Homework2

5.1.2 : $f(t) = t e^{-a|t|}, \quad a > 0.$ $\boxed{4 \text{ points}}$

$$F(s) = \int_{-\infty}^{\infty} t e^{-a|t|} e^{-ist} dt = \int_{-\infty}^{0} t e^{at} e^{-ist} dt + \int_{0}^{\infty} t e^{-at} e^{-ist} dt$$

$$= \int_{-\infty}^{0} t e^{t(a-is)} dt + \int_{0}^{\infty} t e^{-t(a+is)} dt$$

t	$e^{t(a-is)}$
1	$\frac{e^{t(a-is)}}{a-is}$
0	$\frac{e^{t(a-is)}}{(a-is)^2}$

$$= \left[\frac{t e^{t(a-is)}}{a-is} - \frac{e^{t(a-is)}}{(a-is)^2} \right]_{-\infty}^{0}$$

$$+ \left[-\frac{t e^{-t(a+is)}}{a+is} - \frac{e^{-t(a+is)}}{(a+is)^2} \right]_{0}^{\infty}$$

$$= \frac{-1}{(a-is)^2} + \frac{1}{(a+is)^2} = \frac{-(a+is)^2 + (a-is)^2}{((a-is)(a+is))^2} = \frac{-4ais}{(a^2+s^2)^2}.$$

5.1.7 : $f(t) = \begin{cases} \sin(t) & 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$ $\boxed{4 \text{ points}}$

$$F(s) = \int_0^1 \sin t \, e^{-ist} dt = \int_0^1 \sin t \left(\cos(st) - i \sin(st) \right) dt$$

$$= \int_0^1 \cancel{\sin(t+st)} \left[\sin t \cos(st) - i \sin(t) \sin(st) \right] dt$$

$$= \frac{1}{2} \int_0^1 \left[\sin(t+st) + \sin(t-st) - i\left(\cos(t-st) - \cos(t+st) \right) \right] dt$$

$$= \frac{1}{2} \left[-\frac{\cos(t(1+s))}{1+s} - \frac{\cos(t(1-s))}{1-s} - i\left(\frac{\sin(t(1-s))}{1-s} - \frac{\sin(t(1+s))}{1+s} \right) \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1-\cos(1+s)}{1+s} + \frac{1-\cos(1-s)}{1-s} - i\left(\frac{\sin(1-s)}{1-s} - \frac{\sin(1+s)}{1+s} \right) \right]$$

$$\cancel{= \frac{1}{2}\left[\frac{1-\cos(s-1)}{s-1} + \frac{1-\cos(1+s)}{1+s} \right] - \frac{i}{2}\left[\frac{\sin(s-1)}{s-1} - \frac{\sin(s+1)}{s+1} \right]}$$

$$= -\frac{1}{2}\left[\frac{\cos(1+s)-1}{1+s} + \frac{1-\cos(s-1)}{s-1} \right] - \frac{i}{2}\left[\frac{\sin(s-1)}{s-1} - \frac{\sin(s+1)}{s+1} \right].$$

5.3.2 : $f(t) = \cos(at)/_{1+t^2} = \frac{1}{2}\left(e^{iat} + e^{-iat}\right) \cdot \frac{1}{1+t^2}$.

$\cancel{F(t)}\ \mathcal{F}(f(t)) = \frac{1}{2}\mathcal{F}\left(\frac{e^{iat}}{1+t^2}\right) + \frac{1}{2}\mathcal{F}\left(\frac{e^{-iat}}{1+t^2}\right)$, let $g(t) = \frac{1}{1+t^2}$

so

$\cancel{= \frac{1}{2}\mathcal{F}(f}$ $\qquad G(s) = \pi e^{-|s|}$

$\boxed{2 \text{ points}}$

$$= \frac{1}{2} G(s-a) + \frac{1}{2} G(s+a)$$

$$= \frac{\pi}{2}\left[e^{-|s-a|} + e^{-|s+a|} \right]$$

5.3.4 : Let $f(t) = \frac{1}{t^2+a^2}$. By Parseval's equality,

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2 ds \quad \text{so}$$ $\boxed{2 \text{ points}}$

