An ADI Petrov-Galerkin method with quadrature for parabolic problems

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Abstract

We propose and analyze a fully discrete Laplace modified alternating direction implicit quadrature Petrov-Galerkin (ADI-QPG) method for solving parabolic initial-boundary value problems on rectangular domains. We prove optimal order convergence results for a restricted class of the associated elliptic operator and demonstrate accuracy of our scheme with numerical experiments for some parabolic problems with variable non-smooth coefficients.

Keywords: Parabolic initial–boundary value problems, Laplace modified, ADI, Petrov-Galerkin, splines, Gauss quadrature.

1 Introduction

We consider the parabolic initial-boundary value problem

$$u_t + Lu = f(x, y, t), \qquad (x, y, t) \in \Omega \times [0, T], \tag{1.1}$$

$$u(x, y, 0) = g(x, y), \qquad (x, y) \in \overline{\Omega}, \tag{1.2}$$

$$u(x, y, t) = 0, \qquad (x, y, t) \in \partial\Omega \times [0, T], \tag{1.3}$$

where Ω is a rectangular domain and the elliptic operator in (1.1) is given by

$$Lu = -a\ \Delta u + \mathbf{b} \cdot \nabla u + c\ u, \tag{1.4}$$

with known functions $a, \mathbf{b}, c$ defined on $\overline{\Omega}$ and f defined on $\overline{\Omega} \times [0, T]$. The non-divergence form elliptic operator in (1.4) includes the standard divergence form differential operator $-\nabla \cdot (a\ \nabla) + c\ I$. For such non-divergence forms (occurring for example in biological

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applications) due to the need to apply the second-order differential operator on basis functions, it is appropriate to use a finite element method (FEM) with trial space to be a subspace of H^2 as opposed to the standard H_0^1 trial space requirement for partial differential equations (PDEs) in divergence form. The rectangular domain requirement is standard in ADI schemes for which computational gains are substantial (over schemes that allow general domains) due to computational complexity essentially dominated by solutions of a collection of discretized one dimensional PDEs. In image processing with partial differential equations (for example, to restore noisy medical images), rectangular domains are standard and several ground water flow simulations are based on data obtained using rectangular tank experiments.

In this paper, we propose a fully discrete three-time level Laplace modified ADI scheme [8, Page 241] to solve (1.1)–(1.4) with spatial discretization given by a quadrature Petrov-Galerkin (QPG) scheme introduced and analyzed in [3] for elliptic problems. The Laplace modified approach further reduces the computational complexity of the standard ADI schemes by the need to solve only a collection of discretized one dimensional constant coefficient Helmholtz-type equations for general PDEs of the type (1.1)–(1.4). The trial space for the QPG consists of C^2 splines of degree $r \geq 3$, the test space is generated by C^0 splines of degree $r-2$, and the quadrature is the composite $r-1$ point Gauss rule. (It is easy to construct C^2 splines on rectangular domains, through tensor products of one-dimensional C^2 splines.)

This work is motivated by several H^2 trial space ADI (C^1 and C^2) spline collocation methods to solve parabolic PDEs, see, for example, the survey paper [2] and references therein. In particular, the C^1 Laplace modified ADI *orthogonal spline collocation* (OSC) method was investigated for parabolic PDEs in [1]. The C^2 *cubic nodal spline collocation* (NSC) seems to be gaining popularity in computational physics and engineering over the C^0 and C^1 spline methods [2, Page 76]. This interest is due to requirement in many applications to compute C^2 approximate solutions to high accuracy for second order problems. Further, C^2 spline methods have the marked advantage of obtaining smoother approximate solutions with fewer unknowns compared to the OSC and standard C^0 FEMs for the same degree splines.

However, the NSC and ADI-NSC in two space dimensions are currently restricted to cubic splines on uniform partitions [9, 10, 11, 12]. In our earlier work on elliptic problems [3], we avoided such restrictions by proposing and analyzing a fully discrete QPG using C^2 cubic and higher order splines on quasi-uniform partitions. In [4, 5], we generalized the method in [3] to parabolic and hyperbolic problems, using non-ADI schemes that require solutions of a collection of discretized two-dimensional elliptic PDEs.

The purpose of the present paper is to (i) computationally demonstrate that our ADI-QPG solutions are second order accurate in time (that is, $O(\tau^2)$) and optimal order accurate in space (that is, $O(h^{r+1-k})$ in the H^k norm, for $k = 0, 1, 2$) for (1.1)–(1.4) with variable coefficients; and (ii) prove the second order in time and optimal order H^2 norm error bounds in space for the special case of elliptic operator $-\Delta + c\, I$. As we shall see, error analysis for this special case is technically involved due to inclusion of the quadrature error and additional ADI related perturbations terms. Unlike our earlier non-ADI scheme analysis for variable coefficient parabolic problems in [4], the mathematical analysis in this paper does not extend to the general elliptic operator (1.4).

The outline of this paper is as follows. In the next section, we describe the full implementation details of the Laplace modified QPG scheme for (1.1)–(1.4). In Section 3, we recall some results from [3, 4] and prove additional properties of the trial and test spaces. In Section 4, we prove optimal order accurate in time and H^2 convergence in space of the Laplace modified ADI-QPG scheme by restricting the elliptic operator to the special case. Numerical experiments in Section 5 confirm the main theoretical result in Section 4 and demonstrate the applicability of the fully discrete quadrature scheme for constant and variable coefficients test parabolic problems to obtain optimal order accurate solutions in the H^k norm for $k = 0, 1, 2$.

2 ADI-QPG Algorithm and Implementation

For notational convenience and to directly apply analysis results from the literature (where scaling was assumed to reduce rectangular domains to the unit square), throughout the paper we choose $\Omega = (0,1) \times (0,1)$.

For a positive integer N_x, let $\Pi_x = \{x_k\}_{k=0}^{N_x}$ be a partition of $[0,1]$ in the x-direction such that

$$0 = x_0 < x_1 < \cdots < x_{N_x} = 1.$$

Let $\Pi_y = \{y_l\}_{l=0}^{N_y}$ be the corresponding partition of $[0,1]$ in the y-direction. We introduce

$$h_x^k = x_k - x_{k-1}, \quad k = 1, \ldots, N_x, \quad h_y^l = y_l - y_{l-1}, \quad l = 1, \ldots, N_y,$$

$$h_x = \max_{1 \le k \le N_x} h_x^k, \qquad h_y = \max_{1 \le l \le N_y} h_y^l, \qquad h = \max(h_x, h_y).$$

For $r \ge 3$, let

$$S_x = \{v \in C^2[0,1] : v|_{[x_{k-1},x_k]} \in P_r, \quad k = 1, \ldots, N_x, v(0) = v(1) = 0\},$$

$$T_x = \{v \in C[0,1] : v|_{[x_{k-1},x_k]} \in P_{r-2}, \quad k = 1, \ldots, N_x\},$$

where P_n is the space of all polynomials of degree at most n. Let S_y and T_y be the corresponding spline spaces in the y-direction. The trial and test spaces for the Laplace modified ADI-QPG scheme are defined by

$$S_h = S_x \otimes S_y, \quad T_h = T_x \otimes T_y.$$

For implementation and analysis in FEM, it is important to consider discretization of spatial Galerkin-type integrals in the scheme. To this end, we choose $\{w_j\}_{j=1}^{r-1}$ and $\{\sigma_j\}_{j=1}^{r-1}$ to be respectively the weights and nodes of the $r-1$-point Gauss quadrature for $[0,1]$, and let

$$(v, z)_h = \sum_{k=1}^{N_x} \sum_{l=1}^{N_y} h_x^k h_y^l \sum_{m=1}^{r-1} \sum_{n=1}^{r-1} w_m w_n (vz)(x_{k,m}, y_{l,n}), \quad \|v\|_h^2 = (v, v)_h, \tag{2.1}$$

where $x_{k,m} = x_{k-1} + h_x^k \sigma_m$ and $y_{l,n} = y_{l-1} + h_y^l \sigma_n$.

For an integer $N_t \geq 2$, let $\Pi_t = \{t_n\}_{n=0}^{N_t}$ be a partition of $[0, T]$ such that $t_n = n\tau$, where $\tau = T/N_t$. For a function ϕ defined on Π_t, we use the following notation throughout the paper:

$$\phi^n = \phi(t_n), \quad \partial_t \phi^n = \phi^{n+1} - \phi^n, \quad \tilde{\partial}_t \phi^n = \frac{\phi^{n+1} - \phi^{n-1}}{2\tau}, \quad \partial_t^2 \phi^n = \phi^{n+1} - 2\phi^n + \phi^{n-1}. \tag{2.2}$$

The Laplace modified ADI-QPG scheme for (1.1)–(1.4) is: compute $U : \Pi_t \to S_h$ such that, for $n = 1, \dots, N_t - 1$

$$\left(\tilde{\partial}_t U^n, v\right)_h + (LU^n, v)_h - \lambda(\partial_t^2 \Delta U^n, v)_h + 2\lambda^2 \tau (\partial_t^2 U^n_{xxyy}, v)_h = (f^n, v)_h, \ v \in T_h, \tag{2.3}$$

where λ is a parameter.

