A second-order accurate numerical method for a semilinear integro-differential equation with a weakly singular kernel

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We study a generalized extrapolated Crank–Nicolson scheme for the time discretization of a semilinear integro-differential equation with a weakly singular kernel, in combination with a space discretization by linear finite elements. The scheme uses variable grids in time to compensate for the singular behaviour of the exact solution at \( t = 0 \). With appropriate assumptions on the data and assuming that the spatial domain is convex or smooth, we show that the error is of order \( k^2 + h^2 \), where \( k \) and \( h \) are the parameters for the time and space meshes, respectively. The results of numerical computations demonstrate the convergence of our scheme.

**Keywords:** integro-differential equation; weakly singular kernel; nonuniform time steps; quadrature error; finite elements; Gronwall’s lemma.

1. Introduction

In this paper we study the time and space discretization for semilinear parabolic integro-differential equations of the form

\[
\frac{\partial u}{\partial t}(x, t) + Au(x, t) + \mathcal{F}_\alpha Bu(x, t) = f(x, t, u(t, x))
\]

with \( u(x, 0) = u_0(x) \) for \( (x, t) \in \Omega \times (0, T] \),

subject to the Dirichlet-type boundary condition \( u(x, t) = 0 \) for \( (x, t) \in \partial \Omega \times (0, T] \). Here \( \mathcal{F}_\alpha \) is the Riemann–Liouville fractional integration operator of order \( 0 < \alpha < 1 \), defined by

\[
\mathcal{F}_\alpha v(t) = \int_0^t \beta(t - s)v(s)ds, \quad \text{where} \quad \beta(t) = \frac{t^{\alpha - 1}}{\Gamma(\alpha)}.
\]

By \( \Gamma \) we denote the usual gamma function, \( A \) and \( B \) are linear second-order elliptic differential operators and \( \Omega \) is a bounded domain subset of \( \mathbb{R}^d \).

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The solution \( u(\cdot, t) \) and the source term \( f(\cdot, t, u(\cdot, t)) \) take values in \( L_2(\Omega) \), and the initial data \( u_0 \) is an element of \( L_2(\Omega) \). Throughout the paper we write \( u(t) = u(x, t), \ f(t, u(t)) = f(x, t, u(x, t)) \) and \( u_0 = u_0(x) \). Equations of type (1.1) may be thought of as a model problem occurring in the theory of heat conduction in materials with memory, population dynamics and viscoelasticity (e.g., Friedman & Shinbrot, 1967; Heard, 1982; Renardy et al., 1987).

Many authors have considered numerical methods for a linear problem of the form (1.1), i.e., when \( f(t, u) = f(t) \). Typically, the time discretization is affected by a combination of finite difference and quadratures. Finite difference in time and finite elements in space have been discussed in the case of a smooth kernel (e.g., Sloan & Thomée, 1986; Cannon & Lin, 1988, 1990; Yanik & Fairweather, 1988; Thomée & Zhang, 1989; Lin et al., 1991; Zhang, 1993). For the nonsmooth kernel case we refer to Chen et al. (1992) and Larsson et al. (1998).

In contrast, over the last three decades different numerical methods and techniques have been applied widely to problems of the form (1.1) when \( A = 0 \) and the source term \( f \) is independent of the unknown solution \( u \) (e.g., Lubich, 1988; Fairweather, 1994; McLean et al., 1996; Adolfsson et al., 2003; López-Fernández & Palencia, 2004; Cuesta et al., 2006; McLean & Mustapha, 2007 and the references therein).

The purpose of this paper is to generalize the extrapolated Crank–Nicolson time discretization of the semilinear problem (1.1) using variable time steps combined with appropriate quadrature rules approximating the integral. Also, we will consider the discretization in space by the finite elements, which will then define a fully discrete method for (1.1) in this case. In addition, we present numerical evidence that our error bounds are sharp. We assume that the operators \( A \) and \( B \) are strictly positive definite. However, our approach can be extended with some modifications to cover the case that \( A \) and \( B \) are non-negative, e.g., consider (1.1) subject to homogenous Neumann boundary conditions with \( A = B = -\nabla^2 \).

To define our time-stepping scheme we introduce the time levels \( 0 = t_0 < t_1 < \cdots < t_n < \cdots \leq T \) and set \( t_n := t_n - t_{n-1} \) and \( t_{n-1/2} := \frac{t_{n-1} + t_{n-1}}{2} \). Given a grid function \( \zeta^n = \zeta(t_n) \), we write

\[
\hat{\zeta}^n = \frac{\zeta^n - \zeta^{n-1}}{k_n}, \quad \zeta^{n-1/2} = \frac{\zeta^n + \zeta^{n-1}}{2}, \quad \zeta^n_{n-1/2} = \frac{3\zeta^{n-1} - \zeta^{n-2}}{2}
\]

and define a piecewise constant approximation

\[
\bar{\zeta}(t) := \begin{cases} 
\zeta^1 & \text{for } t_0 < t < t_1, \\
\zeta^{n-1/2} & \text{for } t_{n-1} < t < t_n \text{ and } n \geq 2.
\end{cases}
\]

The modification on the first subinterval in the above piecewise-constant approximation ensures that \( \bar{\zeta}(t) \) does not depend on \( \zeta^0 \), which is necessary for our numerical scheme in cases when \( u_0 \) is not sufficiently regular. Using \( \bar{\zeta} \), we define a discrete fractional integral

\[
\mathcal{I}_{\alpha}^{1/2} \zeta = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{0}^{t} \beta(t-s) \bar{\zeta}(s) \, ds \, dt = \omega_{n,1} \zeta^1 k_1 + \sum_{j=2}^{n} \omega_{n,j} \zeta^{j-1/2} k_j,
\]

where

\[
\omega_{n,j} = \frac{1}{k_n k_j} \int_{t_{n-1}}^{t_n} \int_{t_{j-1}}^{\min(t, t_j)} \beta(t-s) \, ds \, dt > 0.
\]
When the source term \( f \) in (1.1) is independent of \( u \), i.e., \( f(t, u(t)) = f(t) \), starting from an approximation \( U^0 \approx u_0 \) to the initial data, we consider a Crank–Nicolson time discretization

\[
\partial U^n + AU^{n-1/2} + \mathcal{R}_\alpha^{n-1/2} BU = f(t_{n-1/2}) \quad \text{for } n \geq 1
\]  

(1.2)

that generates an approximate solution \( U^n \approx u(t_n) \). To apply the above scheme to problem (1.1) we replace \( f(t_{n-1/2}) \) with \( f(t_{n-1/2}, U^{n-1/2}) \), which leads to solving a nonlinear algebraic system at each time step. To avoid this, following Cannon & Lin (1988, 1990), we generalize the extrapolated Crank–Nicolson method (2.9) in Dupont et al. (1974) for semilinear parabolic problems without a memory term. So the numerical scheme takes the form

\[
\partial U^n + AU^{n-1/2} + \mathcal{R}_\alpha^{n-1/2} BU = f(t_{n-1/2}, \hat{U}^{n-1/2}) \quad \text{for } n \geq 2,
\]  

(1.3)

which requires selections of \( U^0 \) and \( U^1 \). For a given \( U^0 = V^0 \approx u_0 \), we select \( U^1 \) using a predictor–corrector method:

\[
\partial V^1 + AV^1 + \mathcal{R}_\alpha^{1/2} BV = f(t_{1/2}, V^0),
\]  

(1.4)

\[
\partial U^1 + AU^1 + \mathcal{R}_\alpha^{1/2} BU = f(t_{1/2}, V^{1/2}).
\]  

(1.5)

The above scheme is implicit because \( AU^{n-1/2} \) and \( \mathcal{R}_\alpha^{n-1/2} BU \) depend on \( U^n \). Of course, unlike the case of semilinear parabolic problems without the memory term, at each step we must compute a sum involving the solution at all previous time levels.

Although the extrapolated Crank–Nicolson scheme (1.3)–(1.5) is formally second-order accurate in time, the error in \( U^n \) generally fails to be \( \mathcal{O}(k^2) \) if we use a uniform time step \( k \). The reason for this is the lack of regularity in time of the exact solution \( u(t) \) near \( t = 0 \), which is present even in the case of smooth initial data. Since our main goal of the current paper is to show that the above scheme is second-order accurate in time under suitable assumptions, we therefore employ a family of nonuniform meshes that concentrate the time levels near \( t = 0 \). We therefore assume, for a fixed \( \gamma \geq 1 \), that

\[
k_n \leq C_\gamma k \min(1, t_n^{1-1/\gamma}) \quad \text{and} \quad t_n \leq C_\gamma t_{n-1} \quad \text{for } 2 \leq n \leq N,
\]  

(1.6)

with

\[
c_\gamma k^\gamma \leq k_1 \leq C_\gamma k^\gamma.
\]  

(1.7)

Use of the mesh properties (1.6) and (1.7) is standard for integral equations in which the solution \( u(s) \) possesses a fixed singularity at \( s = 0 \) (see, e.g., te Riele, 1982; Chandler & Graham, 1988; Brunner et al., 2001). In our case the integrand \( \beta(t - s)u(s) \) also possesses a moving singularity at \( s = t \) that the quadrature approximation handles using product integration. It turns out that for our error analysis we require a further, and more restrictive, assumption on the time mesh:

\[
0 \leq k_{n+1} - k_n \leq C_\gamma k^2 \min(1, t_n^{1-2/\gamma}) \quad \text{for } n \geq 2.
\]  

(1.8)

Thus the step size must increase monotonically with no abrupt changes from one time level to the next. A typical example of a mesh satisfying all three assumptions (1.6)–(1.8) for an interval \([0, T]\) is

\[
t_n = (nk)^\gamma \quad \text{for } 0 \leq n \leq N \quad \text{with} \quad k = \frac{T^{1/\gamma}}{N}.
\]  

(1.9)
It is important to note that, if the domain $\Omega$ is not convex or $C^{1,1}$, then in practice the exact solution will generally not possess sufficient regularity to achieve second-order accuracy for the space discretization.

