IMA Journal of Numerical Analysis Advance Access published February 26, 2009

IMA Journal of Numerical Analysis Page 1 of 24
doi:10.1093/imanum/drn075

A second-order accurate numerical method for a semilinear integro-differential equation with a weakly singular kernel

KASSEM MUSTAPHA†
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

AND

HUSSEIN MUSTAPHA‡
Department of Mining and Materials Engineering, McGill University, Montreal, Canada

[Received on 30 August 2007; revised on 1 May 2008]

We study a generalized extrapolated Crank–Nicolson scheme for the time discretization of a semilinear integro-differential equation with a weakly singular kernel, in combination with a space discretization by linear finite elements. The scheme uses variable grids in time to compensate for the singular behaviour of the exact solution at $t = 0$. With appropriate assumptions on the data and assuming that the spatial domain is convex or smooth, we show that the error is of order $k^2 + h^2$, where k and h are the parameters for the time and space meshes, respectively. The results of numerical computations demonstrate the convergence of our scheme.

Keywords: integro-differential equation; weakly singular kernel; nonuniform time steps; quadrature error; finite elements; Gronwall's lemma.

1. Introduction

In this paper we study the time and space discretization for semilinear parabolic integro-differential equations of the form

$$\frac{\partial u}{\partial t}(x,t) + Au(x,t) + \mathscr{F}_\alpha Bu(x,t) = f(x,t,u(t,x))$$
$$\text{with } u(x,0) = u_0(x) \quad \text{for } (x,t) \in \Omega \times (0,T], \tag{1.1}$$

subject to the Dirichlet-type boundary condition $u(x,t) = 0$ for $(x,t) \in \partial\Omega \times (0,T]$. Here $\mathscr{F}_\alpha$ is the Riemann–Liouville fractional integration operator of order $0 < \alpha < 1$, defined by

$$\mathscr{F}_\alpha v(t) = \int_0^t \beta(t-s)v(s)\mathrm{d}s, \quad \text{where } \beta(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

By Γ we denote the usual gamma function, A and B are linear second-order elliptic differential operators and Ω is a bounded domain subset of $\mathbb{R}^{\mathrm{d}}$.

†Corresponding author. Email: kassem@kfupm.edu.sa
‡Email: hussein.mustapha@mcgill.ca

The solution $u(\cdot,t)$ and the source term $f(\cdot,t,u(\cdot,t))$ take values in $L_2(\Omega)$, and the initial data u_0 is an element of $L_2(\Omega)$. Throughout the paper we write $u(t) = u(x,t)$, $f(t,u(t)) = f(x,t,u(x,t))$ and $u_0 = u_0(x)$. Equations of type (1.1) may be thought of as a model problem occurring in the theory of heat conduction in materials with memory, population dynamics and viscoelasticity (e.g., Friedman & Shinbrot, 1967; Heard, 1982; Renardy *et al.*, 1987).

Many authors have considered numerical methods for a linear problem of the form (1.1), i.e., when $f(t,u) = f(t)$. Typically, the time discretization is affected by a combination of finite difference and quadratures. Finite difference in time and finite elements in space have been discussed in the case of a smooth kernel (e.g., Sloan & Thomée, 1986; Cannon & Lin, 1988, 1990; Yanik & Fairweather, 1988; Thomée & Zhang, 1989; Lin *et al.*, 1991; Zhang, 1993). For the nonsmooth kernel case we refer to Chen *et al.* (1992) and Larsson *et al.* (1998).

In contrast, over the last three decades different numerical methods and techniques have been applied widely to problems of the form (1.1) when $A = 0$ and the source term f is independent of the unknown solution u (e.g., Lubich, 1988; Fairweather, 1994; McLean *et al.*, 1996; Adolfsson *et al.*, 2003; López-Fernández & Palencia, 2004; Cuesta *et al.*, 2006; McLean & Mustapha, 2007 and the references therein).