For computation of $U^{n+1}, n = 1, \dots, N_t - 1$ using the three-time level scheme (2.3), we require U^0 and U^1. We compute $U^0, U^1 \in S_h$ by solving the following fully discrete linear system, with the given data $g, f^0 := f(x, y, 0)$ in (1.1)–(1.4):

$$(LU^0, v)_h = (Lg, v)_h, \qquad (LU^1, v)_h = (L[g + \tau(f^0 - Lg)], v)_h, \quad v \in T_h. \tag{2.4}$$

It is useful to note that the above scheme does not use the divergence form assumption required in standard variational forms used in FEM. We prove and demonstrate in Sections 4 and 5, respectively, that solutions of the linear systems (2.3)–(2.4) yield approximate solutions that converge with second order in time and optimal order in space. For implementation of the above scheme, it is sufficient that the coefficient functions $a, \mathbf{b}, c$ and the source term f^n be defined at the quadrature points and are not required to be continuous on $\overline{\Omega}$.

Next we show that the perturbation (third and fourth) terms in the left hand side of (2.3) are crucial to reduce the complexity of solving a collection of discretized two-dimensional spatial problems (to compute $U^{n+1}, \ n = 1, \cdots, N_t - 1$) to that of a collection of systems in x and y directions separately. To this end, let

$$D^x = (r-2)N_x + 1 = \dim S_x = \dim T_x \qquad \text{and} \qquad D^y = (r-2)N_y + 1 = \dim S_y = \dim T_y.$$

Let $\{\phi_p^x\}_{p=1}^{D^x}$, $\{\phi_q^y\}_{q=1}^{D^y}$ and $\{\psi_i^x\}_{i=1}^{D^x}$, $\{\psi_j^y\}_{j=1}^{D^y}$ be finite element bases for S_x, S_y and T_x, T_y, respectively.

After solving for U^0 and U^1, the main task, for each $n = 2, \dots, N_t$, is to compute the unknown coefficients in the approximate solution

$$U^n(x, y) = \sum_{p=1}^{D^x} \sum_{q=1}^{D^y} u_{p,q}^n \phi_p^x(x) \phi_q^y(y).$$

The unknown coefficients are collected as vectors

$$\mathbf{u}^n = [u_{1,1}^n, \dots, u_{1,D^y}^n, \dots, u_{D^x,1}^n, \dots, u_{D^x,D^y}^n]^T, \qquad n = 2, \dots, N_t. \tag{2.5}$$

For each $n = 1, \dots, N_t - 1$, having computed the vectors $\mathbf{u}^{n-1}$ and $\mathbf{u}^n$, using (2.2)–(2.3), it is easy to see that the unknown vector $\mathbf{u}^{n+1}$ in $\mathbf{z}^{n+1} := \mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}$ satisfy the linear system

$$(2\tau)^{-1} B^x \otimes B^y (\mathbf{u}^{n+1} - \mathbf{u}^{n-1}) - \lambda[A^x \otimes B^y + B^x \otimes A^y + 2\lambda^2 \tau A^x \otimes A^y] \mathbf{z}^{n+1} = \mathbf{g}^n, \tag{2.6}$$

where

$$\mathbf{g}^n = [g^n_{1,1}, \dots, g^n_{1,D^y}, \dots, g^n_{D^x,1}, \dots, g^n_{D^x,D^y}]^T,$$
$$g^n_{i,j} = (f^n, \psi^x_i(x)\psi^y_j(y))_h - (LU^n, \psi^x_i(x)\psi^y_j(y))_h,$$
$$A^x = (a_{i,p})^{D^x}_{i,p=1}, \quad a_{i,p} = \sum_{k=1}^{N_x} h^k_x \sum_{m=1}^{r-1} w_m((\phi^x_p)''\psi^x_i)(x_{k,m}),$$
$$B^x = (b_{j,q})^{D^x}_{j,q=1}, \quad b_{j,q} = \sum_{k=1}^{N_x} h^k_x \sum_{m=1}^{r-1} w_m(\phi^x_q\psi^y_j)(x_{k,m}),$$

and similarly, we define A^y and B^y in the y direction.

Let $\mathbf{v}^n := \mathbf{u}^n - \mathbf{u}^{n-1}$. Then it is easy to see that $\mathbf{z}^{n+1} = \mathbf{v}^{n+1} - \mathbf{v}^n$ and $\mathbf{u}^{n+1} - \mathbf{u}^{n-1} = \mathbf{z}^{n+1} + 2\mathbf{v}^n$. Hence, multiplying (2.6) by 2τ, (2.6) can be written as

$$[B^x \otimes B^y - 2\lambda\tau(A^x \otimes B^y + B^x \otimes A^y) + 4\lambda^2\tau^2 A^x \otimes A^y]\mathbf{z}^{n+1} = \mathbf{f}^n, \tag{2.7}$$

where $\mathbf{f}^n = 2\tau\mathbf{g}^n - 2B^x \otimes B^y\mathbf{v}^n$. Thus, using $\mathbf{u}^{n+1} = \mathbf{z}^{n+1} + \mathbf{v}^n + \mathbf{u}^n$, the main task is reduced to computing $\mathbf{z}^{n+1}$ for $n = 1, \cdots, N_t - 1$, by solving the tensor product of one-dimensional linear systems

$$(B^x - 2\lambda\tau A^x) \otimes (B^y - 2\lambda\tau A^y)\mathbf{z}^{n+1} = \mathbf{f}^n, \qquad n = 1, \cdots, N_t - 1. \tag{2.8}$$

It is sufficient, for example, to compute sparse LU factorization of $B^x - 2\lambda\tau A^x$ and $B^y - 2\lambda\tau A^y$ corresponding to the one-dimensional Helmholtz-type operators $I - 2\lambda\tau\frac{d^2}{dx^2}$ and $I - 2\lambda\tau\frac{d^2}{dy^2}$ once and then, for each $n = 1, \cdots, N_t - 1$, solve the resulting lower and upper triangular systems by simple substitution techniques.

3 Preliminary Results for Convergence Analysis

For a nonnegative integer m, the standard norm in the Sobolev space $H^m(\Omega)$ is denoted by $\|\cdot\|_m$. (Note that $H^0(\Omega) = L^2(\Omega)$ and that $\|\cdot\|_0 = \|\cdot\|_{L^2(\Omega)}$.) For nonnegative integers l and s, let $C^l([0,T], H^s(\Omega))$ be the space of all functions v defined on $\Omega \times [0,T]$ such that $v(\cdot,t) \in H^s(\Omega)$, $t \in [0,T]$, and $\|v\|_s \in C^l[0,T]$.

By $(\cdot,\cdot)_x$ and $\|\cdot\|_x$, we denote respectively the discrete inner product and the discrete norm on S_x, that is,

$$(v,z)_x = \sum_{k=1}^{N_x} h^k_x \sum_{m=1}^{r-1} w_m(vz)(x_{k,m}), \quad \|v\|^2_x = (v,v)_x; \tag{3.1}$$

the discrete inner product $(\cdot,\cdot)_y$ and the discrete norm $\|\cdot\|_y$ on S_y are defined in a similar way. For $v \in H^3(\Omega)$, we introduce

$$\|\nabla v_{xy}\|^2_0 = \|v_{xxy}\|^2_0 + \|v_{xyy}\|^2_0.$$

Let Q be the collection of the rectangles $\rho = (x_{k-1}, x_k) \times (y_{l-1}, y_l)$, $k = 1, \dots, N_x$, $l = 1, \dots, N_y$. As required in [3], in our convergence analysis we assume that (2.3) is solved on a quasi-uniform collection of partitions $\Pi_x \times \Pi_y$ corresponding to a sequence of values (N_x, N_y). Throughout the paper, C denotes a generic positive constant which may depend on r, but which is independent of h and τ.

We assume that the unique solution u of (1.1)–(1.3) is such that

$$u,\, u_t \in C^0([0,T], H^{r+3}(\Omega)), \quad u_{tt} \in C^0([0,T], H^{\max(r,4)+3}(\Omega)), \quad u_{ttt} \in C^0([0,T], H^4(\Omega)).$$

The standard optimal spatial regularity assumption in Galerkin-type FEM analysis *without quadrature* (using splines of degree $r \geq 3$ to obtain $O(h^{r+1})$ error in L_2 norm) is $H^{r+1}(\Omega)$. The above additional two order smoothness assumption is standard *with quadrature* error analysis. Numerical experiments in [3] demonstrate that such extra regularity assumptions may not be necessary in practice and also that our QPG scheme yields $O(h^{\min(r+1,k)})$ error in L_2 norm, where $u(\cdot,t) \in H^k(\Omega)$ for $k \geq 2$. The assumption $u(\cdot,t) \in H^{r+3}(\Omega)$ is also standard in OSC and NSC error analysis [2].

For our convergence analysis, in this and next section, we assume that

$$Lu = -\Delta u + c\, u, \tag{3.2}$$

and that the coefficient function c in (3.2) is in $C^{r+1}(\overline{\Omega})$ and satisfies

$$c(x,y) \geq 0, \quad (x,y) \in \overline{\Omega}. \tag{3.3}$$

Numerical experiments in Section 5 suggest that such smoothness assumptions may not be necessary. Due to the nature of quadrature error analysis, it is difficult to prove results without such assumptions.

Following [3], we introduce the comparison function $W : [0,T] \to S_h$ satisfying the fully discrete linear system

$$(LW, v)_h = (Lu, v)_h, \quad v \in T_h, \quad t \in [0,T], \tag{3.4}$$

Throughout the paper, we use the notation $\eta = u - W$. In Lemmas 3.1 and 3.3, we bound η and its derivatives.

