HW5 (Heat equation).

Q1 Use separation of variables method to solve:

$$\begin{cases} u_t = 4u_{xx}, & 0 < x < \pi, \quad t > 0 \\ u(x,0) = 2, & 0 < x < \pi. \\ u(0,t) = u_x(\pi,t) = 0, & t > 0 \end{cases}$$

Q2 Use Laplace transform to solve:

$$\begin{cases} u_t = u_{xx}, & x > 0, \quad t > 0 \\ u(0,t) = 1, & t > 0 \\ u(x,0) = 2 \text{ and } \lim_{x \to \infty} u(x,t) = 2. \end{cases}$$

Q3 problem # 11.5.2 from the text book.

HW5

Q1 Use separation of variables method to solve:

$$\begin{cases} u_t = 4u_{xx} & 0<x<\pi, \quad t>0 \quad (1) \\ u(x,0) = 2 & 0<x<\pi \quad (2) \\ u(0,t) = u_x(\pi,t) = 0, & t>0 \quad (3) \end{cases}$$

solution: Let $u = X(x)T(t)$. $(u \neq 0)$

From (1), $XT' = 4X''T \Rightarrow \frac{X''}{X} = \frac{T'}{4T} = \lambda$.

$\Rightarrow X'' - \lambda X = 0$, & $T' - 4\lambda T = 0$.

From (3), $u(0,t) = 0 \Rightarrow X(0) = 0$.

$u_x(\pi,t) = 0 \Rightarrow X'(\pi) = 0$.

Thus, we have the following BVP & DE:

$$\begin{cases} X'' - \lambda X = 0 & (4) \\ X(0) = X'(\pi) = 0 & (5) \end{cases} \qquad T' - 4\lambda T = 0 \quad (6)$$

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Case 1 $\lambda = 0$. $X = ax + b$. $X(0) = 0 \Rightarrow b = 0$. So, $X = ax$, $X' = a$

$X'(\pi) = 0 \Rightarrow a = 0$. Hence $X = 0$. we reject the case 1

Case 2 $\lambda > 0$. check by your self that $X = 0$. 1

Case 3 $\lambda < 0$. Let $\lambda = -k^2$. From (4), $X = A\cos(kx) + B\sin(kx)$

From (5), $X(0) = 0 \Rightarrow \boxed{A = 0}$. So, $X = B\sin(kx)$. $\Rightarrow X' = kB\cos(kx)$.

$X'(\pi) = 0 \Rightarrow kB\cos(k\pi) = 0$. Assume $B \neq 0$, so ~~we choo~~.

$\cos(k\pi) = 0 \Rightarrow k = k_n = \frac{n}{2}$ for $n = 1, 3, 5, 7, \ldots$

Thus, $X_n = B_n \sin\left(\frac{n}{2}x\right)$. From (6), 2

$T_n' - 4\lambda_n T_n = 0 \Rightarrow T_n' + n^2 T_n = 0 \Rightarrow T_n = D_n e^{-n^2 t}$. 1

$\therefore$ $u_n = X_n T_n = E_n \sin\left(\frac{nx}{2}\right) e^{-n^2 t}$ for $n = 1, 3, 5, 7, \ldots$

are solutions of ~~(1)-(3)~~ the problem (1) & (3).

It is clear that u_n does not satisfy the initial condition $u_n(x,0)=2$. For this reason, we look for a solution of (1)-(3) of the form

$$u = \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} E_n \sin\left(\frac{nx}{2}\right) e^{-n^2 t}. \quad (1)$$

u satisfies (1) & (3). We need to find E_n such that u satisfies (2).

$$u(x,0)=2 \Rightarrow \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} E_n \sin\left(\frac{nx}{2}\right) = 2. \quad (1)$$

Recall that, the sine series expansion of 2 on $(0,2\pi)$ is:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{nx}{2}\right) = 2. \quad \text{where } b_n = \frac{2}{2\pi}\int_0^{2\pi} 2\sin\left(\frac{nx}{2}\right)dx$$

$$= \begin{cases} \frac{8}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \quad (1)$$

Thus, $2 = \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{8}{n\pi}\sin\left(\frac{nx}{2}\right)$

$\therefore \quad E_n = \frac{8}{n\pi}$. Finally, $u = \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{8}{n\pi}\sin\left(\frac{nx}{2}\right) e^{-n^2 t}$

Q2 : $\begin{cases} u_t = u_{xx} & x>0,\ t>0 \\ u(0,t) = 1, & t>0 \\ u(x,0) = 2, & \lim_{x\to\infty} u(x,t) = 2. \end{cases}$

solution Let $U(x,s) = \mathcal{L}(u(x,t))$ be the Laplace transform of u w.r.t. t.