The purpose of this paper is to generalize the extrapolated Crank–Nicolson time discretization of the semilinear problem (1.1) using variable time steps combined with appropriate quadrature rules approximating the integral. Also, we will consider the discretization in space by the finite elements, which will then define a fully discrete method for (1.1) in this case. In addition, we present numerical evidence that our error bounds are sharp. We assume that the operators A and B are strictly positive definite. However, our approach can be extended with some modifications to cover the case that A and B are non-negative, e.g., consider (1.1) subject to homogenous Neumann boundary conditions with $A = B = -\nabla^2$.

To define our time-stepping scheme we introduce the time levels $0 = t_0 < t_1 < \cdots < t_n < \cdots \leqslant T$ and set $k_n := t_n - t_{n-1}$ and $t_{n-1/2} = \dfrac{t_{n-1}+t_{n-1}}{2}$. Given a grid function $\zeta^n = \zeta(t_n)$, we write

$$\partial\zeta^n = \frac{\zeta^n - \zeta^{n-1}}{k_n}, \quad \zeta^{n-1/2} = \frac{\zeta^n + \zeta^{n-1}}{2}, \quad \widehat{\zeta}^{n-1/2} = \frac{3\zeta^{n-1} - \zeta^{n-2}}{2}$$

and define a piecewise constant approximation

$$\overline{\zeta}(t) := \begin{cases} \zeta^1 & \text{for } t_0 < t < t_1, \\ \zeta^{n-1/2} & \text{for } t_{n-1} < t < t_n \text{ and } n \geqslant 2. \end{cases}$$

The modification on the first subinterval in the above piecewise-constant approximation ensures that $\overline{\zeta}(t)$ does not depend on ζ^0, which is necessary for our numerical scheme in cases when u_0 is not sufficiently regular. Using $\overline{\zeta}$, we define a discrete fractional integral

$$\mathscr{I}_\alpha^{n-1/2}\zeta = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s)\overline{\zeta}(s)\mathrm{d}s\,\mathrm{d}t = \omega_{n,1}\zeta^1 k_1 + \sum_{j=2}^{n}\omega_{n,j}\zeta^{j-1/2}k_j,$$

where

$$\omega_{n,j} = \frac{1}{k_n k_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{\min(t,t_j)} \beta(t-s)\mathrm{d}s\,\mathrm{d}t > 0.$$

When the source term f in (1.1) is independent of u, i.e., $f(t, u(t)) = f(t)$, starting from an approximation $U^0 \approx u_0$ to the initial data, we consider a Crank–Nicolson time discretization

$$\partial U^n + AU^{n-1/2} + \mathscr{F}_\alpha^{n-1/2} BU = f(t_{n-1/2}) \quad \text{for } n \geqslant 1 \tag{1.2}$$

that generates an approximate solution $U^n \approx u(t_n)$. To apply the above scheme to problem (1.1) we replace $f(t_{n-1/2})$ with $f(t_{n-1/2}, U^{n-1/2})$, which leads to solving a nonlinear algebraic system at each time step. To avoid this, following Cannon & Lin (1988, 1990), we generalize the extrapolated Crank–Nicolson method (2.9) in Dupont *et al.* (1974) for semilinear parabolic problems without a memory term. So the numerical scheme takes the form

$$\partial U^n + AU^{n-1/2} + \mathscr{F}_\alpha^{n-1/2} BU = f(t_{n-1/2}, \widehat{U}^{n-1/2}) \quad \text{for } n \geqslant 2, \tag{1.3}$$

which requires selections of U^0 and U^1. For a given $U^0 = V^0 \approx u_0$, we select U^1 using a predictor–corrector method:

$$\partial V^1 + AV^1 + \mathscr{F}_\alpha^{1/2} BV = f(t_{1/2}, V^0), \tag{1.4}$$

$$\partial U^1 + AU^1 + \mathscr{F}_\alpha^{1/2} BU = f(t_{1/2}, V^{1/2}). \tag{1.5}$$

The above scheme is implicit because $AU^{n-1/2}$ and $\mathscr{F}_\alpha^{n-1/2} BU$ depend on U^n. Of course, unlike the case of semilinear parabolic problems without the memory term, at each step we must compute a sum involving the solution at all previous time levels.

