Homework 3:

6.2.10 $f(t) = \begin{cases} t & 0 \leq t \leq a \\ a e^{-(t-a)} & t \geq a \end{cases}$

$$= t\,H(t-0) + (a e^{-(t-a)} - t)\,H(t-a) = t + (a e^{-(t-a)} - t)\,H(t-a)$$

for $t \geq 0$.

6.3.41:

$$y'' + 3y' + 2y = t + [a e^{-(t-a)} - t]\,H(t-a), \quad y(0) = y'(0) = 0.$$

Applying the Laplace transform to both sides: $(Y(s) = \mathcal{L}(y(t)))$.

$$s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)} + 3(sY(s) - \cancel{y(0)}) + 2Y(s) = \mathcal{L}(t) + \mathcal{L}((a e^{-(t-a)} - t)\,H(t-a))$$

$\Rightarrow$

$$Y(s)(s^2 + 3s + 2) = \frac{1}{s^2} + a\mathcal{L}(e^{-(t-a)}H(t-a)) - \mathcal{L}(tH(t-a)) \Rightarrow$$

$$Y(s)(s^2 + 3s + 2) = \frac{1}{s^2} + a\frac{e^{-as}}{s+1} - \mathcal{L}((t-a)H(t-a)) - a\mathcal{L}(H(t-a)) \Rightarrow$$

$$Y(s) = \left(\frac{1}{s^2} + a\frac{e^{-as}}{s+1} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}\right) \Big/ (s^2 + 3s + 2)$$

6.4.2 Let $g(t) = \begin{cases} \sin(t) & 0 \leq t \leq \pi \\ 0 & t \geq \pi. \end{cases}$

$$\mathcal{L}(f(t)) = \frac{\mathcal{L}(g(t))}{1 - e^{-sT}}. \quad \text{Here } T = 2\pi.$$

$$\mathcal{L}(g(t)) = \int_0^\infty g(t) e^{-st}\,dt = \int_0^\pi \sin(t) e^{-st}\,dt$$

You should know how to complete the solution.

6.5.4: $F(s) = \dfrac{s-3}{(s^2+4)(s+1)} = \dfrac{s-3}{(s+2i)(s-2i)(s+1)} = \dfrac{A}{s+2i} + \dfrac{B}{s-2i} + \dfrac{C}{s+1}$.

$$= \frac{(s-2i)(s+1)A + (s+2i)(s+1)B + C(s+2i)(s-2i)}{(s+2i)(s-2i)(s+1)}$$

so,

$$A = \lim_{s \to -2i} \frac{(s+2i)(s-3)}{(s^2+4)(s+1)} = \lim_{s \to -2i} \frac{(s-3)}{(s-2i)(s+1)} = \frac{-2i-3}{-4i(-2i+1)}$$

Hence $A = \dfrac{2i+3}{4(2+i)}$.

$$B = \lim \frac{(\cancel{s-2i})(s-3)}{\ldots} = \lim \frac{s-3}{\ldots} = \frac{2i-3}{\ldots} = \frac{2i-3}{4(-2+i)}$$

$$C = \lim_{s \to -1} \frac{\cancel{(s+1)}(s-3)}{(s^2+4)\cancel{(s+1)}} = \frac{-4}{5}.$$

$$\therefore \quad F(s) = \frac{(2i+3)}{4(2+i)} \frac{1}{s+2i} + \frac{(2i-3)}{4(i-2)} \cdot \frac{1}{s-2i} - \frac{4}{5} \cdot \frac{1}{s+1}$$

Consequently,

$$f(t) = \frac{(2i+3)}{4(2+i)} e^{-2it} + \frac{(2i-3)}{4(i-2)} e^{2it} - \frac{4}{5} e^{-t}.$$

6.6.10: Let $f_1(t) = t * (H(t) - H(t-2))$ & $f_2(t) = \frac{t^2}{2} - \frac{(t-2)^2}{2} H(t-2)$.

First, we want to verify that $f_1(t) = f_2(t)$.

$$f_1(t) = \int_0^t (t-q)(H(q) - H(q-2))\, dq = \begin{cases} \int_0^t (t-q)\, dq & \text{if } t \leq 2 \\ \int_0^2 (t-q)(H(q)-H(q-2))\, dq & \text{if } t \geq 2 \end{cases}$$

$$= \begin{cases} -\frac{(t-q)^2}{2}\Big|_0^t & \text{if } t \leq 2 \\ \int_0^2 (t-q)\, dq & \text{if } t \geq 2 \end{cases} = \begin{cases} \frac{t^2}{2} & \text{if } t \leq 2 \\ -\frac{(t-q)^2}{2}\Big|_0^2 & \text{if } t \geq 2 \end{cases}$$

$$= \begin{cases} \frac{t^2}{2} & \text{if } t \leq 2 \\ -\frac{(t-2)^2}{2} + \frac{t^2}{2} & \text{if } t \geq 2 \end{cases} = \frac{t^2}{2} + \left(-\frac{(t-2)^2}{2} + \frac{t^2}{2} - \frac{t^2}{2} \right) H(t-2)$$

$$= \frac{t^2}{2} - \frac{(t-2)^2}{2} H(t-2). \quad \therefore f_1(t) = f_2(t).$$