An outline of the paper is as follows. In Section 2 we study some regularity properties of the exact solution that are needed in our convergence analysis. It is important to mention that, in the presence of a weakly singular kernel, the exact solution of (1.1) is not smooth. In our approach we recall some properties of the analytic semigroup generated by the differential operator $-A$ from Pazy (1983). Also, we borrow some results from Larsson et al. (1998) where the regularity properties of problem (1.1) were studied when $f \equiv 0$. Since the source term $f$ depends on the unknown solution, the level of complexity for deriving the desired regularity of the exact solution of problem (1.1) will increase substantially. In our analysis we assume that the source term is Lipschitz continuous. The main regularity results are stated in Theorems 2.4 and 2.5. The stability of the scheme (1.3)–(1.5) is studied in Section 3. In our proof we use the positivity for the discrete analogue of the fractional integral term which was presented in McLean & Mustapha (2007) for the use of the proved bound to some of these terms. For deriving the desired regularity of the exact solution of problem (1.1) will increase substantially. In our approach we recall some recent work of McLean & Mustapha (2007) for the use of the proved bound to some of these terms. In Section 5 we describe our fully discrete scheme for problem (1.1). We use the extrapolated Crank–Nicolson method for the time discretization and Galerkin finite elements for the space discretization. In other words, we apply a spatial discrete version of (1.3)–(1.5) employing linear finite elements. We show that the error is $O(k^2 + h^2 |\log k|)$, so that we achieve essentially second-order accuracy in space as well as in time. Finally, Section 6 presents some numerical studies that demonstrate our theoretical convergence results.

2. Regularity of the exact solution

In this section we derive some regularity estimates of the exact solution $u(t)$ of the problem (1.1). More precisely, we seek to show that the solution of (1.1) satisfies

$$t^{1-\alpha} \| Au \| + \| Au'(t) \| + \| Bu'(t) \| + t \| Au''(t) \| + t \| Bu''(t) \| \leq O(t^{\alpha-1})$$

as $t \to 0$,

which eventually allows us to bound the error arising from the time and space discretization. Here, by $u'(t)$ and $u''(t)$ we denote the first and second derivatives of $u(t)$ with respect to time, and $\| \cdot \|$ denotes the norm on the Sobolev space $L^2(\Omega)$. For later use, $\| \cdot \|_{L_q}$ denotes the norm on the Sobolev space $L^q(\Omega)$ for $1 < q \leq \infty$. For a function $g(t, u(t))$, by $D_t g(t, u(t))$ and $g_1(t, u(t))$ we denote the first total and partial time derivatives, respectively. Finally, by $D_{tt} g(t, u(t))$ and $g_{tt}(t, u(t))$ we denote the second total and partial time derivatives, respectively.

The regularity of the exact solution for problems of the form (1.1) was studied in Larsson et al. (1998) when $f(t, u(t)) = 0$. Some of the results of this section are essentially present in Larsson et al. (1998), but because of their importance we give the complete proof. For the sake of simplicity, throughout this section we assume that the operators $A$ and $B$ in (1.1) are equal.

Throughout the paper we assume that the source term $f$ is Lipschitz continuous, i.e.,

$$\| f(t, w_1) - f(t, w_2) \|_{L_q} \leq L \| w_1 - w_2 \|_{L_q} \quad \text{for } w_1, w_2 \in L_q(\Omega) \quad \text{with } 1 < q < \infty.$$  \hspace{1cm} (2.1)
Split the solution $u(t)$ of (1.1) as $u(t) = v(t) + w(t)$, with $v(t)$ and $w(t)$ solving

$$v'(t) + Av(t) = F(t) \quad \text{for } t > 0 \text{ with } v(0) = u_0$$

and

$$w'(t) + Aw(t) = - \int_0^t (t-s)^{a-1} Au(s) ds + G(t, u(t)) \quad \text{for } t > 0 \text{ with } w(0) = 0,$$

respectively, where $F(t) := f(t, u_0)$ and $G(t, u(t)) := f(t, u(t)) - f(t, u_0)$.

The differential equation (2.2) is purely linear parabolic and the solution can be written as

$$v(t) = E(t)u_0 + \int_0^t E(t-s)F(t) ds,$$

where $E(t) = \exp(-tA)$ is the analytic semigroup in $L_q(\Omega)$ generated by $-A$ (see Pazy, 1983), so that for $0 < t < T$ we have

$$\|E(t)v\|_{L_q} + t\|AE(t)v\|_{L_q} + t^2\|A^2E(t)v\|_{L_q} \leq C\|v\|_{L_q} \quad \text{for all } v \in L_q(\Omega). \quad (2.5)$$

**Lemma 2.1** Let $1 < q < \infty$. For $t > 0$, let $\psi(t) \in L_q(\Omega)$ satisfy $\|\psi(t)\|_{L_q} + t\|\psi'(t)\|_{L_q} \leq Ct^{1-a}$. Then

$$\left\| \int_0^t AE(t-s)\psi(s) ds \right\|_{L_q} \leq Ct^{1-a}.$$

**Lemma 2.2** Let $1 < q < \infty$ and let $\psi(t) \in L_q(\Omega)$ for $t > 0$, then

$$\left\| \int_0^t AE(t-s) \int_0^s (s-\sigma)^{a-1} \psi(\sigma) d\sigma ds \right\|_{L_q} \leq C \int_0^t (t-\sigma)^{a-1} \|\psi(\sigma)\|_{L_q} d\sigma.$$

For $q = 2$ the proof of the above lemmas can be found in Larsson et al. (1998). For $q \neq 2$ the proof is similar.

**Lemma 2.3** Let $1 < q < \infty$. For $t > 0$ assume that $\|Au_0 - F(0)\|_{L_q} + \|F(t)\|_{L_q} \leq C$ and $\|E'(t)(Au_0 - F(0))\|_{L_q} + \|F'(t)\|_{L_q} + t\|F''(t)\|_{L_q} \leq Ct^{1-a}$. Then the solution $v$ of (2.2) satisfies

$$\|Av(t)\|_{L_q} + t^{1-a}\|Av'(t)\|_{L_q} \leq C \quad \text{and} \quad \|v'(t)\|_{L_q} + t^{1-a}\|v''(t)\|_{L_q} \leq C.$$

**Proof.** Using (2.4), $AE(t-s) = D_sE(t-s)$ and integrating by parts, we get

$$Av(t) = E(t)Au_0 + \int_0^t D_s E(t-s) F(s) ds$$

$$= E(t)Au_0 + F(t) - E(t)F(0) - \int_0^t E(t-s)F'(s) ds. \quad (2.6)$$

So (2.5) and the given assumptions yield

$$\|Av(t)\|_{L_q} \leq C \left( \|Au_0 - F(0)\|_{L_q} + \|F(t)\|_{L_q} + \int_0^t \|F'(s)\|_{L_q} ds \right) \leq C.$$
Differentiating both sides of (2.6) with respect to $t$ and using $D_t E(t - s) = -AE(t - s)$, we get

$$Av'(t) = E'(t)(Au_0 - F(0)) + \int_0^t AE(t - s)F'(s)ds,$$

and hence the given assumptions and Lemma 2.1 give

$$\|Av'(t)\|_{L_q} = \|E'(t)(Au_0 - F(0))\|_{L_q} + \left\|\int_0^t AE(t - s)F'(s)ds\right\|_{L_q} \leq Ct^{\alpha-1}.$$  

Thus the proof of the desired inequality is completed. From (2.2), $\|Av(t)\|_{L_q} \leq C$ and the assumption that $\|F(t)\| \leq C$, we get the desired bound of $\|v'(t)\|_{L_q}$. The time differentiation of both sides of (2.2), the proved bound $\|Av'(t)\| \leq Ct^{\alpha-1}$ and also the given assumptions yield $\|v''(t)\|_{L_q} \leq Ct^{\alpha-1}$. Therefore the proof is complete.

**Theorem 2.4** Let $1 < q < \infty$. In addition to the assumptions of Lemma 2.3, we impose that $t^{1-a}\|G\_t(t, u(t))\|_{L_q} + \|f_u(t, u(t))\|_{L_q} \leq C$ for $t > 0$. Then the exact solution $u$ of (1.1) satisfies

$$\|u'(t)\|_{L_q} + \|Au(t)\|_{L_q} \leq C \quad \text{for } t > 0.$$

**Proof.** Using $u = w + v$ and Lemma 2.3, it is sufficient to show that the solution $w$ of (2.3) satisfies

$$\|w'(t)\|_{L_q} + \|Aw(t)\|_{L_q} \leq C t^\alpha \quad \text{for } 0 < t \leq T.$$  

From (2.3) we note that

$$\|w'(t)\|_{L_q} \leq \|G(t, u(t))\|_{L_q} + \|Aw(t)\|_{L_q} + \left\|\int_0^t (t - s)^{\alpha-1}Au(s)ds\right\|_{L_q}.  

(2.9)$$

Using Duhamel’s principle, we have

$$w(t) = -\int_0^t E(t - s)\int_0^s (s - \sigma)^{\alpha-1}Au(\sigma)d\sigma ds + \int_0^t E(t - s)G(s, u(s))ds = w_1 + w_2.  

text{(2.10)}$$

For the bound of $Aw_1$, Lemma 2.2, $u = v + w$ and Lemma 2.3 give

$$\|Aw_1\|_{L_q} \leq C \int_0^t (t - \sigma)^{\alpha-1}(\|Av(\sigma)\|_{L_q} + \|Aw(\sigma)\|_{L_q})d\sigma$$

$$\leq C \left( t^\alpha + \int_0^t (t - \sigma)^{\alpha-1}\|Aw(\sigma)\|_{L_q} d\sigma \right).  

