)$. 2pts

8. (12-points) The displacement (in meters) of a particle moving in a straight line is given by $s = \dfrac{1}{\sqrt{5-t}}$ where t is measured in seconds. Use limits to find the instantaneous velocity of the particle when $t = 1$.

The required velocity $= \lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h}$ 2 pts

$= \lim\limits_{h \to 0} \dfrac{1}{h}\left[\dfrac{1}{\sqrt{4-h}} - \dfrac{1}{\sqrt{4}}\right]$ 2 pts

$= \lim\limits_{h \to 0} \dfrac{1}{h}\left(\dfrac{2 - \sqrt{4-h}}{2\sqrt{4-h}}\right)$ 2 pts

$= \lim\limits_{h \to 0} \dfrac{1}{h}\left(\dfrac{4 - (4-h)}{2\sqrt{4-h}\,(2+\sqrt{4-h})}\right)$ 2 pts

$= \lim\limits_{h \to 0} \dfrac{1}{2\sqrt{4-h}\,(2+\sqrt{4-h})}$ 2 pts

$= \lim\limits_{h \to 0} \dfrac{1}{(2)(2)(2+2)} = \dfrac{1}{16}$ m/sec. 2 pts

9. (12 points) Find the values of a and b that make the function

$$f(x) = \begin{cases} 3 & \text{if } x = 1 \\ ax^2 - bx + 3 & \text{if } 1 < x < 2 \\ 2x - a + b & \text{if } 2 \le x < 3 \\ 6 & \text{if } x = 3 \end{cases}$$

continuous on the closed interval $[1, 3]$. (Use limits to justify your steps)

We must have:

$\lim_{x\to 1^+} f(x) = f(1) \Rightarrow \lim_{x\to 1^+} (ax^2 - bx + 3) = 3$ 2 pts

$\Rightarrow a - b + 3 = 3 \Rightarrow a - b = 0 \Rightarrow \boxed{a = b}$ (i) 2 pts

and $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$ 2 pts

$\Rightarrow \lim_{x\to 2^-} (ax^2 - bx + 3) = \lim_{x\to 2^+} (2x - a + b)$

$\Rightarrow 4a - 2b + 3 = 4 - a + b \Rightarrow \boxed{5a - 3b = 1}$ (ii) 2 pts

(i) and (ii) $\Rightarrow 5a - 3a = 1 \Rightarrow 2a = 1 \Rightarrow$

$a = b = \frac{1}{2}$ 2 pts

We must make sure that $\lim_{x\to 3^-} f(x) = f(3)$ 2 pt

$\lim_{x\to 3^-} 2x = 6 = f(3)$

10. (4-points) Given the function $f(x) = \dfrac{2x^2 + kx - 14}{x - 2}$, where k is a constant, find k such that $x = 2$ is a removable discontinuity of f. (Give reasons to your steps).

$x - 2$ must be a factor of $2x^2 + kx - 14$ 2 pts

$\Rightarrow 8 + 2k - 14 = 0$ 1 pt

$\Rightarrow 2k - 6 = 0 \Rightarrow k = 3.$ 1 pt