HW7

9.1.13

$$\begin{cases} \frac{d}{dx}(x^3 y') + \lambda x y = 0 & 1 < x < e^{\pi} \quad (1) \\ y(1) = y(e^{\pi}) = 0, & \quad (2) \end{cases}$$

The problem (1)-(2) is a RSL problem.

Expanding (1) as.

$x^3 y'' + 3x^2 y' + \lambda x y = 0$ dividing by x $(x \neq 0)$

$x^2 y'' + 3 x y' + \lambda y = 0$ (3) This is a secondorder DE of Cauchy Euler type.

Hence, we are looking for a solution of it of the form $y = x^m$.

substitute $y = x^m$ in (3), we get, 1

$m(m-1) x^m + 3 m x^m + \lambda x^m = 0$ dividing by x^m,

$m(m-1) + 3m + \lambda = 0 \Rightarrow m^2 + 2m + \lambda = 0 \Rightarrow (m+1)^2 + \lambda - 1 = 0$

$\Rightarrow (m+1)^2 = 1 - \lambda$. 1

Case 1 $\lambda = 1$, $m + 1 = 0 \Rightarrow m = -1$, so $m_1 = m_2 = -1$.

Thus, $y = c_1 x^{-1} + c_2 x^{-1} \ln x$, $y(1) = 0 \Rightarrow c_1 = 0$. 2

$y(e^{\pi}) = 0 \Rightarrow c_2 e^{-\pi} \pi = 0 \Rightarrow c_2 = 0$.

$\boxed{\lambda = 1 \text{ is not an eigenvalue.}}$

Case 2 $\lambda < 1$ so $1 - \lambda > 0$. Let $1 - \lambda = k^2$.

$(m+1)^2 = k^2 \Rightarrow m + 1 = \pm k \Rightarrow m_1 = k - 1$ & $m_2 = -k - 1$.

Thus, $y = c_1 x^{k-1} + c_2 x^{-k-1}$.

$y(1) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$y(e^{\pi}) = 0 \Rightarrow c_1 \left(e^{(k-1)\pi} - e^{(-k-1)\pi}\right) = 0 \Rightarrow c_1 = 0$. 2

$y = 0$., we don't have eigenvalues < 1

Case 3 $1 - \lambda < 0$. Let $1 - \lambda = -k^2$.

$(m+1)^2 = -k^2 \Rightarrow m + 1 = \pm i k \Rightarrow m_1 = -1 + ik$ & $m_2 = -1 - ik$.

$y = x^{-1}\left(A \cos(k \ln x) + B \sin(k \ln x)\right)$ 2

$y(1) = 0 \Rightarrow \boxed{A = 0}$. $y = x^{-1} B \sin(k \ln x)$

$y(e^{\pi}) = 0 \Rightarrow e^{-\pi} B \sin(k\pi) = 0$, Assume $B \neq 0$, so $\sin(k\pi) = 0$

Thus, $k\pi = k_n \pi = n\pi$ so $k = k_n = n,\ n = 1, 2, \ldots$. So, $\lambda_n = 1 + n^2$ (1)

$\therefore$

$y_n = B_n x^{-1} \sin(n \ln x)$ is a nonzero solution of (1)-(2) corresponding to the eigenvalues $\lambda_n = 1+n^2,\ n = 1, 2, \ldots$ (1)

Finally, $\lambda_n = 1+n^2$; $n = 1, 2, \ldots$ are the eigenvalues &

$y_n = x^{-1} \sin(n \ln x)$, $n = 1, 2, \ldots$ are the corresponding eigenfunctions. (2)

9.2.5 $y_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right)$, $r(x) = 1$. (1)

$$\int_0^L r(x) y_n(x) y_m(x)\,dx = \int_0^L \sin\left(\frac{(2n-1)\pi x}{2L}\right)\sin\left(\frac{(2m-1)\pi x}{2L}\right)dx \quad (2)$$

$$= \frac{1}{2}\int_0^L \left[\cos\left(\frac{(2n-1)\pi x}{2L} - \frac{(2m-1)\pi x}{2L}\right) - \cos\left(\frac{(2n-1)\pi x}{2L} + \frac{(2m-1)\pi x}{2L}\right)\right]dx$$

$$= \ldots = 0, \quad n \neq m. \quad (2)$$

9.3.3 (1)

$$f(x) = x = \sum_{n=1}^{\infty} c_n y_n(x), \quad y_n = \sin\left(\frac{(2n-1)\pi x}{2L}\right), \ r(x) = 1$$

$$c_n = \int_0^L r(x) f(x) y_n(x)\,dx \Big/ \int_0^L r(x) y_n^2(x)\,dx \quad (1)$$

$$\int_0^L x \sin\left(\frac{(2n-1)}{2L}\pi x\right)dx = 4L^2 \frac{(-1)^{n+1}}{(2n-1)^2\pi^2} \quad (1)$$

$$\int_0^L \sin^2\left(\frac{(2n-1)}{2L}\pi x\right)dx = \frac{L}{2} \quad (1)$$

$$c_n = \frac{8L(-1)^{n+1}}{\pi^2(2n-1)^2}. \quad \therefore \ f(x) = \sum_{n=1}^{\infty} \frac{8L(-1)^{n+1}}{\pi^2(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

(1)