(2.11)$$

For the bound of $Aw_2$ we use $AE(t - s) = D_s E(t - s)$ and integrating by parts, we get

$$Aw_2 = \int_0^t AE(t - s)G(s, u(s))ds = \int_0^t D_s E(t - s)G(s, u(s))ds$$

$$= G(t, u(t)) - \int_0^t E(t - s)[G_s(s, u(s)) + f_u(s, u(s))u'(s)]ds.$$
The Lipschitz continuity assumption on the source term $f$ implies that
\[
\|G(t, u(t))\|_{L_q} \leq L\|u(t) - u_0\|_{L_q} \leq L \int_0^t \|u'(s)\|_{L_q} \, ds,
\] (2.12)
and hence (2.5) and the given assumptions yield $\|Aw_2\|_{L_q} \leq Ct^\alpha + C \int_0^t \|u'(s)\|_{L_q} \, ds$. Hence, using (2.10), the bound of $\|Aw_2\|_{L_q}$, (2.14), $u = v + w$ and Lemma 2.3, we have
\[
\|Aw(t)\|_{L_q} \leq Ct^\alpha + C \int_0^t \|w'(s)\|_{L_q} \, ds + C \int_0^t (t - \sigma)^{\alpha - 1}\|Aw(\sigma)\|_{L_q} \, d\sigma.
\]
Thus an application of Gronwall’s lemma (see, for example, Lemma 1 in Chen et al., 1992) achieves
\[
\|Aw(t)\|_{L_q} \leq Ct^\alpha + C \int_0^t \|w'(s)\|_{L_q} \, ds.
\] (2.13)
To bound the third term on the right-hand side of (2.9), using (2.13), we find that
\[
\left\| \int_0^t (t - s)^{\alpha - 1}Aw(s)ds \right\|_{L_q} \leq \int_0^t (t - s)^{\alpha - 1}\|Aw(s)\|_{L_q} \, ds
\]
\[
\leq C \int_0^t (t - s)^{\alpha - 1}\left(s^\alpha + \int_0^s \|w'(\sigma)\|_{L_q} \, d\sigma\right) \, ds
\]
\[
\leq Ct^{2\alpha} + Ct^\alpha \int_0^t \|w'(s)\|_{L_q} \, ds.
\] (2.14)
Finally, (2.9) and (2.12)–(2.14) yield $\|w'(t)\|_{L_q} \leq Ct^\alpha + C \int_0^t \|w'(s)\|_{L_q} \, ds$. Hence an application of Gronwall’s lemma gives $\|w'(t)\|_{L_q} \leq Ct^\alpha$. Substituting this bound into (2.13) completes the proof of the inequality (2.8). □

Next we derive other needed bounds of $u(t)$ involving second time derivatives.

**THEOREM 2.5** Let $1 < q < \infty$. In addition to the assumptions of Theorem 2.4, we impose that $\|f_{uu}(t, u(t))\|_{L_\infty} \leq C$ and $t\|G_{tt}(t, u(t))\|_{L_q} + \|f_{tu}(t, u(t))\|_{L_q} \leq Ct^{\alpha - 1}$ for $t > 0$. Then the exact solution $u$ of (1.1) satisfies
\[
\|u''\|_{L_q} + \|Au'(t)\|_{L_q} \leq Ct^{\alpha - 1}.
\]

**Proof.** As in Theorem 2.4, using $u = w + v$ and Lemma 2.3, it is enough to show that the solution $w$ of (2.3) satisfies
\[
\|w''(t)\|_{L_q} + \|Aw'(t)\|_{L_q} \leq Ct^{\alpha - 1} \quad \text{for } 0 < t \leq T.
\] (2.15)
Differentiating (2.10), we get
\[
w'' + Aw' = -t^{\alpha - 1}Au_0 - \int_0^t s^{\alpha - 1}ADt u(t-s)ds + D_t G(t, u(t)) \quad \text{for } t > 0.
\] (2.16)
Note that \( G(0, u(0)) = 0 \) and also \( Aw(0) = 0 \), and hence (2.16) yields \( w(0) = 0 \). So, by Duhamel’s principle and setting \( u = w + v \), we have

\[
\begin{align*}
w'(t) &= \left[ -\int_0^t E(t-s)s^{\alpha-1} Au_0 \, ds - \int_0^t E(t-s) \int_0^s (s-\sigma)\alpha^{-1} Aw'(\sigma) \, d\sigma \, ds \\
&\quad - \int_0^t E(t-s) \int_0^s (s-\sigma)\alpha^{-1} Au'(\sigma) \, d\sigma \, ds + \int_0^t E(t-s) D_s G(s, u(s)) \, ds \right] \\
&= w_1 + w_2 + w_3.
\end{align*}
\]

(2.17)

Lemmas 2.2 and 2.1 give

\[
\|Aw_1\|_{L_q} \leq C \left( \int_0^t t^{\alpha-1} \|Au_0\|_{L_q} \, ds + \int_0^t (t-s)^{\alpha-1} \|Aw'(s)\|_{L_q} \, ds \right) \leq Ct^{\alpha} + \int_0^t (t-s)^{\alpha-1} \|Aw'(s)\|_{L_q} \, ds.
\]

For the bound of \( Aw_2 \), using \( Au'(t) \) as in (2.7), we have

\[
Aw_2 = \int_0^t AE(t-s)g(s)ds - \int_0^t AE(t-s) \int_0^s (s-\sigma)\alpha^{-1} \int_0^\sigma AE(\sigma-\delta)F'(\delta) \, d\delta \, d\sigma \, ds,
\]

where

\[
g(t) := \int_0^t (t-s)^{\alpha-1} AE(s)(Au_0 - F(0)) \, ds = \int_0^t s^{\alpha-1} AE(t-s)(Au_0 - F(0)) \, ds.
\]

We shall apply Lemma 2.1. By this lemma, \( \|g(t)\|_{L_q} \leq Ct^{\alpha-1}\|Au_0 - F(0)\|_{L_q} \leq Ct^{\alpha-1} \). For the estimate required for \( g'(t) \) we write

\[
g(t) = \left( \int_0^{t/2} + \int_{t/2}^t \right) s^{\alpha-1} AE(t-s)(Au_0 - F(0)) \, ds = g_1(t) + g_2(t).
\]

Here, using \( D_sAE(t-s) = -A^2E(t-s) \), we get

\[
g_1'(t) = \left( \int_0^{t/2} + \int_{t/2}^t \right) A s^{\alpha-1} AE(t/2)(Au_0 - F(0)) - \int_0^{t/2} s^{\alpha-1} A^2E(t-s)(Au_0 - F(0)) \, ds
\]

and clearly, using (2.5), \( \|g_1'(t)\|_{L_q} \leq Ct^{\alpha-2}\|Au_0 - F(0)\|_{L_q} \leq Ct^{\alpha-2} \). Further, by using \( AE(t-s) = D_sE(t-s) \), integrating by parts and a change of variables, we have

\[
g_2(t) = t^{\alpha-1}(1 - 2^{1-\alpha}E(t/2))(Au_0 - F(0)) - (\alpha - 1) \int_0^{t/2} (t-s)^{\alpha-2}E(s)(Au_0 - F(0)) \, ds,
\]

and hence (2.5) yields \( \|g_2(t)\|_{L_q} \leq Ct^{\alpha-1}\|Au_0 - F(0)\|_{L_q} \leq Ct^{\alpha-1} \).

Differentiating this expression with respect to time and proceeding as for \( g_1 \) gives the same bound for \( g_2 \) as for \( g_1 \), namely, \( \|g_2'(t)\|_{L_q} \leq Ct^{\alpha-2}\|Au_0 - F(0)\|_{L_q} \leq Ct^{\alpha-2} \). We may then apply
Lemmas 2.1 and 2.2 to obtain
\[ \|A w_2\|_{L_q} \leq C t^{a-1} + C \int_0^t (t-s)^{a-1} \left\| \int_0^s AE(s-\sigma)F'(\sigma) d\sigma \right\|_{L_q} \, ds \]
\[ \leq C t^{a-1} + C \int_0^t (t-s)^{a-1} s^{a-1} \, ds \]
\[ \leq C \left( t^{a-1} + \int_{t/2}^t s^{a-1}(t-s)^{a-1} \, ds + \int_{t/2}^t s^{a-1} (t-s)^{a-1} \, ds \right) \leq C (t^{a-1} + t^{2a-1}). \]

To bound \( A w_3 \), splitting it into two parts and using \( AE(t-s) = D_s E(t-s) \), we have
\[ A w_3 = \int_0^{t/2} AE(t-s)D_s G(s, u(s)) ds + \int_{t/2}^t D_s E(t-s)D_s G(s, u(s)) ds = h_1(t) + h_2(t). \quad (2.18) \]

Integrating by parts and using \( D_s AE(t-s) = A^2 E(t-s) \), (2.5), (2.12) and Theorem 2.4, we find that
\[ \|h_1(t)\|_{L_q} = \left\| \int_0^{t/2} AE(t/2)G(t/2, u(t/2)) - \int_0^{t/2} D_s AE(t-s)G(s, u(s)) ds \right\|_{L_q} \]
\[ \leq C \|G(t/2, u(t/2))\|_{L_q} + \int_0^{t/2} (t-s)^{-2} \|G(s, u(s))\|_{L_q} \, ds \]
\[ \leq C \int_0^{t/2} \|u'(s)\|_{L_q} \, ds + C t^{-2} \int_0^{t/2} \int_0^s \|u'(\sigma)\|_{L_q} \, d\sigma \, ds \leq C. \quad (2.19) \]

For \( h_2(t) \), integrating by parts, we find that
\[ \|h_2(t)\|_{L_q} = \left\| E(0)D_t G(t, u(t)) - E(t/2)D_t G(t/2, u(t/2)) - \int_{t/2}^t E(t-s)D_{ss} G(s, u(s)) ds \right\|_{L_q} \]
\[ \leq \|D_t G(t, u(t))\|_{L_q} + C \|D_t G(t/2, u(t/2))\|_{L_q} + \int_{t/2}^t \|D_{ss} G(s, u(s))\|_{L_q} \, ds. \quad (2.20) \]

The given assumptions, Theorem 2.4, Lemma 2.3 and \( u = v + w \) lead to
\[ \|D_t G(t, u(t))\|_{L_q} = \|G_t(t, u(t)) + f_u(t, u(t))u'(t)\|_{L_q} \]
\[ \leq \|G_t(t, u(t))\|_{L_q} + \|f_u(t, u(t))\|_{L_\infty} \|u'(t)\|_{L_q} \leq C t^{a-1} \quad (2.21) \]
and
\[ \|D_{tt} G(t, u(t))\|_{L_q} = \|G_{tt}(t, u(t)) + 2f_{uu}(t, u(t))u'(t) + f_{uu}(t, u(t))u'(t)^2 + f_u(t, u(t))u''(s)\|_{L_q} \]
\[ \leq C \left( \|G_{tt}(t, u(t))\|_{L_q} + \|f_{uu}(t, u(t))\|_{L_2} \|u'(t)\|_{L_2}^2 + \|u'(t)\|_{L_2}^2 + \|u''(t)\|_{L_q} + \|w''(t)\|_{L_q} \right) \]
\[ \leq C \left( t^{a-2} + \|w''(t)\|_{L_q} \right). \quad (2.22) \]
Hence (2.18)–(2.22) gives \( \|Aw_1\|_{L_q} \leq Ct^{\alpha - 1} + C \int_{t/2}^t \|w''(s)\|_{L_q} \, ds \). Using this and the estimated results of \( \|Aw_1\|_{L_q} \) and \( \|Aw_2\|_{L_q} \) in (2.17), we get

\[
\|Aw'(t)\|_{L_q} \leq Ct^{\alpha - 1} + C \int_{t/2}^t \|w''(s)\|_{L_q} \, ds.
\] (2.23)

