Homework 1. (solution Keys).

4.1.4.

$$f(t) = \begin{cases} \frac{1}{2} + t & -1 \le t \le 0 \\ \frac{1}{2} - t & 0 \le t \le 1. \end{cases}$$

. 5 points

The Fourier series expansion of $f(t)$ is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi t) + b_n \sin(n\pi t)]$$, here $L = 1$.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(t)\,dt = \int_{-1}^{0} (\tfrac{1}{2}+t)\,dt + \int_0^1 (\tfrac{1}{2}-t)\,dt = 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos(n\pi t) = \int_{-1}^{0} (\tfrac{1}{2}+t)\cos(n\pi t) + \int_0^1 (\tfrac{1}{2}-t)\cos(n\pi t)\,dt$$
$$= 2[1-(-1)^n]/n^2\pi^2.$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin(n\pi t) = \int_{-1}^{0} (\tfrac{1}{2}+t)\sin(n\pi t)\,dt + \int_0^1 (\tfrac{1}{2}-t)\sin(n\pi t)\,dt$$
$$= 0$$

- You may need to use an integration by parts to compute a_n and b_n for $n \ge 1$.

4.1.14:

$f(t) = t\cos\left(\frac{\pi t}{L}\right)$, $-L < t < L$. 5 points

$$f(-t) = (-t)\cos\left(\frac{\pi(-t)}{L}\right) = -t\cos\left(\frac{\pi t}{L}\right) = -f(t)$$

so

f is an odd function, then the Fourier series expansion of f is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right).$$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt = \frac{2}{L}\int_0^L t\cos\left(\frac{\pi t}{L}\right)\sin\left(\frac{n\pi t}{L}\right)dt.$$

~~using~~ $$2t\cos\left(\frac{\pi t}{L}\right)\sin\left(\frac{n\pi t}{L}\right) = t\left(\sin\left(\frac{\pi t}{L}+\frac{n\pi t}{L}\right) - \sin\left(\frac{\pi t}{L}-\frac{n\pi t}{L}\right)\right)$$

$$\cancel{= t\left(\sin\left(\frac{nt}{L}(\pi+1)\right) - \sin\left(\frac{nt}{L}(1-\pi)\right)\right)}$$

(2)

$$= t\left(\sin\left(\frac{\pi t}{L}(n+1)\right) - \sin\left(\frac{\pi t}{L}(Ln)\right)\right)$$

Integrating by parts and using $\sin(n\pi)=0$ & $\cos(n\pi)=(-1)^n$, we find that

$$b_n = \frac{-2L}{\pi^2(n^2-1)^2}\left(-n^3(-1)^n\pi + n\pi(-1)^n\right)$$

$$= \frac{2L(-1)^n n(n^2-1)}{\pi(n^2-1)^2} = \frac{2Ln(-1)^n}{\pi(n^2-1)}.$$

$\therefore$

$$f(t) = \cancel{\sum} \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{n^2-1}\sin\left(\frac{n\pi t}{L}\right)$$ (2 points)

4.2.1: $\frac{x\pi^2 - \pi x^2}{8} = \sum_{n=0}^{\infty}\frac{\sin[(2n+1)x]}{(2n+1)^3}$, $0 < x \le \pi$ (given)

Differentiate both sides: (obtained).

$$\frac{\pi^2 - 2\pi x}{8} = \sum_{n=0}^{\infty}\frac{\cos((2n+1)x)}{(2n+1)^2}.$$ This can be done

because $f(x) = \frac{\pi^2 x - \pi x^2}{8}$ is a continuous function on $[0,\pi]$.
and f' is also continuous ~~of~~ on $[0,\pi]$

If $\frac{\pi^2 - 2\pi x}{8} = \sum_{n=0}^{\infty}\frac{\cos[(2n+1)x]}{(2n+1)^2}$ is given.,

integrating both sides, ~~we get~~ from 0 to x, that is,

$$\int_0^x\left(\frac{\pi^2 - 2\pi x}{8}\right)dx = \sum_{n=0}^{\infty}\int_0^x\frac{\cos((2n+1)x)}{(2n+1)^2}\,dx$$

so

$$\frac{\pi^2 x - \pi x^2}{8} = \sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{(2n+1)^3}.$$

4.1.7

$f(t) = \{ \pi^2 - t^2 \quad 0 < t < \pi.$

$\boxed{7 \text{ points}}$

The cosine series expansion of $f(t)$ is:

$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi \frac{t}{L})$, Here $L = \pi$.