Although the extrapolated Crank–Nicolson scheme (1.3)–(1.5) is formally second-order accurate in time, the error in U^n generally fails to be $\mathcal{O}(k^2)$ if we use a uniform time step k. The reason for this is the lack of regularity in time of the exact solution $u(t)$ near $t = 0$, which is present even in the case of smooth initial data. Since our main goal of the current paper is to show that the above scheme is second-order accurate in time under suitable assumptions, we therefore employ a family of nonuniform meshes that concentrate the time levels near $t = 0$. We therefore assume, for a fixed $\gamma \geqslant 1$, that

$$k_n \leqslant C_\gamma k \min(1, t_n^{1-1/\gamma}) \quad \text{and} \quad t_n \leqslant C_\gamma t_{n-1} \quad \text{for } 2 \leqslant n \leqslant N, \tag{1.6}$$

with

$$c_\gamma k^\gamma \leqslant k_1 \leqslant C_\gamma k^\gamma. \tag{1.7}$$

Use of the mesh properties (1.6) and (1.7) is standard for integral equations in which the solution $u(s)$ possesses a fixed singularity at $s = 0$ (see, e.g., te Riele, 1982; Chandler & Graham, 1988; Brunner *et al.*, 2001). In our case the integrand $\beta(t-s)u(s)$ also possesses a moving singularity at $s = t$ that the quadrature approximation handles using product integration. It turns out that for our error analysis we require a further, and more restrictive, assumption on the time mesh:

$$0 \leqslant k_{n+1} - k_n \leqslant C_\gamma k^2 \min(1, t_n^{1-2/\gamma}) \quad \text{for } n \geqslant 2. \tag{1.8}$$

Thus the step size must increase monotonically with no abrupt changes from one time level to the next. A typical example of a mesh satisfying all three assumptions (1.6)–(1.8) for an interval $[0, T]$ is

$$t_n = (nk)^\gamma \quad \text{for } 0 \leqslant n \leqslant N \quad \text{with } k = \frac{T^{1/\gamma}}{N}. \tag{1.9}$$

It is important to note that, if the domain Ω is not convex or $C^{1,1}$, then in practice the exact solution will generally not possess sufficient regularity to achieve second-order accuracy for the space discretization.

An outline of the paper is as follows. In Section 2 we study some regularity properties of the exact solution that are needed in our convergence analysis. It is important to mention that, in the presence of a weakly singular kernel, the exact solution of (1.1) is not smooth. In our approach we recall some properties of the analytic semigroup generated by the differential operator $-A$ from Pazy (1983). Also, we borrow some results from Larsson *et al.* (1998) where the regularity properties of problem (1.1) were studied when $f \equiv 0$. Since the source term f depends on the unknown solution, the level of complexity for deriving the desired regularity of the exact solution of problem (1.1) will increase substantially. In our analysis we assume that the source term is Lipschitz continuous. The main regularity results are stated in Theorems 2.4 and 2.5. The stability of the scheme (1.3)–(1.5) is studied in Section 3. In our proof we use the positivity for the discrete analogue of the fractional integral term which was presented in McLean *et al.* (1996). In Section 4 our error bounds for the time discretization are given in Theorem 4.3. A few lemmas will be given first to reduce the level of technicality in Theorem 4.3. Under appropriate assumptions on the exact solution of problem (1.1) and the time mesh t_n, an $\mathcal{O}(k^2)$ convergence order is achieved. The first step is to make use of the stability results and then bound a number of terms involving the source term f, the differential operators A and B and the weakly singular kernel β. We refer to the recent work of McLean & Mustapha (2007) for the use of the proved bound to some of these terms. In Section 5 we describe our fully discrete scheme for problem (1.1). We use the extrapolated Crank–Nicolson method for the time discretization and Galerkin finite elements for the space discretization. In other words, we apply a spatial discrete version of (1.3)–(1.5) employing linear finite elements. We show that the error is $\mathcal{O}(k^2 + h^2|\log k|)$, so that we achieve essentially second-order accuracy in space as well as in time. Finally, Section 6 presents some numerical studies that demonstrate our theoretical convergence results.