To bound the second term on the right-hand side of (2.16) we use \( u(t) = w(t) + v(t) \), (2.23) and Lemma 3.2 to obtain

\[
\left\| \int_0^t s^{\alpha - 1} AD_t u(t - s) \, ds \right\|_{L_q} \leq C \int_0^t (t - s)^{\alpha - 1} \|Aw'(s)\|_{L_q} \, ds + \int_0^t (t - s)^{\alpha - 1} \|Av'(s)\|_{L_q} \, ds
\]
\[
\leq C \int_0^t (t - s)^{\alpha - 1} s^{\alpha - 1} \, ds + C \int_0^t (t - s)^{\alpha - 1} \int_0^s \|w''(\sigma)\|_{L_q} \, d\sigma \, ds
\]
\[
\leq C \int_{t/2}^t s^{\alpha - 1} (t - s)^{\alpha - 1} \, ds + C \int_{t/2}^t s^{\alpha - 1} (t - s)^{\alpha - 1} \, ds
\]
\[
+ Ct^\alpha \int_0^t \|w''(s)\|_{L_q} \, ds
\]
\[
\leq Ct^\alpha \left( t^{\alpha - 1} + \int_0^t \|w''(s)\|_{L_q} \, ds \right).
\]

Therefore using (2.21) and (2.23) in (2.16) yields \( \|w''(t)\|_{L_q} \leq Ct^{\alpha - 1} + C \int_0^t \|w''(s)\|_{L_q} \, ds \). Finally, an application of Gronwall’s lemma to this inequality follows by substituting the obtained bound of \( \|w''(t)\|_{L_q} \) into (2.23), giving the desired bound of \( \|Aw'(t)\| \). Hence the proof of (2.15) is complete.

In addition to the regularity results that we have shown, we need to prove that \( \|Au''(t)\|_{L_q} \leq Ct^{\alpha - 2} \). This can be done by following the procedure that was used to show that \( \|Au'(t)\|_{L_q} \leq Ct^{\alpha - 1} \) with some modifications and under suitable assumptions.

### 3. Stability of the numerical solution

In this section we study the stability of the generalized extrapolated Crank–Nicolson scheme (1.3)–(1.5). In Lemma 3.2 we derive some important properties of the approximate solutions \( U^n \) and \( V^1 \). A key ingredient is the following discrete version of the positivity property \( \int_0^T v(t) \int_0^t \beta(t - s) v(s) \, ds \) for all \( T > 0 \) and any continuous function \( v: [0, T] \to \mathbb{R} \). For the proof of the next lemma see McLean et al. (1996).

**Lemma 3.1** If \( v^1, \ldots, v^n \) belong to \( H_0^1(\Omega) \) then \( (v^1, \mathcal{F}_0^{1/2} B v) k_1 + \sum_{j=2}^n (v^{j-1/2}, \mathcal{F}_0^{j-1/2} B v) k_j \geq 0 \).

The next result will be used to prove the stability and the convergence of the approximate solution \( U^n \) for \( n \geq 1 \).

**Lemma 3.2** If \( U^n \) and \( V^1 \) are defined by (1.3)–(1.5) with \( U^0 = V^0 \), then

\[
\|V^1\| \leq \|U^0\| + 2k_1 \|f(t_{1/2}, U^0)\|
\]
and
\[ \|U^M\| \leq \|U^0\| + 2 \sum_{n=2}^{M} k_n \|f(t_{n-1/2}, \widehat{U}^{n-1/2})\| + 2k_1\|f(t_{1/2}, V^{1/2})\| \quad \text{for } M = 1, \ldots, N. \]

**Proof.** Taking the inner product of both sides of (1.4) with $2V^1k_1$, we obtain
\[ 2\|V^1\|^2 - 2(V^1, V^0) + 2k_1(AV^1, V^1) + 2k_1(\mathcal{P}_{1/2}^1 BV, V^1) = 2k_1(f(t_{1/2}, V^0), V^1). \]

Since $\|V^1 - V^0\|^2 = \|V^1\|^2 - 2(V^1, V^0) + \|V^0\|^2$, it follows that
\[ \|V^1\|^2 - \|V^0\|^2 + \|V^1 - V^0\|^2 + 2k_1(AV^1, V^1) + 2k_1(\mathcal{P}_{1/2}^1 BV, V^1) = 2k_1(f(t_{1/2}, V^0), V^1). \quad (3.1) \]

Using this, $\|V^1 - V^0\|^2 \geq 0$, $(AV^1, V^1) \geq 0$, $(\mathcal{P}_{1/2}^1 BV, V^1) \geq 0$ by Lemma 3.1 and the Cauchy–Schwarz inequality, we find that

\[ (\|V^1\| + \|V^0\|)(\|V^1\| - \|V^0\|) = \|V^1\|^2 - \|V^0\|^2 \leq 2k_1(f(t_{1/2}, V^0), V^1) \]
\[ \leq 2k_1\|V^1\|\|f(t_{1/2}, V^0)\| \leq 2k_1(\|V^1\| + \|V^0\|)\|f(t_{1/2}, V^0)\|. \]

So $\|V^1\| - \|V^0\| \leq 2k_1(\|f(t, V^0)\|)$ and the first desired result is obtained.

Taking the inner product of both sides of (1.5) with $2U^1k_1$ and following a similar procedure as before, we obtain
\[ \|U^1\|^2 - \|U^0\|^2 + \|U^1 - U^0\|^2 + 2k_1(AU^1, U^1) + 2k_1(\mathcal{P}_{1/2}^1 BU, U^1) = 2k_1(f(t_{1/2}, V^{1/2}), U^1) \quad (3.2) \]

and $\|U^1\| - \|U^0\| \leq 2k_1\|f(t_{1/2}, V^{1/2})\|$. So the second required result is proved for $M = 1$.

For $M \geq 2$ we take the inner product of both sides of (1.3) with $k_n(U^n + U^{n-1})$ and get
\[ \|U^n\|^2 - \|U^{n-1}\|^2 + 2k_n(AU^{n-1/2} + \mathcal{P}_{1/2}^{n-1/2} BU, U^{n-1/2}) = 2k_n(f(t_{n-1/2}, \widehat{U}^{n-1/2}), U^{n-1/2}). \]

Summing from $n = 2$ to $n = M$ gives
\[ \|U^M\|^2 - \|U^1\|^2 + 2 \sum_{n=2}^{M} k_n(AU^{n-1/2}, U^{n-1/2}) + 2 \sum_{n=2}^{M} k_n(\mathcal{P}_{1/2}^{n-1/2} BU, U^{n-1/2}) \]
\[ = 2 \sum_{n=2}^{M} k_n(f(t_{n-1/2}, \widehat{U}^{n-1/2}), U^{n-1/2}). \quad (3.3) \]

Using (3.2), (3.3), $(AU^{n-1/2}, U^{n-1/2}) \geq 0$, $(AU^1, U^1) \geq 0$ and $k_1(\mathcal{P}_{1/2}^1 BU, U^1) + \sum_{n=2}^{M} k_n(\mathcal{P}_{1/2}^{n-1/2} BU, U^{n-1/2}) \geq 0$ from Lemma 3.1, we have
\[ \|U^M\|^2 - \|U^0\|^2 \leq 2 \sum_{n=2}^{M} k_n(f(t_{n-1/2}, \widehat{U}^{n-1/2}), U^{n-1/2}) + 2k_1(f(t_{1/2}, V^{1/2}), U^1), \quad (3.4) \]
and hence the Cauchy–Schwarz inequality gives

\[ \| U^M \|^2 \leq \| U^0 \|^2 + 2 \sum_{n=2}^{M} k_n \| f(t_{n-1/2}, \hat{U}^{n-1/2}) \| \| U^{n-1/2} \| + 2k_1 \| f(t_{1/2}, V^{1/2}) \| \| U^1 \|. \]  

Hence, the standard discrete analogue of Gronwall’s inequality yields the stability

\[ \| U^M \|^2 \leq \| U^0 \|^2 + 2 \sum_{n=2}^{M} k_n \| f(t_{n-1/2}, \hat{U}^{n-1/2}) \| \| U^{n-1/2} \| + 2k_1 \| f(t_{1/2}, V^{1/2}) \| \| U^1 \|. \]  

Let \( M_0 \) be such that \( \| U^{M_0} \| = \max_{2 \leq n \leq M} \| U^n \| \). Then (3.5) and the triangle inequality yield

\[ \| U^{M_0} \|^2 \leq \| U^0 \|^2 + 2 \left( \sum_{n=2}^{M_0} k_n \| f(t_{n-1/2}, \hat{U}^{n-1/2}) \| + k_1 \| f(t_{1/2}, V^{1/2}) \| \right) \| U^{M_0} \|. \]

Thus for \( M = 2, \ldots, N \) we have

\[ \| U^M \| \leq \| U^{M_0} \| \leq \| U^0 \| + 2 \sum_{n=2}^{M_0} k_n \| f(t_{n-1/2}, \hat{U}^{n-1/2}) \| + 2k_1 \| f(t_{1/2}, V^{1/2}) \|. \]

**Lemma 3.3** The approximate solution \( U^n \) defined by (1.3)–(1.5) satisfies

\[ \| U^n \| \leq C_{\gamma, L} \| U^0 \| + C_{\gamma, L} \sum_{j=1}^{n} k_j \| f(t_{j-1/2}, U^0) \| \quad \text{for } n = 1, \ldots, N. \]