Lemma 3.1 *If h is sufficiently small, then W exists and unique for each $t \in [0,T]$, and, for each $t \in [0,T]$ and $k = 0,1,2$,*

$$\|\eta\|_2 \leq Ch^{r-1} \|u\|_{r+2}, \tag{3.5}$$

$$\|\eta_{xxyy}\|_0 \leq Ch^{r-2} \|u\|_{r+2}, \quad \|\eta_{ttxxy}\|_0 \leq Ch^{r-1} \|u_{tt}\|_{r+2}. \tag{3.6}$$

Proof. The existence and uniqueness of W follow from (2.1)–(2.2) and Theorem 3.4 in [3]. The inequality in (3.5) follows from Corollary 3.5 in [3] and Theorem 4.1.

Next we prove the first inequality of (3.6) and the second one is obtained by using similar arguments. It follows from Theorem 3.3 in [8] that, for each $t \in [0,T]$, there exists a spline $z \in H^{r+2}(\Omega)$ such that

$$\|u - z\|_k \leq Ch^{r+2-k} \|u\|_{r+2}, \quad k = 0, \ldots, r+2. \tag{3.7}$$

Using $\eta = u - W$, the triangle and inverse inequalities, (3.7), and Theorem 3.4 in [3], we obtain

$$\begin{aligned}\|\eta_{xxyy}\|_0 &\le \|(u-z)_{xxyy}\|_0 + \|(z-W)_{xxyy}\|_0 \\ &\le Ch^{r-2}\|u\|_{r+2} + Ch^{-1}\left(\|z-u\|_3 + \|\eta_{xxy}\|_0\right) \le Ch^{r-2}\|u\|_{r+3}.\end{aligned}$$

□

The next lemma is Lemma 2.4 in [3]

Lemma 3.2 *We have $\|v\|_h \le C\|v\|_0$ for any spline v.*

Lemma 3.3 *If h is sufficiently small, then for each $t \in [0,T]$,*

$$\|\eta\|_h \le Ch^{r+1}\|u\|_{r+3}, \quad \|\eta_t\|_h \le Ch^{r+1}\|u_t\|_{r+3}, \tag{3.8}$$

$$\|\eta_{ttxxyy}\|_h \le Ch^{r-2}\|u_{tt}\|_{r_{\max}+3}, \quad r_{\max} = \max(r,4). \tag{3.9}$$

Proof. Both inequalities in (3.8) are proved in Lemma 2.5 of [4]. Next we prove (3.9). It follows from Theorem 3.3 in [8] that, for each $t \in [0,T]$, there exists a spline $z \in H^{r_{\max}+3}(\Omega)$ such that

$$\|u_{tt} - z\|_k \le Ch^{r_{\max}+3-k}\|u_{tt}\|_{r_{\max}+3}, \quad k = 0,\ldots,r_{\max}+3. \tag{3.10}$$

Using $\eta = u - W$ and the triangle inequality, we have

$$\|\eta_{ttxxyy}\|_h \le \|(u_{tt}-z)_{xxyy}\|_h + \|(z-W_{tt})_{xxyy}\|_h \equiv I_1 + I_2. \tag{3.11}$$

Introducing

$$I_3 = \|(u_{tt}-z)_{xxyy}\|_h^2 - \|(u_{tt}-z)_{xxyy}\|_0^2$$

and using (3.10), we obtain

$$I_1^2 = I_3 + \|(u_{tt}-z)_{xxyy}\|_0^2 \le I_3 + Ch^{2r-2}\|u_{tt}\|_{r_{\max}+3}^2. \tag{3.12}$$

Using the Bramble-Hilbert lemma (see Theorem 2 in [6]) for the quadrature formula, the Cauchy-Schwarz inequalities, and (3.10), we have

$$\begin{aligned} I_3 &\le Ch^3\sum_{\rho\in Q}\left(\left\|\frac{\partial^3}{\partial x^3}[(u_{tt}-z)_{xxyy}]^2\right\|_{L^1(\rho)} + \left\|\frac{\partial^3}{\partial y^3}[(u_{tt}-z)_{xxyy}]^2\right\|_{L^1(\rho)}\right) \\ &\le Ch^3\sum_{\rho\in Q}\left(\|(u_{tt}-z)_{xxyy}\|_{0,\rho}\|(u_{tt}-z)_{xxyy}\|_{3,\rho} + \|(u_{tt}-z)_{xxyy}\|_{1,\rho}\|(u_{tt}-z)_{xxyy}\|_{2,\rho}\right) \\ &\le Ch^3\left(\|(u_{tt}-z)_{xxyy}\|_0\|(u_{tt}-z)_{xxyy}\|_3 + \|(u_{tt}-z)_{xxyy}\|_1\|(u_{tt}-z)_{xxyy}\|_2\right) \\ &\le Ch^{2r-2}\|u_{tt}\|_{r_{\max}+3}^2. \end{aligned} \tag{3.13}$$

Hence, (3.12) and (3.13) give

$$I_1 \le Ch^{r-1}\|u_{tt}\|_{r_{\max}+3}. \tag{3.14}$$

Using Lemma 3.2, the inverse and triangle inequalities, (3.10), and (3.6), we obtain

$$\begin{aligned} I_2 &\le C \left\| (z - W_{tt})_{xxyy} \right\|_0 \le C h^{-1} \left\| (z - W_{tt})_{xxy} \right\|_0 \\ &\le C h^{-1} \left(\left\| (z - u_{tt})_{xxy} \right\|_0 + \left\| (u - W)_{ttxxy} \right\|_0 \right) \le C h^{r-2} \|u_{tt}\|_{r_{\max}+3}. \end{aligned} \tag{3.15}$$

Finally, (3.11), (3.14), and (3.15) imply (3.9). □

The next lemma is a part of Lemma 2.6 in [4].

Lemma 3.4 *If $z, v \in S_h$, then $(z, v_{xxyy})_h = (z_{yy}, v_{xx})_h$.*

The next lemma is Lemma 2.9 in [5].

Lemma 3.5 *If $z, v \in S_h$, then $(\Delta z, v_{xxyy})_h = (z_{xxyy}, \Delta v)_h$.*

The next lemma is a part of Lemma 2.7 in [4].

Lemma 3.6 *We have*

$$(v_{xx}, v_{yy})_h \ge C \|v_{xy}\|_0^2, \quad v \in S_h.$$

The next two lemmas follow respectively from Lemma 2.8 in [4] and Lemma 2.3 in [3].

Lemma 3.7 *We have*

$$\|\nabla v_{xy}\|_0^2 \le C(-\Delta v, v_{xxyy})_h, \quad v \in S_h.$$

Lemma 3.8 *We have*

$$\|v\|_2 \le C \|\nabla v_{xy}\|_0, \quad v \in S_h.$$

The following quadrature error estimate is proved in Lemma 2.6 [3].

Lemma 3.9 *Let integers s and q be such that $s \ge 2$, $q \ge 0$, and $s + q \le 2r - 2$. Then there exists a positive constant C independent of $\rho = (x_{k-1}, x_k) \times (y_{l-1}, y_l)$, $k = 1, \ldots, N_x$, $l = 1, \ldots, N_y$, such that for any $g \in H^s(\rho)$ and $v \in P_q \otimes P_q$,*

$$\left| \int_\rho g v \, dx \, dy - h_x^k h_y^l \sum_{m=1}^{r-1} \sum_{n=1}^{r-1} w_m w_n (gv)(x_{k,m}, y_{l,m}) \right| \le C h^s \left(\left\| \frac{\partial^s g}{\partial x^s} \right\|_{0,\rho} + \left\| \frac{\partial^s g}{\partial y^s} \right\|_{0,\rho} \right) \|v\|_{0,\rho}.$$

The next lemma follows easily from Lemma 2.10 in [5].

Lemma 3.10 *If $z \in H^2(\Omega)$ and $z = 0$ on $\partial\Omega$, then*

$$|(z, v_{xxyy})_h| \le C \|z\|_2 \|v_{xy}\|_0, \quad v \in S_h.$$

4 Convergence Analysis of ADI Scheme

In this section, $\xi^n = U^n - W^n$, $n = 0, \dots, N_t$, where $\{U^n\}_{n=2}^{N_t}$ and W are defined by (2.3) and (3.4) with elliptic operator L as in (3.2).

Lemma 4.1 *For $n = 1, \dots, N_t - 1$, we have*

$$\left(\tilde{\partial}_t \xi^n, v\right)_h + (L\xi^n, v)_h - \lambda(\partial_t^2 \Delta \xi^n, v)_h + 2\lambda^2 \tau(\partial_t^2 \xi^n_{xxyy}, v)_h = (u_t^n - \tilde{\partial}_t u^n, v)_h + (\tilde{\partial}_t \eta^n, v)_h$$
$$+ \lambda(\partial_t^2 \Delta u^n, v)_h - \lambda(c\partial_t^2 \eta^n, v)_h - 2\lambda^2\tau(\partial_t^2 u^n_{xxyy}, v)_h + 2\lambda^2 \tau \left(\partial_t^2 \eta^n_{xxyy}, v\right)_h, \quad v \in T_h. \quad (4.1)$$

Proof. For $v \in T_h$, from $\xi^n = U^n - W^n$, (2.3), $\eta = u - W$, (3.4), and (3.2), we obtain

$$\begin{aligned}
&\left(\tilde{\partial}_t \xi^n, v\right)_h + (L\xi^n, v)_h - \lambda(\partial_t^2 \Delta \xi^n, v)_h + 2\lambda^2 \tau(\partial_t^2 \xi^n_{xxyy}, v)_h = \\
&\left(\tilde{\partial}_t U^n, v\right)_h + (LU^n, v)_h - \lambda(\partial_t^2 \Delta U^n, v)_h + 2\lambda^2 \tau(\partial_t^2 U^n_{xxyy}, v)_h \\
&- \left(\tilde{\partial}_t W^n, v\right)_h - (LW^n, v)_h + \lambda(\partial_t^2 \Delta W^n, v)_h - 2\lambda^2\tau(\partial_t^2 W^n_{xxyy}, v)_h \\
&= (f^n, v)_h - \left(\tilde{\partial}_t u^n, v\right)_h + \left(\tilde{\partial}_t \eta^n, v\right)_h - (Lu^n, v)_h + \lambda(\partial_t^2 \Delta u^n, v)_h - \lambda(c\partial_t^2 \eta^n, v)_h \\
&- 2\lambda^2 \tau(\partial_t^2 u^n_{xxyy}, v)_h + 2\lambda^2\tau(\partial_t^2 \eta^n_{xxyy}, v)_h,
\end{aligned}$$

and hence (4.1) follows from (1.1). □

In the remainder of this section, $\sum_{i=j-1}^{j}{}^{*}$ denotes the usual sum except that for $j = 2$ it reduces to $\sum_{i=j}$. To simplify proof of our main result, we first prove two auxiliary lemmas.