2. Regularity of the exact solution

In this section we derive some regularity estimates of the exact solution $u(t)$ of the problem (1.1). More precisely, we seek to show that the solution of (1.1) satisfies

$$t^{1-\alpha}\|Au\| + \|Au'(t)\| + \|Bu'(t)\| + t\|Au''(t)\| + t\|Bu''(t)\| \leqslant \mathcal{O}(t^{\alpha-1}) \quad \text{as } t \to 0,$$

which eventually allows us to bound the error arising from the time and space discretization. Here, by $u'(t)$ and $u''(t)$ we denote the first and second derivatives of $u(t)$ with respect to time, and $\|\cdot\|$ denotes the norm on the Sobolev space $L^2(\Omega)$. For later use, $\|\cdot\|_{L_q}$ denotes the norm on the Sobolev space $L_q(\Omega)$ for $1 < q \leqslant \infty$. For a function $g(t, u(t))$, by $D_t g(t, u(t))$ and $g_t(t, u(t))$ we denote the first total and partial time derivatives, respectively. Finally, by $D_{tt} g(t, u(t))$ and $g_{tt}(t, u(t))$ we denote the second total and partial time derivatives, respectively.

The regularity of the exact solution for problems of the form (1.1) was studied in Larsson *et al.* (1998) when $f(t, u(t)) = 0$. Some of the results of this section are essentially present in Larsson *et al.* (1998), but because of their importance we give the complete proof. For the sake of simplicity, throughout this section we assume that the operators A and B in (1.1) are equal.

Throughout the paper we assume that the source term f is Lipschitz continuous, i.e.,

$$\|f(t, w_1) - f(t, w_2)\|_{L_q} \leqslant L\|w_1 - w_2\|_{L_q} \quad \text{for } w_1, w_2 \in L_q(\Omega) \quad \text{with } 1 < q < \infty. \tag{2.1}$$

Split the solution $u(t)$ of (1.1) as $u(t) = v(t) + w(t)$, with $v(t)$ and $w(t)$ solving

$$v'(t) + Av(t) = F(t) \quad \text{for } t > 0 \text{ with } v(0) = u_0 \tag{2.2}$$

and

$$w'(t) + Aw(t) = -\int_0^t (t-s)^{\alpha-1} Au(s)\mathrm{d}s + G(t, u(t)) \quad \text{for } t > 0 \text{ with } w(0) = 0, \tag{2.3}$$

respectively, where $F(t) := f(t, u_0)$ and $G(t, u(t)) := f(t, u(t)) - f(t, u_0)$.

The differential equation (2.2) is purely linear parabolic and the solution can be written as

$$v(t) = E(t)u_0 + \int_0^t E(t-s)F(t)\mathrm{d}s, \tag{2.4}$$

where $E(t) = \exp(-tA)$ is the analytic semigroup in $L_q(\Omega)$ generated by $-A$ (see Pazy, 1983), so that for $0 < t < T$ we have

$$\|E(t)v\|_{L_q} + t\|AE(t)v\|_{L_q} + t^2\|A^2E(t)v\|_{L_q} \leqslant C\|v\|_{L_q} \quad \text{for all } v \in L_q(\Omega). \tag{2.5}$$

LEMMA 2.1 Let $1 < q < \infty$. For $t > 0$, let $\psi(t) \in L_q(\Omega)$ satisfy $\|\psi(t)\|_{L_q} + t\|\psi'(t)\|_{L_q} \leqslant Ct^{\alpha-1}$. Then

$$\left\| \int_0^t AE(t-s)\psi(s)\mathrm{d}s \right\|_{L_q} \leqslant Ct^{\alpha-1}.$$

LEMMA 2.2 Let $1 < q < \infty$ and let $\psi(t) \in L_q(\Omega)$ for $t > 0$, then

$$\left\| \int_0^t AE(t-s) \int_0^s (s-\sigma)^{\alpha-1}\psi(\sigma)\mathrm{d}\sigma\, \mathrm{d}s \right\|_{L_q} \leqslant C \int_0^t (t-\sigma)^{\alpha-1}\|\psi(\sigma)\|_{L_q}\, \mathrm{d}\sigma.$$

For $q = 2$ the proof of the above lemmas can be found in Larsson *et al.* (1998). For $q \neq 2$ the proof is similar.