**Proof.** Using Lemma 3.2, (2.1) and our graded mesh assumption (1.7), we obtain

\[ \| U^n \| \leq \| U^0 \| + 2 \sum_{j=2}^{n} k_j \| f(t_{j-1/2}, \hat{U}^{j-1/2}) \| + 2k_1 \| f(t_{1/2}, V^{1/2}) \| \]

\[ \leq \| U^0 \| + 2 \sum_{j=2}^{n} k_j \| f(t_{j-1/2}, \hat{U}^{j-1/2}) - f(t_{j-1/2}, U^0) \| + 2k_1 \| f(t_{1/2}, V^{1/2}) - f(t_{1/2}, U^0) \| \]

\[ + 2 \sum_{j=1}^{n} k_j \| f(t_{j-1/2}, U^0) \| \]

\[ \leq \| U^0 \| + 2L \sum_{j=2}^{n} k_j \| \hat{U}^{j-1/2} - U^0 \| + 2Lk_1 \| V^{1/2} - U^0 \| + 2 \sum_{j=1}^{n} k_j \| f(t_{j-1/2}, U^0) \| \]

\[ \leq \| U^0 \| + L \sum_{j=2}^{n} k_j \| 3U^{j-1} - U^{j-2} \| + 2L \| U^0 \| + Lk_1 (\| V^{1/2} \| + \| U^0 \| ) + 2 \sum_{j=1}^{n} k_j \| f(t_{j-1/2}, U^0) \| \]

\[ \leq (1 + 2L + 2Lk_1 + Lk_2) \| U^0 \| + 2(Lk_1 + 1)2 \sum_{j=1}^{n} k_j \| f(t_{j-1/2}, U^0) \| + C_{\gamma} L \sum_{j=1}^{n-1} k_j \| U^j \|. \]

Therefore an application of the standard discrete analogue of Gronwall’s inequality yields the stability result. \( \square \)
4. Error from the time discretization

In this section we estimate the error \( e^n = U^n - u^n \) (\( u^n := u(t_n) \)), where the approximate solution \( U^n \) is obtained using the extrapolated Crank–Nicolson time discretization defined by (1.3)–(1.5) or equivalently by

\[
U^n - U^{n-1} + k_n AU^{n-1/2} + \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s)B\hat{U}(s)ds \, dt = k_n f(t_{n-1}/2, \hat{U}^{n-1/2}) \quad \text{for } n \geq 2, \tag{4.1}
\]

and for \( n = 1 \) by

\[
U^1 - U^0 + k_1 AU^1 + \int_0^{t_1} \int_0^t \beta(t-s)BU^1 ds \, dt = k_1 f(t_{n-1}/2, V^{1/2}), \tag{4.2}
\]

where \( V^0 = U^0 \approx u(t_0) \) and \( V^1 \) is defined by (1.4).

For comparison, integrating (1.1) from \( t = t_{n-1} \) to \( t = t_n \) shows that the exact solution satisfies

\[
u(t_n) - u(t_{n-1}) + \int_{t_{n-1}}^{t_n} \left( Au(t) + \int_0^t \beta(t-s)Bu(s)ds \right) dt = \int_{t_{n-1}}^{t_n} f(t, u(t))dt \quad \text{for } n \geq 1. \tag{4.3}
\]

Hence the error satisfies

\[
e^n - e^{n-1} + k_n A e^{n-1/2} + \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s)B\bar{v}(s)ds \, dt = \eta^{n-1/2} \quad \text{for } n \geq 2, \tag{4.4}
\]

and for \( n = 1 \) we have

\[
e^1 - e^0 + k_1 A e^1 + \int_0^{t_1} \int_0^t \beta(t-s)B\bar{v}(s)ds \, dt = \eta^{1/2}, \tag{4.5}
\]

where \( \eta^{n-1/2} = \eta_1^{n-1/2} + \eta_2^{n-1/2} + \eta_3^{n-1/2} \) with

\[
\eta_1^{1/2} = \int_{t_{n-1}}^{t_n} \left[ f(t_{1/2}, V^{1/2}) - f(t, u(t)) \right] dt,
\]

\[
\eta_1^{n-1/2} = \int_{t_{n-1}}^{t_n} \left[ f(t_{n-1}/2, \hat{U}^{n-1/2}) - f(t, u(t)) \right] dt \quad \text{for } n \geq 2,
\]

\[
\eta_2^{n-1/2} = \int_{t_{n-1}}^{t_n} \left[ Au(t) - Au(t_n) \right] dt \quad \text{for } n \geq 1,
\]

\[
\eta_3^{n-1/2} = \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s)[Bu(s) - B\bar{u}(s)]ds \, dt \quad \text{for } n \geq 1. \tag{4.6}
\]

Since (4.4) and (4.5) have the same form as (4.1) and (4.2), respectively, Lemma 3.2 implies that

\[
\|e^n\| \leq \|e^0\| + 2 \sum_{j=1}^{n} \|\eta^{j-1/2}\| \quad \text{for } n \geq 1. \tag{4.7}
\]
Thus our task is reduced to estimating the sum on the right-hand side of this inequality. In our process we allow for the possibility that \( \|Au''(t)\|, \|Bu''(t)\| \) and \( \|u''(t)\| \) are not integrable on the time step size \((0, t_{1/2})\).

Lemma 3.2 and Theorem 3.3 in the recent work of McLean & Mustapha (2007) provide the following estimate of \( \|\eta_3^{j-1/2}\| \):

\[
\sum_{j=1}^{n} \|\eta_3^{j-1/2}\| \leq C_{a,T} \left( \int_{0}^{t_1} s \|Bu'(s)\|ds + k_2 \int_{t_1}^{t_2} \|Bu'(s)\|ds + \sum_{j=2}^{n} k_j^2 \int_{t_{j-1}}^{t_j} \|Bu''(s)\|ds \right) \\
+ C_{a,T} \sum_{j=2}^{n-1} \left( k_j^2 \int_{t_j}^{t_{j+1}} \|Bu'(s)\|ds + (k_{j+1} - k_j) \int_{t_{j-1}}^{t_{j+1}} \|Bu'(s)\|ds \right),
\]  

(4.8)

where

\[
t_j^n = \frac{t_{j+1} + t_{j-1}}{2} \quad \text{for} \quad j \geq 1.
\]

**Lemma 4.1** For \( \eta_1^{j-1/2} \) defined as in (4.6) for \( j \geq 1 \), we have

\[
\sum_{j=1}^{n} \|\eta_1^{j-1/2}\| \leq C_L \left( \sum_{j=2}^{n} k_j \|\hat{e}^{j-1/2}\| + k_1(\|V^1 - u^1\| + ||e^0||) \right) \\
+ \sum_{j=2}^{n} k_j \left( C_L k_j \int_{t_{j-1}}^{t_j} \left( \|u''(t)\| + \|D_{tt} f(t, u(t))\| \right)dt + C_L \int_{t_{j-1}}^{t_j} \|u'(t)\|dt \right) \\
+ \int_{0}^{t_1} [C_L k_1 \|u''(t)\| + t^2 \|D_{tt} f(t, u(t))\|]dt \\
+ k_1^2 \int_{t_{1/2}}^{t_1} [C_L \|u''(t)\| + \|D_{tt} f(t, u(t))\|]dt.
\]  

(4.9)

**Proof.** To show the stated result we make the splitting \( \eta_1^{j-1/2} = \eta_{11}^{j-1/2} + \eta_{12}^{j-1/2} \), where

\[
\eta_{11}^{1/2} = k_1 [f(t_{1/2}, V^{1/2}) - f(t_{1/2}, u(t_{1/2}))],
\]

\[
\eta_{11}^{j-1/2} = k_j [f(t_{j-1/2}, \hat{U}^{j-1/2}) - f(t_{j-1/2}, u(t_{j-1/2}))] \quad \text{for} \quad j \geq 2,
\]

\[
\eta_{12}^{j-1/2} = - \int_{t_{j-1}}^{t_j} [f(t, u(t)) - f(t_{j-1/2}, u(t_{j-1/2}))]dt \quad \text{for} \quad j \geq 1.
\]

To bound \( \|\eta_{11}^{j-1/2}\| \) for \( j \geq 2 \) we use (2.1), and by adding and subtracting \( \hat{\eta}^{j-1/2} \) we get

\[
\|\eta_{11}^{j-1/2}\| \leq C_L k_j \|\hat{U}^{j-1/2} - u(t_{j-1/2})\| \leq C_L k_j (\|\hat{e}^{j-1/2}\| + ||\hat{t}^{j-1/2} - u(t_{j-1/2})||).
\]
Adding and subtracting \(u(t_{j-1}^*)\), we get
\[
\widehat{u}^{j-1/2} - u(t_{j-1/2}) = [u^{j-1/2} - u(t_{j-1/2})] - \left[ \frac{u^j + u^{j-2}}{2} - u(t_{j-1}^*) \right] + [u^{j-1} - u(t_{j-1}^*)],
\]
\[
u^{j-1/2} := \frac{u(t_{j-1}) + u(t_j)}{2}.
\]

For \(\mu_1 < \mu_2\) and \(\mu_{12} = \frac{\mu_1 + \mu_2}{2}\) we observe that
\[
\frac{u(\mu_1)}{2} - u(\mu_{12}) = \frac{1}{2} \left[ \int_{\mu_1}^{\mu_{12}} (s - \mu_1)u''(s)ds - \int_{\mu_{12}}^{\mu_2} (s - \mu_2)u''(s)ds \right].
\]
\[
(4.10)
\]

Since \(k_{j-1} \leq k_j\), we have \(t_{j-1} \leq t_{j-1}^* \leq t_j\), and hence for \(j \geq 2\) it follows that
\[
\|\widehat{\eta}^{j-1/2} - u(t_{j-1/2})\| \leq k_{j} \int_{t_{j-1}}^{t_j} \|u''(s)\|ds + \frac{1}{2} \int_{t_{j-2}}^{t_{j-1}} (s - t_{j-2}) \|u''(s)\|ds + \|u^{j-1} - u(t_{j-1}^*)\|.
\]

Thus for \(j \geq 2\) we have
\[
\|\eta_{11}^{j-1/2}\| \leq C_L k_j \left( \|\widehat{\eta}^{j-1/2}\| + k_{j} \int_{t_{j-1}}^{t_j} \|u''(s)\|ds + \int_{t_{j-2}}^{t_{j-1}} (s - t_{j-2}) \|u''(s)\|ds + \int_{t_{j-1}}^{t_j} \|u'(s)\|ds \right).
\]