Lemma 4.2 *If $v^n \in S_h$, $n = 0, \dots, N_t$, then*

$$(\partial_t^2 \Delta u^n, \tilde{\partial}_t v^n_{xxyy})_h \le C\tau \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|_4 \, ds \left\|\tilde{\partial}_t v^n_{xy}\right\|_0 + J_n, \quad n = 1, \dots, N_t - 1, \quad (4.2)$$

where, for $2 \le j \le N_t$,

$$\begin{aligned}
\tau \sum_{n=1}^{j-1} J_n &\le C\tau^2 \sum_{n=2}^{j-2} \int_{t_{n-2}}^{t_{n+2}} \|u_{ttt}\|_3 \, ds \left\|\nabla v^n_{xy}\right\|_0 \\
&+ C\tau \sum_{i=j-1}^{j}{}^{*} \int_{t_{i-2}}^{t_i} \|u_{tt}\|_3 \, ds \left\|\nabla v^i_{xy}\right\|_0 + C\tau \sum_{i=0}^{1} \int_{t_i}^{t_{i+2}} \|u_{tt}\|_3 \, ds \left\|\nabla v^i_{xy}\right\|_0. \quad (4.3)
\end{aligned}$$

Proof. With $z^n = \tilde{\partial}_t v^n$, we have

$$(\partial_t^2 \Delta u^n, \tilde{\partial}_t v^n_{xxyy})_h = (\partial_t^2 \Delta u^n, z^n_{xxyy})_h = I_n + J_n, \quad (4.4)$$

where

$$I_n = \left(\partial_t^2 \Delta u^n, z^n_{xxyy}\right)_h - \left(\partial_t^2 \Delta u^n, z^n_{xxyy}\right),$$

$$J_n = \left(\partial_t^2 \Delta u^n, z^n_{xxyy}\right). \tag{4.5}$$

Using Lemma 3.9, the inverse and Cauchy-Schwarz inequalities, we have

$$\begin{aligned} I_n &\le Ch^2 \sum_{\rho\in Q} \left(\left\| \frac{\partial^2}{\partial x^2} \partial_t^2 \Delta u^n \right\|_{0,\rho} + \left\| \frac{\partial^2}{\partial y^2} \partial_t^2 \Delta u^n \right\|_{0,\rho} \right) \left\| z^n_{xxyy} \right\|_{0,\rho} \\ &\le C \left\| \partial_t^2 \Delta u^n \right\|_2 \left\| z^n_{xy} \right\|_0 . \end{aligned} \tag{4.6}$$

Since

$$\partial_t^2 u^n = \int_{t_{n-1}}^{t_{n+1}} (\tau - |s - t_n|) u_{tt} \, ds, \tag{4.7}$$

using (4.6), we obtain

$$I_n \le C\tau \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|_4 \; ds \left\| z^n_{xy} \right\|_0 . \tag{4.8}$$

Hence, (4.4), (4.8), and $z^n = \tilde{\partial}_t v^n$ give (4.2) with J_n as in (4.5).

To show (4.3), we first use (4.5), integration by parts, $u^n_{xx}(x,\alpha) = u^n_{yy}(\alpha, y) = 0$, $x, y \in [0,1]$, $\alpha = 0, 1$, to obtain

$$J_n = \left(\partial_t^2 u^n_{xx}, z^n_{xxyy}\right) + \left(\partial_t^2 u^n_{yy}, z^n_{xxyy}\right) = -\left(\partial_t^2 u^n_{xxy}, z^n_{xxy}\right) - \left(\partial_t^2 u^n_{xyy}, z^n_{xyy}\right),$$

which gives, for $2 \le j \le N_t$,

$$\tau \sum_{n=1}^{j-1} J_n = -\tau \sum_{n=1}^{j-1} \left(\partial_t^2 u^n_{xxy}, z^n_{xxy}\right) - \tau \sum_{n=1}^{j-1} \left(\partial_t^2 u^n_{xyy}, z^n_{xyy}\right) \equiv -K_1 - K_2. \tag{4.9}$$

Using $z^n = \tilde{\partial}_t v^n$, we have

$$\begin{aligned} 2K_1 &= \sum_{n=1}^{j-1} \left(\partial_t^2 u^n_{xxy}, v^{n+1}_{xxy}\right) - \sum_{n=1}^{j-1} \left(\partial_t^2 u^n_{xxy}, v^{n-1}_{xxy}\right) = \sum_{n=2}^{j} \left(\partial_t^2 u^{n-1}_{xxy}, v^n_{xxy}\right) - \sum_{n=0}^{j-2} \left(\partial_t^2 u^{n+1}_{xxy}, v^n_{xxy}\right) \\ &= \sum_{n=2}^{j-2} \left(\partial_t^2 u^{n-1}_{xxy}, v^n_{xxy}\right) + \sum_{n=j-1}^{j}{}^{*} \left(\partial_t^2 u^{n-1}_{xxy}, v^n_{xxy}\right) - \sum_{n=0}^{\min(1,j-2)} \left(\partial_t^2 u^{n+1}_{xxy}, v^n_{xxy}\right) - \sum_{n=2}^{j-2} \left(\partial_t^2 u^{n+1}_{xxy}, v^n_{xxy}\right) \\ &= \sum_{n=2}^{j-2} \left(\partial_t^3 u^n_{xxy}, v^n_{xxy}\right) + \sum_{i=j-1}^{j}{}^{*} \left(\partial_t^2 u^{i-1}_{xxy}, v^i_{xxy}\right) - \sum_{i=0}^{\min(1,j-2)} \left(\partial_t^2 u^{i+1}_{xxy}, v^i_{xxy}\right), \end{aligned} \tag{4.10}$$

where

$$\partial_t^3 \phi^n = \partial_t^2 \phi^{n-1} - \partial_t^2 \phi^{n+1} = -\phi^{n+2} + 2\phi^{n+1} - 2\phi^{n-1} + \phi^{n-2}. \tag{4.11}$$

It follows from the Peano Kernel Theorem (see, for example, Theorem 1.5 in [8]) that

$$\partial_t^3 u^n = \int_{t_{n-2}}^{t_{n+2}} K(s) u_{ttt} \, ds, \quad |K(s)| \le C\tau^2, \quad s \in [t_{n-2}, t_{n+2}]. \tag{4.12}$$

Using (4.10), the Cauchy-Schwarz inequality, (4.12), and (4.7), we obtain

$$
\begin{aligned}
-2K_1 \leq & \sum_{n=2}^{j-2} \left\|\partial_t^3 u_{xxy}^n\right\|_0 \left\|v_{xxy}^n\right\|_0 + \sum_{i=j-1}^{j}{}^{*} \left\|\partial_t^2 u_{xxy}^{i-1}\right\|_0 \left\|v_{xxy}^i\right\|_0 + \sum_{i=0}^{1} \left\|\partial_t^2 u_{xxy}^{i+1}\right\|_0 \left\|v_{xxy}^i\right\|_0 \\
\leq & \, C\tau^2 \sum_{n=2}^{j-2} \int_{t_{n-2}}^{t_{n+2}} \|u_{ttt}\|_3 \, ds \left\|\nabla v_{xy}^n\right\|_0 \\
& + C\tau \sum_{i=j-1}^{j}{}^{*} \int_{t_{i-2}}^{t_i} \|u_{tt}\|_3 \, ds \left\|\nabla v_{xy}^i\right\|_0 + C\tau \sum_{i=0}^{1} \int_{t_i}^{t_{i+2}} \|u_{tt}\|_3 \, ds \left\|\nabla v_{xy}^i\right\|_0 . \qquad (4.13)
\end{aligned}
$$

Therefore, (4.3) follows from (4.9), (4.13), and a similar bound for $-2K_2$. □

Lemma 4.3 *Assume that $\lambda > 1/4$, $v^n \in S_h$, $n = 0, \ldots, N_t$, and that, for $2 \leq j \leq N_t$,*

$$
S_1 = \sum_{n=1}^{j-1} \left(\Delta v^n, v_{xxyy}^{n+1} - v_{xxyy}^{n-1}\right)_h, \quad S_2 = \sum_{n=1}^{j-1} \left[(\Delta v^{n-1}, v_{xxyy}^{n-1})_h - (\Delta v^{n+1}, v_{xxyy}^{n+1})_h\right].
$$