LEMMA 2.3 Let $1 < q < \infty$. For $t > 0$ assume that $\|Au_0 - F(0)\|_{L_q} + \|F(t)\|_{L_q} \leqslant C$ and $\|E'(t)(Au_0 - F(0))\|_{L_q} + \|F'(t)\|_{L_q} + t\|F''(t)\|_{L_q} \leqslant Ct^{\alpha-1}$. Then the solution v of (2.2) satisfies

$$\|Av(t)\|_{L_q} + t^{1-\alpha}\|Av'(t)\|_{L_q} \leqslant C \quad \text{and} \quad \|v'(t)\|_{L_q} + t^{1-\alpha}\|v''(t)\|_{L_q} \leqslant C.$$

Proof. Using (2.4), $AE(t-s) = D_sE(t-s)$ and integrating by parts, we get

$$\begin{aligned} Av(t) &= E(t)Au_0 + \int_0^t D_sE(t-s)F(s)\mathrm{d}s \\ &= E(t)Au_0 + F(t) - E(t)F(0) - \int_0^t E(t-s)F'(s)\mathrm{d}s. \end{aligned} \tag{2.6}$$

So (2.5) and the given assumptions yield

$$\|Av(t)\|_{L_q} \leqslant C\left(\|Au_0 - F(0)\|_{L_q} + \|F(t)\|_{L_q} + \int_0^t \|F'(s)\|_{L_q}\, \mathrm{d}s\right) \leqslant C.$$

Differentiating both sides of (2.6) with respect to t and using $D_t E(t-s) = -AE(t-s)$, we get

$$Av'(t) = E'(t)(Au_0 - F(0)) + \int_0^t AE(t-s)F'(s)\mathrm{d}s, \tag{2.7}$$

and hence the given assumptions and Lemma 2.1 give

$$\|Av'(t)\|_{L_q} = \|E'(t)(Au_0 - F(0))\|_{L_q} + \left\| \int_0^t AE(t-s)F'(s)\mathrm{d}s \right\|_{L_q} \leqslant Ct^{\alpha-1}.$$

Thus the proof of the first desired inequality is completed. From (2.2), $\|Av(t)\|_{L_q} \leqslant C$ and the assumption that $\|F(t)\| \leqslant C$, we get the desired bound of $\|v'(t)\|_{L_q}$. The time differentiation of both sides of (2.2), the proved bound $\|Av'(t)\| \leqslant Ct^{\alpha-1}$ and also the given assumptions yield $\|v''(t)\|_{L_q} \leqslant Ct^{\alpha-1}$. Therefore the proof is complete. □

THEOREM 2.4 Let $1 < q < \infty$. In addition to the assumptions of Lemma 2.3, we impose that $t^{1-\alpha}\|G_t(t,u(t))\|_{L_q} + \|f_u(t,u(t))\|_{L_\infty} \leqslant C$ for $t > 0$. Then the exact solution u of (1.1) satisfies

$$\|u'(t)\|_{L_q} + \|Au(t)\|_{L_q} \leqslant C \quad \text{for } t > 0.$$

Proof. Using $u = w + v$ and Lemma 2.3, it is sufficient to show that the solution w of (2.3) satisfies

$$\|w'(t)\|_{L_q} + \|Aw(t)\|_{L_q} \leqslant Ct^\alpha \quad \text{for } 0 < t \leqslant T. \tag{2.8}$$

From (2.3) we note that

$$\|w'(t)\|_{L_q} \leqslant \|G(t,u(t))\|_{L_q} + \|Aw(t)\|_{L_q} + \left\| \int_0^t (t-s)^{\alpha-1}Au(s)\mathrm{d}s \right\|_{L_q}. \tag{2.9}$$