Similarly, for \(j = 1\) we have
\[
\|\eta_{11}^{1/2}\| \leq C_L k_1 \left( \|\widehat{\eta}^{1/2}\| + \|u^{1/2} - u(t_{1/2})\| \right)
\]
\[
\leq C_L k_1 \left( \|V^{1/2} - u^1\| + \|e^0\| + \int_0^{t_{1/2}} s \|u''(s)\|ds + k_1 \int_{t_{1/2}}^{t_1} \|u''(s)\|ds \right)
\]
\[
\leq C_L k_1 \left( \|V^{1/2} - u^1\| + \|e^0\| + \int_0^{t_{1}} s \|u''(s)\|ds + k_1 \int_{t_{1/2}}^{t_1} \|u''(s)\|ds \right).
\]

Therefore
\[
\sum_{j=1}^{n} \|\eta_{11}^{j-1/2}\| \leq C_L \sum_{j=2}^{n} k_j \left( \|\widehat{\eta}^{j-1/2}\| + k_j \int_{t_{j-1}}^{t_j} \|u''(t)\|dt + \int_{t_{j-1}}^{t_{j-1}} \|u'(t)\|dt \right)
\]
\[
+ C_L k_1 \left( \int_0^{t_{1}} s \|u''(s)\|ds + k_1 \int_{t_{1/2}}^{t_1} \|u''(s)\|ds + \|V^{1/2} - u^1\| + \|e^0\| \right). \quad (4.11)
\]
For the estimation of $\eta_{12}^{-1/2}$, using Taylor’s expansion for integrals, splitting, changing the order of integrals and integrating, we get

$$-\eta_{12}^{-1/2} = \int_{t_{j-1}}^{t_j} \left( (t - t_{j-1}/2) D_t f(t_{j-1}/2, u(t_{j-1}/2)) + \int_{t_{j-1}/2}^{t} (t - s) D_{ss} f(s, u(s)) ds \right) dt$$

$$= \int_{t_{j-1}}^{t_j} \int_{t_{j-1}/2}^{t} (t - s) D_{ss} f(s, u(s)) ds \, dt$$

$$= \int_{t_{j-1}}^{t_j} \int_{t_{j-1}/2}^{s} (s - t) D_{ss} f(s, u(s)) ds \, dt + \int_{t_{j-1}}^{t_j} \int_{s}^{t} (t - s) D_{ss} f(s, u(s)) ds \, dt$$

$$= \int_{t_{j-1}}^{t_j} \int_{t_{j-1}/2}^{s} (s - t) D_{ss} f(s, u(s)) ds \, dt + \int_{t_{j-1}}^{t_j} \int_{t_{j-1}/2}^{t} (t - s) D_{ss} f(s, u(s)) ds \, dt$$

$$= \frac{1}{2} \int_{t_{j-1}}^{t_j} (s - t_{j-1})^2 D_{ss} f(s, u(s)) ds + \frac{1}{2} \int_{t_{j-1}/2}^{t_j} (t_j - s)^2 D_{ss} f(s, u(s)) ds.$$

So

$$\sum_{j=1}^{n} \| \eta_{12}^{-1/2} \| \leq \sum_{j=2}^{n} k_j^2 \int_{t_{j-1}}^{t_j} \| D_t f(t, u(t)) \| dt + \frac{1}{2} \left( \int_{0}^{t_{j-1}/2} t^2 \| D_t f(t, u(t)) \| dt \right)$$

$$+ k_1^2 \int_{t_{j-1}/2}^{t_{j-1}} \| D_t f(t, u(t)) \| dt \right).$$

Now, using the estimated bound of $\| \eta_{11}^{-1/2} \|$ and $\| \eta_{12}^{-1/2} \|$, we complete the proof. \[ \square \]

At this stage $\eta_{2}^{-1/2}$ remains to be estimated. Using

$$u(t) - u(t_{j-1}/2) = (t - t_{j-1}/2)u'(t_{j-1}/2) + \int_{t_{j-1}/2}^{t} (t - s)u''(s) ds$$

and (4.10), we get for $j \geq 2$ that

$$\eta_{2}^{-1/2} = \int_{t_{j-1}}^{t_j} A[u(t) - u(t_{j-1}/2)] dt + \int_{t_{j-1}}^{t_j} A[u(t_{j-1}/2) - u^{j-1/2}] dt$$

$$= \int_{t_{j-1}}^{t_j} \int_{t_{j-1}/2}^{s} (s - t) A u''(s) ds \, dt - \frac{k_j}{2} \left( \int_{t_{j-1}/2}^{t} (s - t_{j-1}) A u''(s) ds - \int_{t_{j-1}/2}^{t} (s - t_j) A u''(s) ds \right).$$

For $j = 1$, changing the order of integrals, we have

$$\eta_{2}^{1/2} = -\int_{0}^{t_{1}} \int_{t}^{t_1} A u'(s) ds \, dt = -\int_{0}^{t_{1}} \int_{0}^{s} A u'(s) dt \, ds = -\int_{0}^{t_{1}} s A u'(s) ds.$$

Hence

$$\sum_{j=1}^{n} \| \eta_{2}^{-1/2} \| \leq \int_{0}^{t_{1}} t \| A u'(t) \| dt + \sum_{j=2}^{n} k_j^2 \int_{t_{j-1}/2}^{t_j} \| A u''(t) \| dt. \quad (4.12)$$
To state the error bound from the time discretization we need to estimate the term \( \|V^1 - u^1\| \) that appears in the estimation of \( \|\eta_{1}^{n-1/2}\| \) in Lemma 4.1.

**Lemma 4.2** Let \( u \) be the exact solution of (1.1) and \( V^1 \) be defined by (1.4) with \( V^0 = U^0 \). Then

\[
\|V^1 - u^1\| \leq C_L\|e^0\| + C_{a,T} \int_{0}^{t_{1}} t (\|u'(t)\| + \|Au'(t)\| + \|Bu'(t)\|) dt.
\]

**Proof.** From (1.4) and by using \( V^0 = U^0 \) we observe that

\[
V^1 - U^0 + k_1 AV^1 + \int_{0}^{t_{1}} \int_{0}^{t} \beta(t-s) B \overline{V}(s) ds \, dt = k_1 f(t_{1/2}, U^0).
\]  

(4.13)

Comparing this with (4.3) for \( n = 1 \), we find that

\[
(V^1 - u^1) - e^0 + k_1 A(V^1 - u^1) + \int_{0}^{t_{1}} \int_{0}^{t} \beta(t-s)[B \overline{V}(s) - B \overline{u}(s)] ds \, dt = \eta^1 + \eta_2^{1/2} + \eta_3^{1/2}
\]

(4.14)

for the same \( \eta_2^{1/2} \) and \( \eta_3^{1/2} \) as in (4.6) and where

\[
\eta^1 = \int_{0}^{t_{1}} [f(t_{1/2}, U^0) - f(t, u(t))] dt.
\]

(4.15)

Since (4.14) has the same form as (4.13), the stability of \( V^1 - u^1 \) is obtained from the first result of Lemma 3.2, i.e.,

\[
\|V^1 - u^1\| \leq \|e^0\| + 2\|\eta^1\| + 2\|\eta_2^{1/2}\| + 2\|\eta_3^{1/2}\|.
\]

Using (2.1) and changing the order of integrals yields

\[
\|\eta^1\| \leq L \int_{0}^{t_{1}} \|U^0 - u(t)\| dt \leq L \int_{0}^{t_{1}} \|U^0 - u_0\| dt + L \int_{0}^{t_{1}} \|u_0 - u(t)\| dt
\]

\[
\leq L k_1 e^0 + \int_{0}^{t_{1}} t \|u'(t)\| dt.
\]

(4.16)

Using this and the obtained bounds of \( \|\eta_2^{1/2}\| \) and \( \|\eta_3^{1/2}\| \) in (4.12) and (4.8), we complete the proof. \( \square \)

**Theorem 4.3** Assume that the \( t_n \) satisfy (1.6)–(1.8) and let \( U^n \) be the solution of the discrete-time scheme (1.3)–(1.5). If the exact solution \( u \) of (1.1) and the source term satisfy

\[
\|Au'(t)\| + \|Bu'(t)\| + t\|Au''(t)\| + t\|Bu''(t)\| \leq Mt_{n}^{\alpha-1} \quad \text{and} \quad t\|D_{11} f(t, u(t))\| \leq Mt_{n}^{\alpha-1}
\]

for \( t > 0 \), then, we have for \( 0 \leq t_n \leq T \), we have

\[
\|U^n - u(t_n)\| \leq C\|U^0 - u_0\| + C \times \begin{cases} 
k^{\gamma(1+\alpha)} & \text{if } 1 \leq \gamma < 2/(1+\alpha), 
k^2 \log(t_n/t_1) & \text{if } \gamma = 2/(1+\alpha), 
k^2 & \text{if } \gamma > 2/(1+\alpha), \end{cases}
\]

where \( C \) is a generic positive constant that is independent of \( k_n \) but may depend on \( L, \alpha, \gamma, T \) and \( M \).
Proof. We plan to bound the quantities on the right-hand side of (4.9), (4.12) and (4.8) using the properties of the graded mesh and the given regularity assumptions on $u$ and $f$. Then, by substituting these bounds into (4.7) we obtain the desired result.