Then there exists $\delta \in (0, 1]$ such that

$$
(2\lambda - 1)S_1 + \lambda S_2 \geq \lambda\delta \sum_{i=j-1}^{j} \left(-\Delta v^i, v_{xxyy}^i\right)_h - F(v^0, v^1),
$$

where

$$
F(v^0, v^1) = (2\lambda - 1)\left(\Delta v^1, v_{xxyy}^0\right)_h - \lambda \sum_{i=0}^{1} (\Delta v^i, v_{xxyy}^i)_h. \qquad (4.14)
$$

Proof. Using Lemma 3.5, we have

$$
\begin{aligned}
S_1 &= \sum_{n=1}^{j-1} (\Delta v^n, v_{xxyy}^{n+1})_h - \sum_{n=0}^{j-2} (\Delta v^{n+1}, v_{xxyy}^n)_h = \sum_{n=1}^{j-1} (\Delta v^n, v_{xxyy}^{n+1})_h - \sum_{n=0}^{j-2} (v_{xxyy}^{n+1}, \Delta v^n)_h \\
&= \left(\Delta v^{j-1}, v_{xxyy}^j\right)_h - \left(\Delta v^1, v_{xxyy}^0\right)_h. \qquad (4.15)
\end{aligned}
$$

Clearly,

$$
S_2 = \sum_{n=0}^{j-2} (\Delta v^n, v_{xxyy}^n)_h - \sum_{n=2}^{j} (\Delta v^n, v_{xxyy}^n)_h = \sum_{i=0}^{1} (\Delta v^i, v_{xxyy}^i)_h - \sum_{i=j-1}^{j} (\Delta v^i, v_{xxyy}^i)_h. \qquad (4.16)
$$

Using (4.15) and (4.16), we have

$$
(2\lambda - 1)S_1 + \lambda S_2 = \lambda \left[(2 - \lambda^{-1}) \left(\Delta v^{j-1}, v_{xxyy}^j\right)_h - \sum_{i=j-1}^{j} (\Delta v^i, v_{xxyy}^i)_h \right] - F(v^0, v^1), \qquad (4.17)
$$

where F is defined in (4.14).

To bound the right-hand side of (4.17), we first note that

$$
(2 - \lambda^{-1}) \left(\Delta v^{j-1}, v_{xxyy}^j\right)_h = \left(2 - \lambda^{-1}\right) \left[\left(v_{xx}^{j-1}, v_{xxyy}^j\right)_h + \left(v_{yy}^{j-1}, v_{xxyy}^j\right)_h \right]. \qquad (4.18)
$$

Using (2.1), (3.1), Lemma 3.1 of [7], the triangle and Cauchy-Schwarz inequalities, we obtain

$$
\begin{aligned}
-\left(v_{xx}^{j-1}, v_{xxyy}^{j}\right)_h &= -\sum_{k=1}^{N_x} h_x^k \sum_{m=1}^{r-1} w_m \left(v_{xx}^{j-1}(x_{k,m},\cdot), v_{xxyy}^{j}(x_{k,m},\cdot)\right)_y \\
&= \int_0^1 \left(v_{xxy}^{j-1}(\cdot,y), v_{xxy}^{j}(\cdot,y)\right)_x dy \\
&+ C_r \sum_{l=1}^{N_y} (h_y^l)^{2r-1} \left(\frac{\partial^r}{\partial y^r} v_{xx}^{j-1}(\cdot,\tilde{y}_l), \frac{\partial^r}{\partial y^r} v_{xx}^{j}(\cdot,\tilde{y}_l)\right)_x \\
&\le (J_1+J_2)/2, \qquad (4.19)
\end{aligned}
$$

where the positive constant C_r depends only on r, $\tilde{y}_l = (y_{l-1}+y_l)/2$, and

$$
J_1 = \sum_{i=j-1}^{j} \int_0^1 \left\|v_{xxy}^i(\cdot,y)\right\|_x^2 dy, \quad J_2 = C_r \sum_{i=j-1}^{j} \sum_{l=1}^{N_y} (h_y^l)^{2r-1} \left\|\frac{\partial^r}{\partial y^r} v_{xx}^i(\cdot,\tilde{y}_l)\right\|_x^2. \qquad (4.20)
$$

In a similar way, we obtain

$$
\left|\left(v_{yy}^{j-1}, v_{xxyy}^{j}\right)_h\right| \le (K_1+K_2)/2, \qquad (4.21)
$$

where

$$
K_1 = \sum_{i=j-1}^{j} \int_0^1 \left(\left\|v_{xyy}^i(x,\cdot)\right\|_y\right)^2 dx, \quad K_2 = C_r \sum_{i=j-1}^{j} \sum_{k=1}^{N_x} (h_x^l)^{2r-1} \left\|\frac{\partial^r}{\partial x^r} v_{yy}^i(\tilde{x}_k,\cdot)\right\|_y^2, \qquad (4.22)
$$

and $\tilde{x}_k = (x_{k-1}+x_k)/2$. The assumption $\lambda > 1/4$ implies $-2 < 2-\lambda^{-1} < 2$ and hence $|2-\lambda^{-1}|/2 < 1$. Thus, there exists $\delta \in (0,1]$, such that

$$
\left|2-\lambda^{-1}\right|/2 = 1-\delta. \qquad (4.23)
$$

Therefore, using (4.18)–(4.23), we have

$$
\left|(2-\lambda^{-1})\left(\Delta v^{j-1}, v_{xxyy}^{j}\right)_h\right| \le (1-\delta)(J_1+J_2+K_1+K_2)
$$

which gives

$$
(2-\lambda^{-1})\left(\Delta v^{j-1}, v_{xxyy}^{j}\right)_h \ge (\delta-1)(J_1+J_2+K_1+K_2). \qquad (4.24)
$$

Using again (2.1), (3.1), and Lemma 3.1 of [7], we have

$$
\begin{aligned}
-\sum_{i=j-1}^{j} (\Delta v^i, v_{xxyy}^i)_h &= -\sum_{i=j-1}^{j} \sum_{k=1}^{N_x} h_x^k \sum_{m=1}^{r-1} w_m \left(v_{xx}^i(x_{k,m},\cdot), v_{xxyy}^i(x_{k,m},\cdot)\right)_y \\
&- \sum_{i=j-1}^{j} \sum_{l=1}^{N_y} h_y^l \sum_{n=1}^{r-1} w_n \left(v_{yy}^i(\cdot,y_{l,n}), v_{xxyy}^i(\cdot,y_{l,n})\right)_x \\
&= J_1+J_2+K_1+K_2, \qquad (4.25)
\end{aligned}
$$

where J_1, J_2, K_1, K_2 are defined in (4.20) and (4.22). Equations (4.24) and (4.25) give

$$(2-\lambda^{-1})(\Delta v^{j-1}, v^j_{xxyy})_h - \sum_{i=j-1}^{j} (\Delta v^i, v^i_{xxyy})_h \geq \delta \sum_{i=j-1}^{j} \left(-\Delta v^i, v^i_{xxyy}\right)_h . \tag{4.26}$$

Hence the desired inequality follows from (4.17) and (4.26). □

Theorem 4.4 *If $\lambda > 1/4$, h is sufficiently small, and $\tau \leq Ch$, then*

$$\max_{0\leq n\leq N_t} (-\Delta\xi^n, \xi^n_{xxyy})_h \leq C\Bigg\{G(\xi^0,\xi^1)+\tau^4\left(\max_{0\leq t\leq T} \|u_{tt}\|^2_{r_{\max}+3} + \max_{0\leq t\leq T} \|u_{ttt}\|^2_4\right) + h^{2r-2}\sum_{i=0}^{1} \max_{0\leq t\leq T} \left\|\frac{\partial^i u}{\partial t^i}\right\|^2_{r+3}\Bigg\},$$

where $r_{\max}$ is defined in (3.9) and

$$G(\xi^0,\xi^1) = \tau \left\|\partial_t \xi^0_{xxyy}\right\|^2_h + |\left(\Delta\xi^1, \xi^0_{xxyy}\right)_h| + \sum_{i=0}^{1} |(\Delta\xi^i, \xi^i_{xxyy})_h|, \tag{4.27}$$

Proof. Taking $v = \tilde{\partial}_t \xi^n_{xxyy}$ in (4.1) and using (3.2), we obtain

$$\left(\tilde{\partial}_t\xi^n, \tilde{\partial}_t\xi^n_{xxyy}\right)_h + (2\lambda-1)\left(\Delta\xi^n, \tilde{\partial}_t\xi^n_{xxyy}\right)_h - \lambda\left(\Delta(\xi^{n+1}+\xi^{n-1}), \tilde{\partial}_t\xi^n_{xxyy}\right)_h + 2\lambda^2\tau\left(\partial^2_t\xi^n_{xxyy}, \tilde{\partial}_t\xi^n_{xxyy}\right)_h = \sum_{i=1}^{7} I^n_i, \quad n = 1,\ldots,N_t-1, \tag{4.28}$$

where

$$I^n_1 = \left(u^n_t - \tilde{\partial}_t u^n, \tilde{\partial}_t\xi^n_{xxyy}\right)_h, \quad I^n_2 = \left(\tilde{\partial}_t\eta^n, \tilde{\partial}_t\xi^n_{xxyy}\right)_h, \quad I^n_3 = \lambda\left(\partial^2_t\Delta u^n, \tilde{\partial}_t\xi^n_{xxyy}\right)_h, \tag{4.29}$$

$$I^n_4 = -\lambda\left(c\partial^2_t\eta^n, \tilde{\partial}_t\xi^n_{xxyy}\right)_h, \quad I^n_5 = -2\lambda^2\tau\left(\partial^2_t u^n_{xxyy}, \tilde{\partial}_t\xi^n_{xxyy}\right)_h, \tag{4.30}$$

$$I^n_6 = 2\lambda^2\tau\left(\partial^2_t\eta^n_{xxyy}, \tilde{\partial}_t\xi^n_{xxyy}\right)_h, \quad I^n_7 = -\left(c\xi^n, \tilde{\partial}_t\xi^n_{xxyy}\right)_h. \tag{4.31}$$