Using Duhamel's principle, we have

$$w(t) = -\int_0^t E(t-s)\int_0^s (s-\sigma)^{\alpha-1}Au(\sigma)\mathrm{d}\sigma\,\mathrm{d}s + \int_0^t E(t-s)G(s,u(s))\mathrm{d}s = w_1 + w_2. \tag{2.10}$$

For the bound of Aw_1, Lemma 2.2, $u = v + w$ and Lemma 2.3 give

$$\begin{aligned}\|Aw_1\|_{L_q} &\leqslant C\int_0^t (t-\sigma)^{\alpha-1}(\|Av(\sigma)\|_{L_q} + \|Aw(\sigma)\|_{L_q})\mathrm{d}\sigma \\ &\leqslant C\left(t^\alpha + \int_0^t (t-\sigma)^{\alpha-1}\|Aw(\sigma)\|_{L_q}\,\mathrm{d}\sigma\right).\end{aligned} \tag{2.11}$$

For the bound of Aw_2 we use $AE(t-s) = D_s E(t-s)$ and integrating by parts, we get

$$\begin{aligned}Aw_2 &= \int_0^t AE(t-s)G(s,u(s))\mathrm{d}s = \int_0^t D_s E(t-s)G(s,u(s))\mathrm{d}s \\ &= G(t,u(t)) - \int_0^t E(t-s)[G_s(s,u(s)) + f_u(s,u(s))u'(s)]\mathrm{d}s.\end{aligned}$$

The Lipschitz continuity assumption on the source term f implies that

$$\|G(t,u(t))\|_{L_q} \leqslant L\|u(t)-u_0\|_{L_q} \leqslant L\int_0^t \|u'(s)\|_{L_q}\,\mathrm{d}s, \tag{2.12}$$

and hence (2.5) and the given assumptions yield $\|Aw_2\|_{L_q} \leqslant Ct^\alpha + C\int_0^t \|u'(s)\|_{L_q}\,\mathrm{d}s$. Hence, using (2.10), the bound of $\|Aw_2\|_{L_q}$, (2.11), $u = v + w$ and Lemma 2.3, we have

$$\|Aw(t)\|_{L_q} \leqslant Ct^\alpha + C\int_0^t \|w'(s)\|_{L_q}\,\mathrm{d}s + C\int_0^t (t-\sigma)^{\alpha-1}\|Aw(\sigma)\|_{L_q}\,\mathrm{d}\sigma.$$

Thus an application of Gronwall's lemma (see, for example, Lemma 1 in Chen *et al.*, 1992) achieves

$$\|Aw(t)\|_{L_q} \leqslant Ct^\alpha + C\int_0^t \|w'(s)\|_{L_q}\,\mathrm{d}s. \tag{2.13}$$

To bound the third term on the right-hand side of (2.9), using (2.13), we find that

$$\begin{aligned}\left\|\int_0^t (t-s)^{\alpha-1}Aw(s)\mathrm{d}s\right\|_{L_q} &\leqslant \int_0^t (t-s)^{\alpha-1}\|Aw(s)\|_{L_q}\,\mathrm{d}s\\ &\leqslant C\int_0^t (t-s)^{\alpha-1}\left(s^\alpha + \int_0^s \|w'(\sigma)\|_{L_q}\,\mathrm{d}\sigma\right)\mathrm{d}s\\ &\leqslant Ct^{2\alpha} + Ct^\alpha\int_0^t \|w'(s)\|_{L_q}\,\mathrm{d}s. \end{aligned} \tag{2.14}$$

Finally, (2.9) and (2.12)–(2.14) yield $\left\|w'(t)\right\|_{L_q} \leqslant Ct^\alpha + C\int_0^t \|w'(s)\|_{L_q}\,\mathrm{d}s$. Hence an application of Gronwall's lemma gives $\|w'(t)\|_{L_q} \leqslant Ct^\alpha$. Substituting this bound into (2.13) completes the proof of the inequality (2.8). □

Next we derive other needed bounds of $u(t)$ involving second time derivatives.