Since $t_{j+1}^* - t_j = \frac{1}{2}(k_j - k_{j-1})$, (4.9), (1.7) and (1.8) imply that

$$
\sum_{j=1}^n \| \eta_j^{i-1/2} \| \leq C \left( \sum_{j=2}^n k_j \| \widehat{\varepsilon}^{i-1/2} \| + k_1 (\| V^1 - u^1 \| + \| e^0 \|) \right)
+ C \sum_{j=2}^n (t_j^{\alpha-2} k_j^3 + k_j (t_{j-1}^* - t_j) t_j^{\alpha-1}) + C M t_1^{1+\alpha} + C k_1 t_1^{\alpha-1}

\leq C \sum_{j=2}^n k_j \| \widehat{\varepsilon}^{i-1/2} \| + C k_1 (\| V^1 - u^1 \| + \| e^0 \|) + C k^2 \sum_{j=2}^n t_j^{\alpha-2/\gamma} k_j + C k^\gamma (1+\alpha).
$$

From (4.12) and the mesh assumption (1.7) we get

$$
\sum_{j=1}^n \| \eta_j^{i-1/2} \| \leq C \int_0^{t_1} t^\alpha \, dt + C \sum_{j=2}^n k_j^3 t_j^{\alpha-2} \leq C k^2 \sum_{j=2}^n t_j^{\alpha-2/\gamma} k_j + C k^\gamma (1+\alpha).
$$

Using (4.8), $t_1^{1-\gamma} k_2 \leq C t_1^{1-1/\gamma} \leq C (t_j^* - t_j = \frac{1}{2}(k_{j+1} - k_j))$ (1.7) and (1.8), we have

$$
\sum_{j=1}^n \| \eta_j^{i-1/2} \| \leq C (t_1^{\alpha+1} + t_1^{\alpha} k_2) + C \sum_{j=2}^n (k_j^2 t_j^{\alpha-2} k_j + k_j (t_j^* - t_j) t_j^{\alpha-1} + (k_{j+1} - k_j) t_j^{\alpha-1} (k_j + k_{j+1}))

\leq C k^\gamma (1+\alpha) + C k^2 \sum_{j=2}^n t_j^{\alpha-2/\gamma} k_j.
$$

From (4.7), the above estimations, Lemma 4.2 and $\int_0^{t_1} t (\| u'(t) \| + \| Au'(t) \| + \| Bu'(t) \|) \, dt \leq C t_1^{\alpha+1} \leq C k^\gamma (1+\alpha)$, we find that

$$
\| U^n - u^n \| \leq C \| U^0 - u_0 \| + C \sum_{j=1}^{n-1} k_j \| U^j - u^j \|
+ C k^2 \sum_{j=2}^n t_j^{\alpha-2/\gamma} k_j + C k^\gamma (1+\alpha)
$$

for $n = 1, \ldots, N$,

$$
\text{for } n = 1, \ldots, N,
$$

and hence an application of the standard discrete analogue of Gronwall’s inequality yields

$$
\| U^n - u^n \| \leq C \| U^0 - u_0 \| + C \left( k^2 \sum_{j=2}^n t_j^{\alpha-2/\gamma} k_j + C k^\gamma (1+\alpha) \right).
$$
Estimating the sum, we obtain
\[
\sum_{j=2}^{n} t_j^{\alpha-2/\gamma} k_j \leq C \int_{t_1}^{t_n} t^{\alpha-2/\gamma} \, dt \leq C \times \begin{cases} t_1^{\alpha-2/\gamma+1}/(2/\gamma - \alpha - 1) & \text{if } \gamma < 2/(\alpha + 1), \\ \log(t_n/t_1) & \text{if } \gamma = 2/(\alpha + 1), \\ t_1^{\alpha-2/\gamma+1}/(\alpha - 2/\gamma + 1) & \text{if } \gamma > 2/(\alpha + 1), \end{cases}
\]
and we now complete the proof by noting that
\[
t_1^{\alpha-2/\gamma+1} = (k_1)^{-(2/\gamma-a-1)} \leq (c_\gamma k_1)^{-(2/\gamma-a-1)} = Ck_1^{(1+a)-2}.
\]

5. Discretization in time and space

Throughout this section we carry out our analysis assuming that \(\Omega\) is a bounded, convex or \(C^{1,1}\) domain and that the differential operators \(A\) and \(B\) in (1.1) are equal.

We choose a finite-dimensional subspace \(S_h \subset H^1_0(\Omega)\) formed of continuous, piecewise-linear finite-element spaces constructed on a triangulation of \(\Omega\) for which \(h\) is the maximum element diameter such that
\[
\min_{\chi \in S_h} (\|\nu - \chi\| + h\|\nabla(\nu - \chi)\|) \leq C h^2\|\nu\|_{H^2(\Omega)} \quad \text{for } \nu \in H^1_0 \cap H^2(\Omega).
\]
Our fully discrete scheme for (1.1) is based on the extrapolated Crank–Nicolson method and the standard Galerkin method for the time and space discretizations, respectively. So we describe our fully discrete scheme by applying spatial Galerkin finite elements to the time discretization scheme (1.3)–(1.5) in which \(U^n_h, V^n_1 \in S_h\) satisfy
\[
\left( \frac{U^n_h - U^{n-1}_h}{k_n}, \chi \right) + A(U^n_h - 1/2, \chi) + \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s)A(U(h)(s), \chi) \, ds \, dt = (f(t_{n-1}/2, \hat{U}^{n-1}_h), \chi) \quad \text{for } n \geq 2,
\]
\[
\left( \frac{V^1_h - V^0_h}{k_1}, \chi \right) + A(V^1_h, \chi) + \frac{1}{k_1} \int_0^{t_1} \int_0^t \beta(t-s)B(V(h)(s), \chi) \, ds \, dt = (f(t_1/2, V^0_h), \chi),
\]
\[
\left( \frac{U^1_h - U^0_h}{k_1}, \chi \right) + A(U^1_h, \chi) + \frac{1}{k_1} \int_0^{t_1} \int_0^t \beta(t-s)B(U(h)(s), \chi) \, ds \, dt = (f(t_1/2, V^{1/2}_h), \chi),
\]
for all \(\chi \in S_h\), with suitable approximations \(U^0_h = V^0_h \approx u_0\). Here \(A(\cdot, \cdot)\) is the bilinear form associated with \(A\).

To state our error bound for the time–space discretizations we let \(R_h: H^1_0(\Omega) \to S_h\) denote the Ritz projector for the operator \(A\). Thus
\[
A(R_h\nu, \chi) = A(\nu, \chi) \quad \text{for } \chi \in S_h,
\]
and we recall that
\[
\|\nu - R_h\nu\| \leq C h^2 \|u(t)\|_2.
\]
The error bound in the next theorem allows for the possibility that \(\|u'(t)\|_2\) is not integrable on the first subinterval \((0, t_1)\).
THEOREM 5.1 Let $U^n_h \in S_h$ be the solution of the discrete-time, discrete-space scheme (5.1)–(5.3). In addition to the assumptions of Theorem 4.3, if the exact solution $u$ of (1.1) satisfies $\| u(t) \|_2 + t \| u'(t) \|_2 \leq Mt^v$ for $t > 0$, with $v \geq 0$, then, for $0 \leq t_n \leq T$, we have

$$
\| U^n_h - u(t_n) \| \leq C \| U^n_h - R_h u_0 \| + C \times \begin{cases} k^{\gamma(1+\alpha)} & \text{if } 1 \leq \gamma < 2/(1+\alpha), \\
k^2 \log(t_n/t_1) & \text{if } \gamma = 2/(1+\alpha), \\
k^2 & \text{if } \gamma > 2/(1+\alpha)
\end{cases} + C h^2 \times \begin{cases} 1 + \log(t_n/t_1) & \text{if } v = 0, \\
1 & \text{if } v > 0,
\end{cases}
$$

where $C$ is a generic positive constant that is independent of $k_n$ and $h$ but may depend on $L, \alpha, \gamma, M, T$ and $v$.

Proof. We split the error into two terms: $U^n_h - u(t_n) = \theta^n + \rho(t_n)$, where

$$
\theta^n = U^n_h - R_h u(t_n) \in S_h \quad \text{and} \quad \rho(t) = R_h u(t) - u(t).
$$

Integrating (1.1) from $t = t_{n-1}$ to $t = t_n$ and then taking the inner product with $\chi \in S_h$, we have

$$
(u(t_n) - u(t_{n-1}), \chi) + \int_{t_{n-1}}^{t_n} \left( A(u(t), \chi) + \int_0^t \beta(t-s)B(u(s), \chi)ds \right) dt = \int_{t_{n-1}}^{t_n} (f(t, u(t)), \chi) dt.
$$

Since

$$
(U^n_h - U^{n-1}_h) - (u(t_n) - u(t_{n-1})) = (\theta^n - \theta^{n-1}) + (\rho(t_n) - \rho(t_{n-1}))
$$

and for $t_{n-1} \leq t \leq t_n$ we have

$$
A(U^n_h, \chi) - A(u(t), \chi) + \int_0^t \beta(t-s)A(U_h(s), \chi)ds - \int_0^t \beta(t-s)A(u(s), \chi)ds
$$

$$
= A(\theta^n, \chi) + A(u(t_n) - u(t), \chi) + \int_0^t \beta(t-s)A(\theta(s), \chi)ds + \int_0^t \beta(t-s)A(\theta(s) - u(s), \chi)ds,
$$

we see that

$$
(\theta^n - \theta^{n-1}, \chi) + k_n A(\theta^n, \chi) + \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s)B(\theta(s), \chi)ds dt = (\eta^{n-1/2}, \chi),
$$

where $\eta^{n-1/2} = \eta_1^{n-1/2} + \eta_2^{n-1/2} + \eta_3^{n-1/2} + \eta_4^{n-1/2}$, for the same $\eta_1^{n-1/2}$, $\eta_2^{n-1/2}$ and $\eta_3^{n-1/2}$ as in (4.6) and for

$$
\eta_4^{n-1/2} = -[\rho(t_n) - \rho(t_{n-1})].
$$

Following the derivation used in Lemma 3.2 and using $\theta^0 = U^0_h - R_h u_0$, we have for $n = 1, \ldots, N$ that

$$
\| \theta^n \| \leq \| U^0_h - R_h u_0 \| + 2 \sum_{j=1}^n \| \eta^{j-1/2} \|. $$
Using the bounds of \( \| \eta_1^{n-1/2} \|, \| \eta_2^{n-1/2} \| \) and \( \| \eta_3^{n-1/2} \| \) that were already dealt with in Theorem 4.3, we have

\[
\sum_{j=1}^{n} (\| \eta_1^{n-1/2} \| + \| \eta_2^{n-1/2} \| + \| \eta_3^{n-1/2} \|)
\leq C \sum_{j=1}^{n-1} k_j \| U_h^j - u^j \| + C k_1 (\| V_h^1 - u(t_1) \| + \| U_h^0 - u_0 \|) + C \left( k^2 \sum_{j=2}^{n} t_j^{a-2/\gamma} k_j + k^{(1+\alpha)} \right).
\] (5.7)

Now it suffices to estimate \( \| \rho(t_n) \|, \| V_h^1 - u(t_1) \| \) and the contribution from \( \eta_4^{n-1/2} \).