$\mathcal{L}(u_t) = \mathcal{L}(u_{xx}) \Rightarrow sU - u(x,0) = U_{xx} \Rightarrow$

$$U_{xx} - sU = -2. \qquad 2$$

The general solution of this DE is $U = U_c + U_p$

where $U_c = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x}$. 1

The form of U_p is:

$U_p = A$. $\quad U_{p_{xx}} - sU_p = -2 \Rightarrow 0 - sA = -2 \Rightarrow A = \frac{2}{s}$.

$\therefore \quad U = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{2}{s}, \quad (*)$ 1

$u(0,t) = 1 \Rightarrow U(0,s) = \frac{1}{s}$.

From $(*)$, $c_1 \cancel{e^{t}} + c_2 + \frac{2}{s} = \frac{1}{s} \Rightarrow \boxed{c_1 + c_2 = -\frac{1}{s}.}$ 1

$\lim_{x\to\infty} u(x,t) = 2 \Rightarrow \lim_{x\to\infty} U(x,s) = \frac{2}{s}$.

From $(*)$ $\lim_{x\to\infty}\left(c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{2}{s}\right) = \frac{2}{s}$

$\Rightarrow \lim_{x\to\infty} c_1 e^{\sqrt{s}\,x} = 0 \Rightarrow c_1 \lim_{x\to\infty} e^{\sqrt{s}\,x} = 0$ this equality is valid ~~iff~~ $\boxed{c_1 = 0}$ 1. Using $c_1 + c_2 = -\frac{1}{s} \Rightarrow c_2 = -\frac{1}{s}$.

Thus, $U = \frac{-e^{-\sqrt{s}\,x}}{s} + \frac{2}{s} = -\mathcal{L}\left(\text{erf}\left(\sqrt{\frac{x^2}{4t}}\right)\right) + 2\mathcal{L}(1)$ 1

Finally,

$$u = -\text{erf}\left(\sqrt{\frac{x^2}{4t}}\right) + 2. \quad 1$$

Q3 : $\begin{cases} u_t = a^2 u_{xx}, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = e^{-b|x|}, & b > 0 \end{cases}$

Solution: Let $u = X(x)T(t)$, $(u \neq 0)$. (We are interested in finding a bounded solution)

$$u_t = a^2 u_{xx} \Rightarrow XT' = a^2 X'' T \Rightarrow \frac{X''}{X} = \frac{T'}{a^2 T} = \lambda. \text{ so}$$

$$X'' - \lambda X = 0 \ \& \ T' - \lambda a^2 T = 0. \quad 2$$

Case 1: $\lambda = 0$. $X = ax + b$ & $T = c \Rightarrow u = XT = \alpha x + \beta$. As $x \to \infty$, $u \to \infty$ if $\alpha \neq 0$. So, we must choose $\alpha = 0$. Hence $u = \beta$. (this case is included in case 3)

Case 2: $\lambda > 0$. Let $\lambda = k^2$ $(k>0)$. So, $X = c_1 e^{kx} + c_2 e^{-kx}$, as $x \to \infty$, $X \to \infty$ if $c_1 \neq 0$. So, choose $c_1 = 0$. As $x \to -\infty$, $X = c_2 e^{-kx} \to \infty$. Because we want X to be bounded then, we choose $c_2 = 0$. $\therefore X \equiv 0$. We reject this choice of λ.

Case 3: $\lambda < 0$. Let $\lambda = -k^2$. So, $X = A\cos(kx) + B\sin(kx)$ & $T = Ce^{-a^2k^2t}$ 1

Since we do not have any boundary conditions, we must include all possible values of k. Thus we sum all the products of X & T, to obtain u:

$$u = \int_0^\infty [D(k)\cos(kx) + E(k)\sin(kx)]\, e^{-k^2a^2t}\, dk \quad 1$$

Using $u(x,0) = e^{-b|x|}$, we get

$$\int_0^\infty [D(k)\cos(kx) + E(k)\sin(kx)]\, dk = e^{-b|x|} \quad 1$$

This is the Fourier integral of $e^{-b|x|}$. So,

$$D(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \left(e^{-b|x|}\cos(kx)\right) dx \ \& \ E(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-b|x|}\sin(kx)\, dx$$

Splitting & integrating by parts twice, we find that

$D(k) = \dfrac{2b}{\pi(b^2+k^2)}$. Since $e^{-b|x|}\sin(kx)$ is an odd function,

$$E(k) = \frac{1}{\pi}\cdot 0 = 0. \quad 2$$

$\therefore$

$$u = \frac{2b}{\pi}\int_0^\infty \frac{\cos(kx)}{b^2+k^2}\, e^{-k^2a^2t}\, dk = \text{---}$$

(You need to integrate by parts twice)