$$\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi^2\left(e^{-|s|}\right)^2 ds = \frac{\pi}{2}\int_{-\infty}^{\infty} e^{-2|s|} ds$$

$$= \frac{\pi}{2}\left[\int_{-\infty}^{0} e^{+2s} ds + \int_{0}^{\infty} e^{-2s} ds\right]$$

$$= \frac{\pi}{2}\left[\frac{e^{2s}}{2}\Big|_{-\infty}^{0} + \frac{e^{-2s}}{-2}\Big|_{0}^{\infty} \right]$$

$$= \frac{\pi}{2}\left[\frac{1}{2} + \frac{1}{2}\right] = \frac{\pi}{2}. \qquad \square$$

5.4.2 : $\boxed{2 \text{ points}}$

$$F(s) = \frac{1}{(1+is)(1+2is)} = \frac{-1}{1+is} + \frac{2}{1+2is} \quad \text{. so,}$$

$$\frac{2}{1+2is} = \frac{1}{\frac{1}{2}+is}$$

$$f(t) = -e^{-t}H(t) + e^{-\frac{t}{2}}H(t).$$

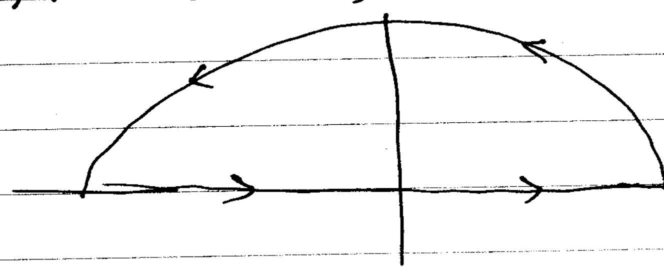
5.4.10: $F(s) = \dfrac{1}{(s^2-3is-3)}$. Find $f(t)$. $\boxed{8 \text{ points}}$

$$f(t) = \frac{1}{2\pi}\int_C e^{izt} g(z)\,dz - \frac{1}{2\pi}\int_{C_R} e^{izt} g(z)\,dz, \quad g(z) = \frac{1}{z^2-3iz-3},$$

where C denotes a closed contour consisting of the ~~and~~ entire real axis plus C_R.

$\lim_{|z|\to\infty} g(z) = 0$, so $\int_{C_R} e^{izt} g(z)\,dz = 0$ for $t > 0$ provided by Jordan's lemma.

that C_R is chosen to be a semi-circle in the upper-half plane with an infinite radius. so, in this case,



$$f(t) = \frac{1}{2\pi}\int_C e^{izt} g(z)\,dz.$$

$$e^{izt} g(z) = \frac{e^{izt}}{(z-\frac{\sqrt{3}}{2}-\frac{3i}{2})(z+\frac{\sqrt{3}}{2}-\frac{3i}{2})}$$

so, we have 2 simple poles inside C which are:

$z_1 = \frac{\sqrt{3}}{2}+\frac{3i}{2}$ & $z_2 = -\frac{\sqrt{3}}{2}+\frac{3i}{2}$. Hence,

$$f(t) = \frac{1}{2\pi}\left[2\pi i \operatorname{Res}(e^{izt}g(z), z_1) + 2\pi i \operatorname{Res}(e^{izt}g(z), z_2)\right)$$

$$= i\left[\lim_{z\to z_1}\frac{e^{izt}}{(z-z_2)} + \lim_{z\to z_2}\frac{e^{izt}}{z-z_1}\right]$$

$$= i\left[\frac{e^{(i\frac{\sqrt{3}}{2}-\frac{3}{2})t}}{z_1-z_2} + \frac{e^{(-i\frac{\sqrt{3}}{2}-\frac{3}{2})t}}{z_2-z_1}\right] = \frac{i\,e^{-\frac{3}{2}t}}{z_1-z_2}\left(e^{i\frac{\sqrt{3}}{2}t} - e^{-i\frac{\sqrt{3}}{2}t}\right)$$

$$= \frac{i\,e^{-\frac{3}{2}t}}{\sqrt{3}}\left(2i\sin(\tfrac{\sqrt{3}}{2}t)\right) = \frac{-2}{\sqrt{3}}e^{-\frac{3}{2}t}\sin(\tfrac{\sqrt{3}}{2}t), \quad \text{if } t>0.$$

~~by Jordan's lemma, C_R is a semi-circle~~

If $t<0$, $\lim_{|z|\to\infty} g(z)=0$. So, by Jordan's lemma, $\int_{C_R} e^{izt} g(z)\,dz = 0$

provided that we choose C_R be a semi-circle with an infinite ~~radu~~ radius in the lower-half plane.