THEOREM 2.5 Let $1 < q < \infty$. In addition to the assumptions of Theorem 2.4, we impose that $\|f_{uu}(t,u(t))\|_{L_\infty} \leqslant C$ and $t\|G_{tt}(t,u(t))\|_{L_q} + \|f_{tu}(t,u(t))\|_{L_q} \leqslant Ct^{\alpha-1}$ for $t > 0$. Then the exact solution u of (1.1) satisfies

$$\|u''\|_{L_q} + \|Au'(t)\|_{L_q} \leqslant Ct^{\alpha-1}.$$

Proof. As in Theorem 2.4, using $u = w + v$ and Lemma 2.3, it is enough to show that the solution w of (2.3) satisfies

$$\|w''(t)\|_{L_q} + \|Aw'(t)\|_{L_q} \leqslant Ct^{\alpha-1} \quad \text{for } 0 < t \leqslant T. \tag{2.15}$$

Differentiating (2.10), we get

$$w'' + Aw' = -t^{\alpha-1}Au_0 - \int_0^t s^{\alpha-1}AD_t u(t-s)\mathrm{d}s + D_t G(t,u(t)) \quad \text{for } t > 0. \tag{2.16}$$

Note that $G(0, u(0)) = 0$ and also $Aw(0) = 0$, and hence (2.16) yields $w(0) = 0$. So, by Duhamel's principle and setting $u = w + v$, we have

$$\begin{aligned} w'(t) = &\left[-\int_0^t E(t-s)s^{\alpha-1}Au_0 \, ds - \int_0^t E(t-s) \int_0^s (s-\sigma)^{\alpha-1} Aw'(\sigma) d\sigma \, ds \right] \\ &- \int_0^t E(t-s) \int_0^s (s-\sigma)^{\alpha-1} Av'(\sigma) d\sigma \, ds + \int_0^t E(t-s) D_s G(s, u(s)) ds \\ &= w_1 + w_2 + w_3. \end{aligned} \tag{2.17}$$

Lemmas 2.2 and 2.1 give

$$\begin{aligned} \|Aw_1\|_{L_q} &\leqslant C \left(\int_0^t t^{\alpha-1} \|Au_0\|_{L_q} \, ds + \int_0^t (t-s)^{\alpha-1} \|Aw'(s)\|_{L_q} \, ds \right) \\ &\leqslant Ct^\alpha + \int_0^t (t-s)^{\alpha-1} \|Aw'(s)\|_{L_q} \, ds. \end{aligned}$$

For the bound of Aw_2, using $Av'(t)$ as in (2.7), we have

$$Aw_2 = \int_0^t AE(t-s)g(s) ds - \int_0^t AE(t-s) \int_0^s (s-\sigma)^{\alpha-1} \int_0^\sigma AE(\sigma-\delta)F'(\delta) d\delta \, d\sigma \, ds,$$

where

$$g(t) := \int_0^t (t-s)^{\alpha-1} AE(s)(Au_0 - F(0)) ds = \int_0^t s^{\alpha-1} AE(t-s)(Au_0 - F(0)) ds.$$

We shall apply Lemma 2.1. By this lemma, $\|g(t)\|_{L_q} \leqslant Ct^{\alpha-1}\|Au_0 - F(0)\|_{L_q} \leqslant Ct^{\alpha-1}$. For the estimate required for $g'(t)$ we write

$$g(t) = \left(\int_0^{t/2} + \int_{t/2}^t \right) s^{\alpha-1} AE(t-s)(Au_0 - F(0)) ds = g_1(t) + g_2(t).$$

Here, using $D_t AE(t-s) = -A^2 E(t-s)$, we get

$$g_1'(t) = (t/2)^{\alpha-1} AE(t/2)(Au_0 - F(0)) - \int_0^{t/2} s^{\alpha-1} A^2 E(t-s)(Au_0 - F(0)) ds$$

and clearly, using (2.5), $\|g_1'(t)\|_{L_q} \leqslant Ct^{\alpha-2}\|Au_0 - F(0)\|_{L_q} \leqslant Ct^{\alpha-2}$. Further, by using $AE(t-s) = D_s E(t-s)$, integrating by parts and a change of variables, we have

$$g_2(t) = t^{\alpha-1}(1 - 2^{1-\alpha} E(t/2))(Au_0 - F(0)) - (\alpha-1) \int_0^{t/2} (t-s)^{\alpha-2} E(s)(Au_0 - F(0)) ds,$$

and hence (2.5) yields $\|g_2(t)\|_{L_q} \leqslant Ct^{\alpha-1}\|Au_0 - F(0)\|_{L_q} \leqslant Ct^{\alpha-1}$.