To bound the first term on the left-hand side of (4.28), we use Lemmas 3.4 and 3.6, to obtain

$$\left(\tilde{\partial}_t\xi^n, \tilde{\partial}_t\xi^n_{xxyy}\right)_h = \left(\tilde{\partial}_t\xi^n_{xx}, \tilde{\partial}_t\xi^n_{yy}\right)_h \geq C\left\|\tilde{\partial}_t\xi^n_{xy}\right\|^2_0. \tag{4.32}$$

For the third term on the left-hand side of (4.28), we use Lemma 3.5, to obtain

$$\begin{aligned}
\lambda\left(\Delta\left(\xi^{n+1}+\xi^{n-1}\right), \tilde{\partial}_t\xi^n_{xxyy}\right)_h &= \frac{\lambda}{2\tau}\left(\Delta\left(\xi^{n+1}+\xi^{n-1}\right), \xi^{n+1}_{xxyy} - \xi^{n-1}_{xxyy}\right)_h \\
&= \frac{\lambda}{2\tau}\left[\left(\Delta\xi^{n+1}, \xi^{n+1}_{xxyy}\right)_h - \left(\Delta\xi^{n+1}, \xi^{n-1}_{xxyy}\right)_h + \left(\Delta\xi^{n-1}, \xi^{n+1}_{xxyy}\right)_h - \left(\Delta\xi^{n-1}, \xi^{n-1}_{xxyy}\right)_h\right] \\
&= \frac{\lambda}{2\tau}\left[\left(\Delta\xi^{n+1}, \xi^{n+1}_{xxyy}\right)_h - \left(\Delta\xi^{n-1}, \xi^{n-1}_{xxyy}\right)_h\right].
\end{aligned} \tag{4.33}$$

For the fourth term on the left-hand side of (4.28), we have

$$
\begin{aligned}
2\lambda^2\tau\left(\partial_t^2\xi_{xxyy}^n, \tilde{\partial}_t\xi_{xxyy}^n\right)_h &= \lambda^2\left(\partial_t\xi_{xxyy}^n - \partial_t\xi_{xxyy}^{n-1}, \partial_t\xi_{xxyy}^n + \partial_t\xi_{xxyy}^{n-1}\right)_h \\
&= \lambda^2\left(\|\partial_t\xi_{xxyy}^n\|_h^2 - \|\partial_t\xi_{xxyy}^{n-1}\|_h^2\right). \qquad (4.34)
\end{aligned}
$$

Next we bound terms on the right-hand side of (4.28). Since

$$
u_t^n - \tilde{\partial}_t u^n = -\frac{1}{4\tau}\left[\int_{t_{n-1}}^{t_n}(s-t_{n-1})^2 u_{ttt}\, ds + \int_{t_n}^{t_{n+1}}(s-t_{n+1})^2 u_{ttt}\, ds\right],
$$

using (4.29), Lemma 3.10, the ϵ and Cauchy-Schwarz inequalities, we have

$$
I_1^n \le C\tau\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|_2\, ds\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0 \le \epsilon_1\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0^2 + C(\epsilon_1)\tau^3\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|_2^2\, ds. \qquad (4.35)
$$

Next we bound I_2^n given in (4.29). Using $\tilde{\partial}_t\eta^n = (2\tau)^{-1}\int_{t_{n-1}}^{t_{n+1}}\eta_t\, ds$ and (3.8), we have

$$
\|\tilde{\partial}_t\eta^n\|_h \le \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t\|_h\, ds \le C\tau^{-1}h^{r+1}\int_{t_{n-1}}^{t_{n+1}}\|u_t\|_{r+3}\, ds. \qquad (4.36)
$$

Using Lemma 3.2 and the inverse inequality, we have

$$
\left\|\tilde{\partial}_t\xi_{xxyy}^n\right\|_h \le C\left\|\tilde{\partial}_t\xi_{xxyy}^n\right\|_0 \le Ch^{-2}\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0. \qquad (4.37)
$$

Hence (4.29), the Cauchy-Schwarz inequality, (4.36), (4.37), and the ϵ and Cauchy-Schwarz inequalities give

$$
\begin{aligned}
I_2^n &\le \|\tilde{\partial}_t\eta^n\|_h\left\|\tilde{\partial}_t\xi_{xxyy}^n\right\|_h \le C\tau^{-1}h^{r-1}\int_{t_{n-1}}^{t_{n+1}}\|u_t\|_{r+3}\, ds\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0 \\
&\le \epsilon_2\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0^2 + C(\epsilon_2)\tau^{-1}h^{2r-2}\int_{t_{n-1}}^{t_{n+1}}\|u_t\|_{r+3}^2\, ds. \qquad (4.38)
\end{aligned}
$$

Using (4.29), Lemma 4.2, the ϵ and Cauchy-Schwarz inequalities, we have

$$
I_3^n \le \epsilon_3\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0^2 + C(\epsilon_3)\tau^3\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_4^2\, ds + CJ_n, \qquad (4.39)
$$

where for $2 \le j \le N_t$,

$$
\begin{aligned}
\tau\sum_{n=1}^{j-1}J_n &\le C\tau^2\sum_{n=2}^{j-2}\int_{t_{n-2}}^{t_{n+2}}\|u_{ttt}\|_3\, ds\left\|\nabla\xi_{xy}^n\right\|_0 \\
&+ C\tau\sum_{i=j-1}^{j}{}^{*}\int_{t_{i-2}}^{t_i}\|u_{tt}\|_3\, ds\left\|\nabla\xi_{xy}^i\right\|_0 + C\tau\sum_{i=0}^{1}\int_{t_i}^{t_{i+2}}\|u_{tt}\|_3\, ds\left\|\nabla\xi_{xy}^i\right\|_0. \qquad (4.40)
\end{aligned}
$$

Next we bound I_4^n defined in (4.30). Using the triangle inequality and (3.8), we have

$$\|c\partial_t^2\eta^n\|_h \leq C\sum_{i=n-1}^{n+1}\|\eta^i\|_h \leq Ch^{r+1}\sum_{i=n-1}^{n+1}\|u^i\|_{r+3}. \tag{4.41}$$

Hence (4.30), the Cauchy-Schwarz inequality, (4.41), (4.37), and the ϵ and Cauchy-Schwarz inequalities give

$$\begin{aligned} I_4^n &\leq C\|c\partial_t^2\eta^n\|_h \left\|\tilde{\partial}_t\xi_{xxyy}^n\right\|_h \leq Ch^{r-1}\sum_{i=n-1}^{n+1}\|u^i\|_{r+3}\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0 \\ &\leq \epsilon_4\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0^2 + C(\epsilon_4)h^{2r-2}\sum_{i=n-1}^{n+1}\left\|u^i\right\|_{r+3}^2. \end{aligned} \tag{4.42}$$

Next, we bound I_6^n of (4.31). Using

$$\partial_t^2\eta^n = \int_{t_{n-1}}^{t_{n+1}}(\tau - |s-t_n|)\eta_{tt}\,ds$$

and (3.9), we obtain

$$\left\|\partial_t^2\eta_{xxyy}^n\right\|_h \leq C\tau\int_{t_{n-1}}^{t_{n+1}}\|\eta_{ttxxyy}\|_h\,ds \leq C\tau h^{r-2}\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_{r_{\max}+3}\,ds. \tag{4.43}$$

Hence, (4.31), the Cauchy-Schwarz inequality, (4.43), (4.37), $r \geq 3$, $\tau \leq Ch$, and the ϵ and Cauchy-Schwarz inequalities give

$$\begin{aligned} I_6^n &\leq C\tau\left\|\partial_t^2\eta_{xxyy}^n\right\|_h\left\|\tilde{\partial}_t\xi_{xxyy}^n\right\|_h \leq C\tau^2h^{-1}\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_{r_{\max}+3}\,ds\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0 \\ &\leq \epsilon_6\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0^2 + C(\epsilon_6)\tau^3\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_{r_{\max}+3}^2\,ds. \end{aligned} \tag{4.44}$$

Using (4.31), Lemmas 3.10 and 3.8, and the ϵ inequality, we have

$$I_7^n \leq C\|\xi^n\|_2\|\tilde{\partial}_t\xi_{xy}^n\|_0 \leq C\|\nabla\xi_{xy}^n\|_0\|\tilde{\partial}_t\xi_{xy}^n\|_0 \leq \epsilon_7\left\|\tilde{\partial}_t\xi_{xy}^n\right\|_0^2 + C(\epsilon_7)\|\nabla\xi_{xy}^n\|_0^2. \tag{4.45}$$