We see from (5.4) that \( \| \rho(t) \| \leq C h^2 \| u(t) \|_2 \), so

\[
\| \rho(t_n) \| \leq C h^2 \left( \| u(t_1) \|_2 + \int_{t_1}^{t_n} \| u'(t) \|_2 \, dt \right) \quad \text{for } n \geq 1.
\] (5.8)

To deal with \( \eta_4^{j-1/2} \) we note that

\[
\sum_{j=1}^{n} \| \eta_4^{j-1/2} \| = \| \rho(t_1) - \rho(t_0) \| + \sum_{j=2}^{n} \left\| \int_{t_{j-1}}^{t_j} \rho'(t) \, dt \right\|
\leq \| (R_h - I)[u(t_1) - u_0] \| + \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \| \rho'(t) \| \, dt
\leq C h^2 \| u(t_1) - u_0 \|_2 + \int_{t_1}^{t_n} \| \rho'(t) \| \, dt,
\]

and by (5.4) we observe that \( \| \rho'(t) \| = \| (R_h - I)u'(t) \| \leq C h^2 \| u'(t) \|_2 \). Thus

\[
\sum_{j=1}^{n} \| \eta_4^{j-1/2} \| \leq C h^2 \left( \| u(t_1) - u_0 \|_2 + \int_{t_1}^{t_n} \| u'(t) \|_2 \, dt \right).
\] (5.9)

To estimate \( \| V_h^1 - u(t_1) \| \) we split it as \( V_h^1 - u(t_1) = \tilde{\theta} + \rho(t_1) \) with \( \tilde{\theta} = V_h^1 - R_h u(t_1) \). Since

\[
(V_h^1 - U_h^0) - (u(t_1) - u_0) = (\tilde{\theta} - \theta^0) + (\rho(t_1) - \rho(t_0))
\]

and for \( 0 \leq t \leq t_1 \) we have

\[
A(V_h^1, \chi) - A(u(t), \chi) + \int_{0}^{t} \beta(t-s) A(V_h(s), \chi) \, ds - \int_{0}^{t} \beta(t-s) A(u(s), \chi) \, ds
= A(\tilde{\theta}, \chi) + A(u(t_1) - u(t), \chi) + \int_{0}^{t} \beta(t-s) A(\tilde{\theta}, \chi) \, ds + \int_{0}^{t} \beta(t-s) A(u(t_1) - u(s), \chi) \, ds,
\]

we see that

\[
(\tilde{\theta} - \theta^0, \chi) + k_1 A(\tilde{\theta}, \chi) + \int_{0}^{t_n} \int_{0}^{t} \beta(t-s) B(\tilde{\theta}, \chi) \, ds \, dt = (\eta^{1/2}, \chi),
\]
where \( \eta^{1/2} = \eta^1 + \eta_2^{1/2} + \eta_3^{1/2} + \eta_4^{1/2} \), for the same \( \eta^1, \eta_2^{1/2} \) and \( \eta_3^{1/2} \) as in (4.15) and (4.6) and for \( \eta_4^{1/2} \) as in (5.6). Following the derivation used in Lemma 3.2 to show the stability of \( V^1 \), we have

\[
\|\tilde{\theta}\| \leq \|\theta^0\| + 2 \sum_{j=1}^{n} \|\eta_j^{1/2}\|.
\]

Using the obtained estimations of \( \|\eta^1\|, \|\eta_2^{1/2}\|, \|\eta_3^{1/2}\| \) and \( \|\eta_4^{1/2}\| \) from (4.16), (4.12), (4.8) and (5.9), respectively, we have

\[
\|\tilde{\theta}\| \leq \|\theta^0\| + Lk_1\|U_h^0 - u_0\| + Ch^2\|u(t_1) - u_0\|_2 + C \int_{0}^{t_1} t\|u'(t)\| + \|Au'(t)\|\,dt,
\]

and hence

\[
\|V_h^1 - u(t_1)\| \leq \|\tilde{\theta}\| + \|\rho(t_1)\|
\leq \|\theta^0\| + Lk_1\|U_h^0 - u_0\| + Ch^2(\|u(t_1)\|_2 + \|u_0\|_2) + C \int_{0}^{t_1} t\|Au'(t)\|\,dt. \tag{5.10}
\]

Using the stability of \( \theta^n \), combining (5.7)–(5.10), we find that

\[
\|U^n_h - u(t_n)\| \leq \|\theta^n\| + \|\rho(t_n)\|
\leq (1 + Ck_1)\|\theta^0\| + C \left( \sum_{j=1}^{n-1} k_j\|U_h^j - u^j\| + k^2 \sum_{j=2}^{n} t_{j-1}^{\alpha-2/\gamma} k_j + k^{\gamma (1+\alpha)} \right)
\]

\[
+ Ch^2 \left( \|u(t_1)\| + \|u_0\|_2 + \int_{t_1}^{t_n} \|u'(t)\|_2\,dt \right) + C \int_{0}^{t_1} t\|Au'(t)\|\,dt.
\]

### Table 1 Errors in \( \|U_h - u\|_{0,\infty} \) and apparent convergence rates for three different mesh gradings when \( \alpha = 0.2 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \gamma = 1 )</th>
<th>( \gamma = 5/3 )</th>
<th>( \gamma = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>(2.13 \times 10^{-3})</td>
<td>(7.94 \times 10^{-4})</td>
<td>(8.18 \times 10^{-4})</td>
</tr>
<tr>
<td>120</td>
<td>(8.78 \times 10^{-4})</td>
<td>(1.28)</td>
<td>(1.99)</td>
</tr>
<tr>
<td>240</td>
<td>(3.92 \times 10^{-4})</td>
<td>(1.16)</td>
<td>(4.97 \times 10^{-5})</td>
</tr>
<tr>
<td>480</td>
<td>(1.79 \times 10^{-4})</td>
<td>(1.13)</td>
<td>(1.24 \times 10^{-5})</td>
</tr>
<tr>
<td>960</td>
<td>(8.12 \times 10^{-5})</td>
<td>(1.13)</td>
<td>(2.89 \times 10^{-6})</td>
</tr>
</tbody>
</table>

### Table 2 Errors in \( \|U_h - u\|_{0,\infty} \) and apparent convergence rates for three different mesh gradings when \( \alpha = 0.6 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \gamma = 1 )</th>
<th>( \gamma = 5/4 )</th>
<th>( \gamma = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>(7.94 \times 10^{-4})</td>
<td>(8.22 \times 10^{-4})</td>
<td>(9.54 \times 10^{-4})</td>
</tr>
<tr>
<td>120</td>
<td>(1.99 \times 10^{-4})</td>
<td>(1.99)</td>
<td>(2.05 \times 10^{-4})</td>
</tr>
<tr>
<td>240</td>
<td>(4.99 \times 10^{-5})</td>
<td>(1.99)</td>
<td>(5.14 \times 10^{-5})</td>
</tr>
<tr>
<td>480</td>
<td>(1.66 \times 10^{-5})</td>
<td>(1.58)</td>
<td>(1.28 \times 10^{-5})</td>
</tr>
<tr>
<td>960</td>
<td>(5.89 \times 10^{-6})</td>
<td>(1.49)</td>
<td>(2.99 \times 10^{-6})</td>
</tr>
</tbody>
</table>
Therefore, noting that \( \theta^0 = U_h^0 - R_h u_0 \) and following the steps after (4.18) in Theorem 4.3 yields the desired error bound. \( \square \)

6. Numerical experiment

In this section we apply our fully discrete extrapolated Crank–Nicolson scheme (5.1)–(5.3) to a concrete problem of the form (1.1) in one space dimension, taking \( T = 1 \) and \( Au = Bu = -\partial^2_x u \) on the interval \( \Omega = (0, 1) \) with homogeneous Dirichlet boundary conditions. We employ a time mesh of the form (1.9) with various choices of \( \gamma \geq 1 \). In each case we use a uniform spatial mesh with \( N \) subintervals of length \( h = 1/N \). Since \( N \) is the same for the space and time meshes, this means \( h = k \). For the finite-element space \( S_h \) we choose the space of continuous, piecewise-linear functions, and for \( U_h^0 \) we choose the elliptic projection of the initial data \( u_0 \) onto \( S_h \). The data are chosen so that the exact solution is

\[
  u(x, t) = \sin(\pi x) - t^{1+\alpha}\sin(2\pi x)
\]

and the inhomogeneous term is \( f(x, t, u) = t^\alpha u^2(x, t) + f_1 \). So we put \( u(x, 0) = \sin(\pi x) \) and

\[
  f_1 = \left( \pi^2 \left( 1 + \frac{t^\alpha}{\Gamma(1+\alpha)} \right) - t^\alpha \sin(\pi x) + 2t^{1+2\alpha}\sin(2\pi x) \right) \sin(\pi x)
  - t^\alpha \left( 1 + \alpha \right) + 4\pi^2 t^2 \left( 1 + \frac{t^\alpha \Gamma(2+\alpha)}{\Gamma(2+2\alpha)} \right) + t^{2+2\alpha}\sin(2\pi x) \right) \sin(2\pi x).
\]

For \( t > 0 \) it is clear that

\[
  \|u(t)\|_2 + t\|u'(t)\|_2 \leq M, \quad \|Au'(t)\| + t\|Au''(t)\| \leq Mt^\alpha \quad \text{and} \quad t\|D_{tt}f(t, u(t))\| \leq Mt^{\alpha - 1}.
\]

So the assumptions of Theorem 5.1 are satisfied. Tables 1 and 2 show the error \( \|U_h - u\|_{0,\infty} = \max_{0 \leq n \leq N} \|U_h^n - u(t_n)\|_{L^2(0,1)} \). The observed convergence rate is almost \( O(k^{1+\alpha} + h^2 |\log k|) = O(N^{-1-\alpha}) \) for uniform time steps, i.e., for \( \gamma = 1 \). This rate improves to \( O(N^{-2\log N}) \) if \( \gamma = 2/(\alpha + 1) \). For \( \gamma = 2 \) the mesh is over-graded: theoretically, we have a slightly better convergence rate \( O(N^{-2}) \), but in practice the errors are larger when \( \gamma = 2/(\alpha + 1) \).

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