So, $\cancel{f(t)=\frac{1}{2\pi}\int_C}$ $f(t) = \frac{1}{2\pi}\int_C e^{izt} g(z)\,dz$

$$= \frac{1}{2\pi}\left(-2\pi i \sum_{i=1}^{n} \operatorname{Res}(e^{izt} g(z), z_i)\right)$$

where $z_1, \dots, z_n$ are the poles of $e^{izt} g(z)$ inside C.
where C is the entire real axis pluse C_R.
since we don't have any pole ~~and~~ inside C, then

$f(t) = 0$ for $t<0$.

$$\therefore \quad f(t) = \begin{cases} 0 & t<0 \\ \frac{-2}{\sqrt{3}} e^{-\frac{3}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) & \text{if } t>0 \end{cases}$$

$$= \frac{-2}{\sqrt{3}} e^{-\frac{3}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) H(t).$$

5.5.3: $\underbrace{e^{-t}H(t) * e^{-2t}H(t)}_{LHS} \overset{??}{=} \underbrace{(e^{-t}-e^{-2t})H(t)}_{RHS}$. $\boxed{2 \text{ points}}$

$$\mathcal{F}(LHS) = \mathcal{F}(e^{-t}H(t))\,\mathcal{F}(e^{-2t}H(t)) = \frac{1}{1+is}\cdot\frac{1}{2+is}.$$

$$\mathcal{F}(RHS) = \mathcal{F}(e^{-t}H(t)) - \mathcal{F}(e^{-2t}H(t)) = \frac{1}{1+is} - \frac{1}{2+is} = \frac{2+is-1-is}{(1+is)(2+is)}$$

$$= \frac{1}{(1+is)(2+is)}.$$

Then, $\mathcal{F}(RHS) = \mathcal{F}(LHS)$ which implies that
$RHS = LHS$.

$\boxed{5 \text{ points}}$ 5

5.6.3: ~~solve~~ $y'' + 4y' + 4y = \frac{1}{2} e^{-|t|}$.

Applying the Fourier transform to both sides, we get.

$\mathcal{F}(y'') + 4\mathcal{F}(y') + 4\mathcal{F}(y) = \frac{1}{2}\mathcal{F}(e^{-|t|}) \Rightarrow$ let $Y(s) = \mathcal{F}(y(t))$.

$(is)^2 Y(s) + 4(is)Y(s) + 4Y(s) = \frac{1}{2} \cdot \frac{2}{1+s^2} \Rightarrow$

$Y(s)[-s^2 + 4is + 4] = \frac{1}{1+s^2} \Rightarrow Y(s) = 1/(1+s^2)(-s^2+4is+4)$.

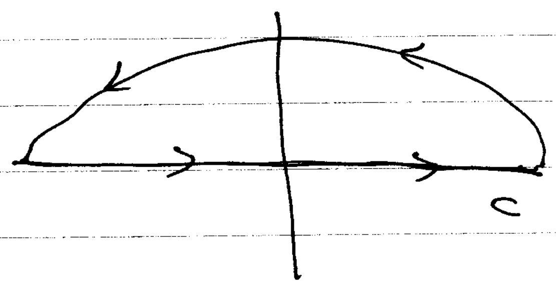
$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(s) e^{ist}\, ds = \frac{1}{2\pi}\int_C Y(z) e^{izt}\, dz - \frac{1}{2\pi}\int_{C_R} Y(z) e^{izt}\, dz$$

where C is a closed contour consists of the entire real axis plus C_R.