Now, we want to verify the convolution theorem.

$$\mathcal{L}(f_1(t)) = \cancel{\mathcal{L}(f_2(t))} \; \mathcal{L}(t)\, \mathcal{L}(H(t) - H(t-2)) = \frac{1}{s^2}\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right). \text{ (by convolution theorem)}$$

$$\mathcal{L}(f_2(t)) = \mathcal{L}\left(\frac{t^2}{2}\right) - \mathcal{L}\left(\frac{(t-2)^2}{2} H(t-2)\right)$$

$$= \frac{1}{2}\left(\frac{2}{s^3}\right) - \frac{1}{2} e^{-2s}\left(\frac{2}{s^3}\right) = \frac{1}{s^2}\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right) = \mathcal{L}(f_1(t))$$

So the convolution theorem is verified.

6.7.6 :

$$f(t) = 8t^2 - 3\int_0^t f(x)\sin(t-x)\,dx .$$ Let $F(s) = \mathcal{L}(f(t))$.

so $\mathcal{L}(f(t)) = \mathcal{L}\left(8t^2 - 3\underbrace{\int_0^t f(x)\sin(t-x)\,dx}_{f * \sin(t)}\right) \Rightarrow$

$$F(s) = 8\cdot\frac{2}{s^3} - 3\,F(s)\cdot\frac{1}{s^2+1} \Rightarrow$$

$$F(s)\left(1+\frac{3}{s^2+1}\right) = \frac{16}{s^3} \Rightarrow F(s) = \frac{16(s^2+1)}{s^3(s^2+4)} .$$

~~By partial fract~~ Here, we may use partial fraction to find $f(t)$. That is, writing $F(s)$ as:

$$F(s) = \frac{16(s^2+1)}{s^3(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+4} .$$

$$= \frac{As^2(s^2+4) + Bs(s^2+4) + C(s^2+4) + Ds^4 + Es^3}{s^3(s^2+4)} .$$

so

$$16(s^2+1) = As^4 + 4As^2 + Bs^3 + 4Bs + Cs^2 + 4C + Ds^4 + Es^3 ,$$

By equating both sides (comparing LHS with RHS),

$A + D = 0 \rightarrow$ using $A=-3$, we get $\boxed{D=3}$.

$B + E = 0 \longrightarrow$ using $B=0$, we get $\boxed{E=0}$

$4A + C = 16 \longrightarrow$ using $C=4$, we get $\boxed{A=-3}$

$4B = 0 \Rightarrow \boxed{B=0}$

$4C = 16 \Rightarrow \boxed{C=4}$

$\therefore F(s) = \frac{-3}{s} + \frac{4}{s^3} + \frac{3s}{s^2+4}$. Thus, $f(t) = -3 + 2t^2 + 3\cos(2t)$.

6.8.22 : $y'' - 5y' + 4y = \delta(t-1)$, $y(0) = y'(0) = 0$. Let $\boxed{Y(s) = \mathcal{L}(y(t))}$ (4)

so $\mathcal{L}(y'' - 5y' + 4y) = \mathcal{L}(\delta(t-1)) \Rightarrow$

$s^2 Y(s) - 5sY(s) + 4Y(s) = e^{-s} \Rightarrow$

$Y(s)(s^2 - 5s + 4) = e^{-s} \Rightarrow Y(s) = \frac{e^{-s}}{(s^2-5s+4)} = \frac{e^{-s}}{(s-1)(s-4)}$

$\therefore$ $Y(s) = -\frac{e^{-s}}{3}\left[\frac{1}{s-1} - \frac{1}{s-4}\right]$

Thus, $y(t) = -\frac{1}{3}(e^{(t-1)} H(t-1)) + \frac{1}{3}(e^{4(t-1)} H(t-1))$

$= \frac{1}{3}\left(e^{4(t-1)} - e^{(t-1)}\right) H(t-1)$.

6.2.28 $x' + 3x - y = 1$ $x(0) = 2, y(0) = 0$.
$x' + y' + 3x = 0$

Let $X(s) = \mathcal{L}(x(t))$ & $Y(s) = \mathcal{L}(y(t))$.

Taking the Laplace transform of both sides of above DEs:

$$\begin{cases} sX(s) - x(0) + 3X(s) - Y(s) = \frac{1}{s} \\ sX(s) - x(0) + sY(s) - y(0) + 3X(s) = 0 \end{cases} \Rightarrow \begin{cases} sX(s) + 3X(s) - Y(s) = \frac{1}{s} + 2 & (1) \\ sX(s) + 3X(s) + sY(s) = 2 & (2) \end{cases}$$

~~$X(s)(2s+6) = \frac{1}{s} + 4$~~

(2) - (1), $Y(s)(s+1) = -\frac{1}{s} \Rightarrow Y(s) = \frac{-1}{s(s+1)} = \frac{1}{s+1} - \frac{1}{s}$

so $y(t) = e^{-t} - 1$.

$s \times (1) + (2)$, $s(s+3)X(s) + (s+3)X(s) = 1 + 2s + 2 \Rightarrow$

$X(s)(s+3)(s+1) = 2s+3 \Rightarrow X(s) = \frac{2s+3}{(s+3)(s+1)}$

& now you need to find $x(t)$ by partial fraction.

Good luck.