Using (4.28), (4.32)–(4.34), (4.35), (4.38), (4.39), (4.42), (4.44), (4.45), taking ϵ_i, $i = 1,2,3,4,6,7$, sufficiently small, and multiplying through by 2τ, we obtain, for $n = 1,\ldots,N_t-1$,

$$\begin{aligned} &(2\lambda-1)\left(\Delta\xi^n, \xi_{xxyy}^{n+1} - \xi_{xxyy}^{n-1}\right)_h + \lambda\left[(\Delta\xi^{n-1}, \xi_{xxyy}^{n-1})_h - (\Delta\xi^{n+1}, \xi_{xxyy}^{n+1})_h\right] \\ &+ 2\lambda^2\tau\left(\|\partial_t\xi_{xxyy}^n\|_h^2 - \|\partial_t\xi_{xxyy}^{n-1}\|_h^2\right) \leq C\left\{\tau^4\alpha_n(u) + h^{2r-2}\beta_n(u) + \tau(J_n + I_5^n + \|\nabla\xi_{xy}^n\|_0^2)\right\}, \end{aligned} \tag{4.46}$$

where

$$\alpha_n(u) = \int_{t_{n-1}}^{t_{n+1}} \left(\|u_{ttt}\|_2^2 + \|u_{tt}\|_{r_{\max}+3}^2\right) ds,\ \beta_n(u) = \int_{t_{n-1}}^{t_{n+1}} \|u_t\|_{r+3}^2 \, ds + \tau \sum_{i=n-1}^{n+1} \left\|u^i\right\|_{r+3}^2. \tag{4.47}$$

Summing both sides of (4.46) for $n = 1, \dots, j-1$, where $2 \le j \le N_t$, and using (4.47), we obtain

$$\begin{aligned}(2\lambda - 1)S_1 + \lambda S_2 \le C \Bigg\{ &\tau^4 C_1^j(u) + h^{2r-2} C_2^j(u) + \tau \sum_{n=1}^{j-1} J_n + \tau \sum_{n=1}^{j-1} I_5^n \\ &+ \tau \sum_{n=1}^{j-1} \|\nabla \xi_{xy}^n\|_0^2 + \tau \|\partial_t \xi_{xxyy}^0\|_h^2 \Bigg\}, \quad j = 2, \dots, N_t,\end{aligned} \tag{4.48}$$

where

$$S_1 = \sum_{n=1}^{j-1} \left(\Delta \xi^n, \xi_{xxyy}^{n+1} - \xi_{xxyy}^{n-1}\right)_h, \quad S_2 = \sum_{n=1}^{j-1} \left[(\Delta \xi^{n-1}, \xi_{xxyy}^{n-1})_h - (\Delta \xi^{n+1}, \xi_{xxyy}^{n+1})_h\right],$$

and

$$C_1^j(u) = \int_0^{t_j} \|u_{ttt}\|_2^2 \, ds + \int_0^{t_j} \|u_{tt}\|_{r_{\max}+3}^2 \, ds, \quad C_2^j(u) = \int_0^{t_j} \|u_t\|_{r+3}^2 \, ds + \max_{0 \le t \le t_j} \|u\|_{r+3}^2. \tag{4.49}$$

Using (4.48), Lemma 4.3, (4.14), and Lemma 3.7, we obtain

$$\begin{aligned}\sum_{i=j-1}^{j} (-\Delta \xi^i, \xi_{xxyy}^i)_h \le C \Bigg[&\tau^4 C_1^j(u) + h^{2r-2} C_2^j(u) + \tau \sum_{n=1}^{j-1} J_n + \tau \sum_{n=1}^{j-1} I_5^n \\ &+ \tau \sum_{n=1}^{j-1} (-\Delta \xi^n, \xi_{xxyy}^n)_h + \tau \|\partial_t \xi_{xxyy}^0\|_h^2 + \left|\left(\Delta \xi^1, \xi_{xxyy}^0\right)_h\right| + \sum_{i=0}^{1} |(\Delta \xi^i, \xi_{xxyy}^i)_h| \Bigg].\end{aligned} \tag{4.50}$$

To bound $\tau \sum_{n=1}^{j-1} I_5^n$, we first use (4.30) to obtain

$$\begin{aligned}\sum_{n=1}^{j-1} I_5^n &= -\lambda^2 \left[\sum_{n=1}^{j-1} \left(\partial_t^2 u_{xxyy}^n, \xi_{xxyy}^{n+1}\right) - \sum_{n=1}^{j-1} \left(\partial_t^2 u_{xxyy}^n, \xi_{xxyy}^{n-1}\right)\right] \\ &= -\lambda^2 \left[\sum_{n=2}^{j} \left(\partial_t^2 u_{xxyy}^{n-1}, \xi_{xxyy}^n\right) - \sum_{n=0}^{j-2} \left(\partial_t^2 u_{xxyy}^{n+1}, \xi_{xxyy}^n\right)\right] \\ &= -\lambda^2 \left[\sum_{n=2}^{j-2} \left(\partial_t^3 u_{xxyy}^n, \xi_{xxyy}^n\right) + \sum_{i=j-1}^{j}{}^{*} \left(\partial_t^2 u_{xxyy}^{i-1}, \xi_{xxyy}^i\right) - \sum_{i=0}^{\min(1,j-2)} \left(\partial_t^2 u_{xxyy}^{i+1}, \xi_{xxyy}^i\right)\right],\end{aligned} \tag{4.51}$$

where $\partial_t^3\phi^n$ is defined in (4.11). Using (4.51), the Cauchy-Schwarz and inverse inequalities, the assumption $\tau \le Ch$, (4.12), and (4.7), we have

$$\tau\sum_{n=1}^{j-1} I_5^n \le C\tau h^{-1}\left[\sum_{n=2}^{j-2}\left\|\partial_t^3 u_{xxyy}^n\right\|_0\left\|\xi_{xxy}^n\right\|_0+\sum_{i=j-1}^{j}{}^{*}\left\|\partial_t^2 u_{xxyy}^{i-1}\right\|_0\left\|\xi_{xxy}^i\right\|_0\right.$$
$$\left.+\sum_{i=0}^{1}\left\|\partial_t^2 u_{xxyy}^{i+1}\right\|_0\left\|\xi_{xxy}^i\right\|_0\right] \le C\tau^2\sum_{n=2}^{j-2}\int_{t_{n-2}}^{t_{n+2}}\|u_{ttt}\|_4\,ds\left\|\nabla\xi_{xy}^n\right\|_0$$
$$+C\tau\sum_{i=j-1}^{j}{}^{*}\int_{t_{i-2}}^{t_i}\|u_{tt}\|_4\,ds\left\|\nabla\xi_{xy}^i\right\|_0+C\tau\sum_{i=0}^{1}\int_{t_i}^{t_{i+2}}\|u_{tt}\|_4\,ds\left\|\nabla\xi_{xy}^i\right\|_0. \qquad (4.52)$$

It follows from (4.40), (4.52), the ϵ and Cauchy-Schwarz inequalities, and Lemma 3.7 that

$$\tau\sum_{n=1}^{j-1} J_n+\tau\sum_{n=1}^{j-1} I_5^n \le C\sum_{n=2}^{j-2}\tau^{3/2}\int_{t_{n-2}}^{t_{n+2}}\|u_{ttt}\|_4\,ds\,\tau^{1/2}\left\|\nabla\xi_{xy}^n\right\|_0$$
$$+C\sum_{i=j-1}^{j}{}^{*}\tau\int_{t_{i-2}}^{t_i}\|u_{tt}\|_4\,ds\left\|\nabla\xi_{xy}^i\right\|_0+C\sum_{i=0}^{1}\tau\int_{t_i}^{t_{i+2}}\|u_{tt}\|_4\,ds\left\|\nabla\xi_{xy}^i\right\|_0$$
$$\le C\tau^4\int_0^{t_j}\|u_{ttt}\|_4^2\,ds+C\tau\sum_{n=2}^{j-2}(-\Delta\xi^n,\xi_{xxyy}^n)_h+\epsilon\sum_{i=j-1}^{j}(-\Delta\xi^i,\xi_{xxyy}^i)_h$$
$$+C(\epsilon)\tau^4\max_{0\le t\le t_j}\|u_{tt}\|_4^2+C\tau^4\max_{0\le t\le t_j}\|u_{tt}\|_4^2+C\sum_{i=0}^{1}(-\Delta\xi^i,\xi_{xxyy}^i)_h. \qquad (4.53)$$

Using (4.50), (4.53) and taking ϵ sufficiently small, we have, for $2 \le j \le N_t$,

$$(-\Delta\xi^j,\xi_{xxyy}^j)_h \le C\left[G(\xi^0,\xi^1)+\tau^4\tilde{C}_1^j(u)+h^{2r-2}C_2^j(u)\right]+C\tau\sum_{n=1}^{j-1}(-\Delta\xi^n,\xi_{xxyy}^n)_h, \quad (4.54)$$

where G is defined in (4.27) and

$$\tilde{C}_1^j(u)=C_1^j(u)+\int_0^{t_j}\|u_{ttt}\|_4^2\,ds+\max_{0\le t\le t_j}\|u_{tt}\|_4^2. \qquad (4.55)$$

It follows from (4.27) that (4.54) holds for $0 \le j \le N_t$ and hence the discrete analogue of Gronwall's inequality (see, for example, Lemma 4.7 in [8]), (4.49), and (4.55) give the desired result. $\square$

Theorem 4.5 *If $\lambda > 1/4$, $G(\xi^0,\xi^1) \le C\tau^4\max_{0\le t\le t_1}\|u_{tt}\|_{r+3}^2$, h is sufficiently small, and $\tau \le Ch$, then*

$$\max_{0\le n\le N_t}\|u^n-U^n\|_2 \le C\left\{\tau^2\left(\max_{0\le t\le T}\|u_{tt}\|_{r_{\max}+3}+\max_{0\le t\le T}\|u_{ttt}\|_4\right)+h^{r-1}\sum_{i=0}^{1}\max_{0\le t\le T}\left\|\frac{\partial^i u}{\partial t^i}\right\|_{r+3}\right\}.$$

Proof. Using Lemmas 3.8 and 3.7, Theorem 4.4, we have

$$\|\xi^n\|_2 \le C(-\Delta\xi^n, \xi^n_{xxyy})_h^{1/2} \le C\Bigg\{\tau^2\left(\max_{0\le t\le T}\|u_{tt}\|_{r_{\max}+3} + \max_{0\le t\le T}\|u_{ttt}\|_4\right) + h^{r-1}\sum_{i=0}^{1}\max_{0\le t\le T}\left\|\frac{\partial^i u}{\partial t^i}\right\|_{r+3}\Bigg\} \tag{4.56}$$

for $n = 0, \ldots, N_t$. Since $u^n - U^n = \eta^n - \xi^n$, the desired result follows from the triangle inequality, (3.5), and (4.56). $\square$

Remark 4.6 The assumption that $\tau \le Ch$ for some positive constant C required in our analysis is not restrictive. Due to higher order convergence in spatial variable, in practice it is reasonable to take, for example, $\tau^2 \le h^{r-1}$ to achieve $O(h^{r-1})$ convergence in the H^2 norm.