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(t)\,dt = \frac{4}{3}\pi^2$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos(n\pi \frac{t}{L})\,dt = \frac{2}{\pi} \int_0^{\pi} (\pi^2 - t^2) \cos(nt)\,dt$

$= \frac{-4(-1)^n}{n^2}$

$\therefore$

$f(t) = \cancel{\frac{4}{3}} \; \frac{2}{3}\pi^2 - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt).$

The sine series expansion is:

$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$, $\quad b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin(nt)\,dt$

integrating by parts, we get

$b_n = \frac{2}{\pi} \left[\frac{\pi^2}{n} + \frac{2(1-(-1)^n)}{n^3} \right] = \frac{2\pi}{n} + \frac{4(1-(-1)^n)}{\pi n^3}.$

$\therefore$

$f(t) = \sum_{n=1}^{\infty} \left(\frac{2\pi}{n} + \frac{4}{\pi n^3} (1-(-1)^n) \right) \sin(nt).$

(4)

4.4.3 $f(t) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt)$. (3 points)

(3)

We can write f in terms of a cosine series with phase angle φ_n as:

$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nt + \varphi_n)$.

Here, $a_0 = 0$, $a_n = 0$ for $n \geq 1$, $b_n = \frac{-2(-1)^n}{n}$, $A_n = \sqrt{a_n^2 + b_n^2} = \frac{2}{n}$.

$\varphi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) = \tan^{-1}(\infty)$ so, $\varphi_n = \pm\frac{\pi}{2}$.

φ_n must satisfies the identities:

$a_n = A_n \cos(\varphi_n) = 0 = 0$

$b_n = -A_n \sin(\varphi_n) \Rightarrow (-1)^n = \sin(\varphi_n)$ so, $\varphi_n = (-1)^n \frac{\pi}{2}$.

$\therefore$

$f(t) = \sum_{n=1}^{\infty} \frac{2}{n} \cos\left(nt + (-1)^n \frac{\pi}{2}\right)$.

4.5.4: $f(t) = t^2$, $-\pi < t \leq \pi$. (4 points)

The complex Fourier series of $f(t)$ is:

$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi t/L}$, where $C_n = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-in\pi t/L} dt$.

Here, $L = \pi$, so, $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 e^{-int} dt$ $\cancel{= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 (\cos nt)}$

so

~~Integrating by parts twice, we find that $C_n =$~~

$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 \cos(nt)\, dt - \frac{i}{2\pi} \underbrace{\int_{-\pi}^{\pi} t^2 \sin(nt)\, dt}_{}$

"0 because $t^2 \sin(nt)$ is an odd funct.

$\Rightarrow$

$C_n = \frac{1}{2\pi} \left[\frac{t^2}{n} \cancel{\sin(nt)}\Big|_{-\pi}^{\pi} + \frac{2t}{n^2} \cos(nt)\Big|_{-\pi}^{\pi} - \frac{t}{n^3} \cancel{\sin(nt)}\Big|_{-\pi}^{\pi} \right] = \frac{2}{n^2}(-1)^n$, $n \neq 0$

t^2	$\cos nt$
$2t$	$\frac{\sin nt}{n}$
t	$-\frac{\cos nt}{n^2}$
0	$-\frac{\sin nt}{n^3}$

For $n=0$, $C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3}$.

$\therefore$

$$f(t) = \frac{\pi^2}{3} + 2\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n^2} e^{int}.$$

4.6.5 $f(t) = |t|$. We want to use complex Fourier series to solve the DE: $y'' + 4y = |t|$, $-\pi \le t \le \pi$. $\boxed{6 \text{ points}}$

The general solution of this DE is: $y = y_c + y_p$ where y_c is the general solution of the associated homogeneous DE & y_p is a particular solution.

$y_c = A\cos(2t) + B\sin(2t)$.

To find y_p using Fourier complex series, we need to find first the complex Fourier series of $f(t) = |t|$.

Refer to our text book,

$$f(t) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2}.$$

so, the form of y_p is

$$y_p = A_0 + \sum_{n=-\infty}^{\infty} K_n e^{i(2n-1)t}. \quad \text{so}$$

$$y_p'' + 4y_p = 4A_0 + \sum_{n=-\infty}^{\infty} K_n\left[-(2n-1)^2 + 4\right] e^{i(2n-1)t} \overset{\text{From the DE}}{=}$$

$$\frac{\pi}{2} \cancel{3} + \sum_{n=-\infty}^{\infty} \left(\frac{-2}{\pi(2n-1)^2}\right) e^{i(2n-1)t}.$$

By comparison, we find that

$A_0 = \frac{\pi}{8}$, & $K_n\left(4 - (2n-1)^2\right) = \frac{-2}{\pi(2n-1)^2}$, so

$$K_n = \frac{-2}{\pi(2n-1)^2(4-(2n-1)^2)}. \quad \text{Finally,}$$

$$y_p = \frac{\pi}{8} - \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2(4-(2n-1)^2)}.$$