$\lim_{|z|\to\infty} Y(z) = 0$. Hence by Jordan's lemma, for $t > 0$,

$\int_{C_R} Y(z) e^{izt} dz = 0$ if C_R is chosen to be a semi-circle in the upper half plane with an infinite radius.

Thus, $y(t) = \frac{1}{2\pi}\int_C Y(z) e^{izt} dt$.



$\cancel{Y(z) = \frac{1}{(z+i)(z+i)}}$

$(1+z^2)(-z^2+4iz+4) = (z+i)(z-i)(iz+2)^2$

so the poles of $e^{izt} Y(z)$ are $z_1 = i$, $z_2 = -i$,

$z_3 = z_4 = +2i$ ($z_3 = 2i$ is a double pole)

$z_1 = i$ & $z_3 = 2i$ are the ~~only~~ pole in the interior of C, so,

$$y(t) = \frac{1}{2\pi}\left(2\pi i \operatorname{Res}(Y(z)e^{izt}, z_1)\right) + \frac{1}{2\pi}\left(2\pi i \operatorname{Res}(Y(z)e^{izt}; z_3)\right)$$

$\cancel{= i \lim_{z\to i} (z-i) Y(z) e^{izt} = i \lim \frac{e^{izt}}{(z+i)(iz+2)^2} = i \frac{e^{-t}}{2i(-1+2)^2}}$

So, $y(t) = i\left[\lim_{z\to i} Y(z)e^{izt}(z-i) + \lim_{z\to 2i} \frac{d}{dz}\left(Y(z)e^{izt}(z-2i)^2\right)\right]$

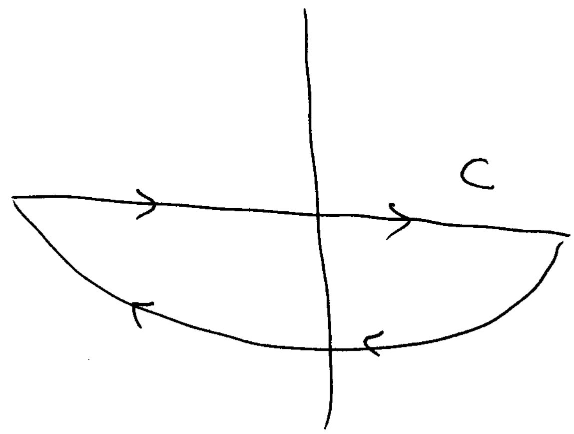
$$= i\left[\lim_{z\to i}\frac{e^{izt}}{(z+i)(iz+2)^2} + \lim_{z\to 2i}\frac{d}{dz}\left(\frac{e^{izt}}{(z^2+1)i^2}\right)\right]$$

$$= i\left[\frac{e^{-t}}{2i(-1+2)^2} - \lim_{z\to 2i}\left(\frac{ite^{izt}(z^2+1) - 2ze^{izt}}{(z^2+1)^2}\right)\right]$$

$$= \frac{e^{-t}}{2} - \frac{3t\,e^{-2t}}{9} - \frac{4}{9}e^{-2t} = -\frac{1}{9}e^{-2t}(3t+4) + \frac{e^{-t}}{2} \quad \text{if } t>0.$$

If $t<0$, by Jordan's lemma, $\int_{C_R} e^{izt}Y(z)\,dz = 0$ provided that we choose C_R to be a semi-circle in the lower-half plane with an infinite radius.

In this case,



$$f(t) = \frac{1}{2\pi}\int_C e^{izt}Y(z)\,dz$$

$$= \frac{1}{2\pi}\left[-2\pi i\,\mathrm{Res}\left(e^{izt}Y(z), z_2\right)\right]$$

$$= -i\lim_{z\to -i}\frac{e^{izt}\cancel{(z+i)}}{\cancel{(z+i)}(z-i)(iz-2)^2}$$

$$= -i\,\frac{e^{t}}{-2i(-1-2)^2} = \frac{e^{t}}{18}.$$

So,

$$f(t) = \begin{cases} \dfrac{e^{t}}{18} & \text{if } t<0, \\[2mm] \dfrac{e^{-t}}{2} - \dfrac{1}{9}e^{-2t}(3t+4) & \text{if } t>0. \end{cases}$$