Remark 4.7 First deriving an error bound in the H^2 norm (rather than the H^1 or L^2norms) appears to be natural for our Petrov-Galerkin method (cf. also [3]) due to the selection of test function $v = \tilde{\partial}_t\xi^n_{xxyy}$ in the proof of Theorem 4.4. Difficulty of obtaining optimal bounds in the H^1 and L^2 norms with this choice of test function is essentially due to the term h^{-2} in the estimate (4.37). This term needs to be h^{-1} for the optimal H^1 error and a constant for optimal L^2 error. Selection of such a test function and hence deriving optimal order bounds in the H^1 and L^2 norms for the ADI Petrov-Galerkin method remains an open problem.

In the next lemma, we select the initial approximate solutions U^0 and U^1 in S_h to obtain the bound on $G(\xi^0, \xi^1)$ assumed in Theorem 4.5.

Lemma 4.8 *Let U^0 and U^1 in S_h satisfy (2.4). Then* $G(\xi^0, \xi^1) \le C\tau^4 \max_{0\le t\le t_1}\|u_{tt}\|^2_{r+3}$.

Proof. First, from (2.4), (3.4), and (1.2) we observe that $\xi^0 = 0$. Hence, (4.27), the Cauchy-Schwarz inequality, and Lemma 3.2, yield

$$G(\xi^0, \xi^1) \le C\left(\|\xi^1_{xxyy}\|_h^2 + \|\Delta\xi^1\|_h^2\right) \le C\left(\|\xi^1_{xxyy}\|_0^2 + \|\xi^1\|_2^2\right). \tag{4.57}$$

Using (2.4) and (3.4), we obtain

$$(L\xi^1, v)_h = (L(g^* - u^1), v)_h, \qquad v \in T_h, \qquad g^* := g + \tau(f^0 - Lg). \tag{4.58}$$

Hence from (1.1) and (1.2),

$$g^* - u^1 = u^0 + \tau u_t^0 - u^1 \in H^{r+2}(\Omega). \tag{4.59}$$

So, (4.58) and Lemma 3.1 with u and W replaced by $g^* - u^1$ and ξ^1, respectively, yield

$$\left\|\left(g^* - u^1 - \xi^1\right)_{xxyy}\right\|_0^2 + \left\|g^* - u^1 - \xi^1\right\|_2^2 \le C\|g^* - u^1\|^2_{r+2}. \tag{4.60}$$

Using repeated integration by parts on $\int_0^{t_1}(t_1-s)u_{tt}(x,y,s)\,ds$, it is easy to verify that for $(x,y)\in\Omega$, we have

$$u(x,y,\tau)=u(x,y,0)+\tau u_t(x,y,0)+\int_0^{t_1}(t_1-s)u_{tt}(x,y,s)\,ds,$$

and hence using (4.59), we obtain

$$\|g^*-u^1\|_{r+2}^2=\left\|\int_0^{t_1}(t_1-s)u_{tt}\,ds\right\|_{r+2}^2\le C\tau^4\max_{0\le t\le t_1}\|u_{tt}\|_{r+2}^2\,. \tag{4.61}$$

Therefore, using (4.57), $\xi^1=(\xi^1-g^*+u^1)+(g^*-u^1)$, the triangle inequality, (4.60), and (4.61), we complete the proof. □

5 Numerical Experiments

In this section, using three test examples, we demonstrate the applicability of the Laplace modified ADI-QPG Petrov-Galerkin scheme for (1.1)–(1.4) with some constant and variable coefficient cases in divergence and non-divergence forms.

In our numerical experiments, for several values of $N_x=N_y=N$, we used uniform partitions in the x- and y- directions and C^2 cubic splines ($r=3$). For uniform time discretization, we chose $\tau=h^2$ so that the expected convergence rate in H^k norm is given by $O(\tau^2+h^{r+1-k})=O(h^{4-k})$, for $k=0,1,2$. In order to check errors and demonstrate the rate of convergence, for all the three test cases, we chose the exact solution to be

$$u(x,y,t)=[\sin(t)+\cos(t)]\sin(\pi x)\sin(\pi y). \tag{5.1}$$

For $k=0,1,2$, we calculated the errors

$$\epsilon_{k,\infty}=\max_{0\le n\le N_t}\|u^n-U^n\|_k,\quad k=0,1,2,$$

by approximating each integral in $\|u^n-U^n\|_k$ with 25 translated Gauss quadrature points on each cell of the 64×64 uniform partition of $\Omega=(0,1)\times(0,1)$. The estimated convergence rate ($R(H^k)$) for the H^k norm was computed in the usual way.

The three test case PDEs, defined for $(x,y,t)\in\Omega\times[0,1]$, are

$$u_t(x,y)-\Delta u(x,y)+c(x,y)u(x,y) = f_1(x,y,t), \tag{5.2}$$
$$u_t(x,y)-\nabla\cdot[a(x,y)\nabla u(x,y)]+c(x,y)u(x,y) = f_2(x,y,t), \tag{5.3}$$
$$u_t(x,y)-a(x,y)\Delta u+\mathbf{b}(x,y)\cdot\nabla u(x,y)+c(x,y)u(x,y) = f_3(x,y,t), \tag{5.4}$$

where

$$\begin{aligned}c(x,y) &= \exp(x+y)\sin(x)\cos(y)\\ a(x,y) &= \exp(x)\cos(y),\\ \mathbf{b}(x,y) &= \left(\frac{\partial a}{\partial x}(x,y)+c(x,y),\frac{\partial a}{\partial y}(x,y)+c(x,y)\right),\end{aligned}$$

$f_i, i = 1, 2, 3$ and the initial and boundary conditions are chosen for the PDEs in conjunction with the test solution (5.1).

In our implementation of the ADI-QPG scheme (2.3) for the three test cases, we chose $\lambda = 1$. This choice was motivated by Theorem 4.5 for (5.2) and by [8, Theorem 6.2] for (5.3) and (5.4). Numerical results in the following three tables demonstrate that the quality of ADI-QPG solutions are similar for all the three test cases. The optimal order theoretical convergence result in the previous section is applicable only for the PDE (5.2).

N	$\epsilon_{0,\infty}$	$R(L_2)$	$\epsilon_{1,\infty}$	$R(H^1)$	$\epsilon_{2,\infty}$	$R(H^2)$
4	4.3681 e-03		2.1537 e-02		2.5208 e-01	
8	1.7348 e-04	4.6541	1.3806 e-03	3.9634	5.7816 e-02	2.1243
16	9.7011 e-06	4.1605	1.4490 e-04	3.2522	1.4249 e-02	2.0206
32	5.8859 e-07	4.0428	1.7343 e-05	3.0627	3.5495 e-03	2.0051
64	3.6621 e-08	4.0065	2.1446 e-06	3.0156	8.8656 e-04	2.0013

Table 1: Optimal order convergence of ADI-QPG solutions for (5.2)

N	$\epsilon_{0,\infty}$	$R(L_2)$	$\epsilon_{1,\infty}$	$R(H^1)$	$\epsilon_{2,\infty}$	$R(H^2)$
4	3.0571 e-03		1.6405 e-02		2.4714 e-01	
8	1.2116 e-04	4.6572	1.2623 e-03	3.7000	5.7776 e-02	2.0968
16	6.7519 e-06	4.1655	1.4147 e-04	3.1576	1.4249 e-02	2.0197
32	4.0921 e-07	4.0444	1.7238 e-05	3.0368	3.5495 e-03	2.0051
64	2.5084 e-08	4.0280	2.1412 e-06	3.0090	8.8656 e-04	2.0013

Table 2: Optimal order convergence of ADI-QPG solutions for (5.3)

N	$\epsilon_{0,\infty}$	$R(L_2)$	$\epsilon_{1,\infty}$	$R(H^1)$	$\epsilon_{2,\infty}$	$R(H^2)$
4	3.2412 e-03		1.7151 e-02		2.4848 e-01	
8	1.2530 e-04	4.6931	1.2726 e-03	3.7523	5.7781 e-02	2.1045
16	6.9782 e-06	4.1663	1.4175 e-04	3.1665	1.4249 e-02	2.0198
32	4.2285 e-07	4.0446	1.7246 e-05	3.0389	3.5495 e-03	2.0051
64	2.5767 e-08	4.0365	2.1414 e-06	3.0096	8.8656 e-04	2.0013

Table 3: Optimal order convergence of ADI-QPG solutions for (5.4)

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