Differentiating this expression with respect to time and proceeding as for g_1 gives the same bound for g_2 as for g_1, namely, $\|g_2'(t)\|_{L_q} \leqslant Ct^{\alpha-2}\|Au_0 - F(0)\|_{L_q} \leqslant Ct^{\alpha-2}$. We may then apply

Lemmas 2.1 and 2.2 to obtain

$$\begin{aligned}\|Aw_2\|_{L_q} &\leqslant Ct^{\alpha-1} + C\int_0^t (t-s)^{\alpha-1}\left\|\int_0^s AE(s-\sigma)F'(\sigma)\mathrm{d}\sigma\right\|_{L_q} \mathrm{d}s\\ &\leqslant Ct^{\alpha-1} + C\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\,\mathrm{d}s\\ &\leqslant C\left(t^{\alpha-1} + \int_0^{t/2} s^{\alpha-1}(t-s)^{\alpha-1}\,\mathrm{d}s + \int_{t/2}^t s^{\alpha-1}(t-s)^{\alpha-1}\,\mathrm{d}s\right) \leqslant C(t^{\alpha-1}+t^{2\alpha-1}).\end{aligned}$$

To bound Aw_3, splitting it into two parts and using $AE(t-s) = D_sE(t-s)$, we have

$$Aw_3 = \int_0^{t/2} AE(t-s)D_sG(s,u(s))\mathrm{d}s + \int_{t/2}^t D_sE(t-s)D_sG(s,u(s))\mathrm{d}s = h_1(t)+h_2(t). \quad (2.18)$$

Integrating by parts and using $D_sAE(t-s) = A^2E(t-s)$, (2.5), (2.12) and Theorem 2.4, we find that

$$\begin{aligned}\|h_1(t)\|_{L_q} &= \left\|AE(t/2)G(t/2,u(t/2)) - \int_0^{t/2} D_sAE(t-s)G(s,u(s))\mathrm{d}s\right\|_{L_q}\\ &\leqslant C\|G(t/2,u(t/2))\|_{L_q} + \int_0^{t/2}(t-s)^{-2}\|G(s,u(s))\|_{L_q}\,\mathrm{d}s\\ &\leqslant C\int_0^{t/2}\|u'(s)\|_{L_q}\,\mathrm{d}s + Ct^{-2}\int_0^{t/2}\int_0^s \|u'(\sigma)\|_{L_q}\,\mathrm{d}\sigma\,\mathrm{d}s \leqslant C. \end{aligned}\quad (2.19)$$

For $h_2(t)$, integrating by parts, we find that

$$\begin{aligned}\|h_2(t)\|_{L_q} &= \left\|E(0)D_tG(t,u(t)) - E(t/2)D_tG(t/2,u(t/2)) - \int_{t/2}^t E(t-s)D_{ss}G(s,u(s))\mathrm{d}s\right\|_{L_q}\\ &\leqslant \|D_tG(t,u(t))\|_{L_q} + C\|D_tG(t/2,u(t/2))\|_{L_q} + C\int_{t/2}^t \|D_{ss}G(s,u(s))\|_{L_q}\,\mathrm{d}s. \end{aligned}\quad (2.20)$$

The given assumptions, Theorem 2.4, Lemma 2.3 and $u = v + w$ lead to

$$\begin{aligned}\|D_tG(t,u(t))\|_{L_q} &= \|G_t(t,u(t)) + f_u(t,u(t))u'(t)\|_{L_q}\\ &\leqslant \|G_t(t,u(t))\|_{L_q} + \|f_u(t,u(t)\|_{L_\infty}\|u'(t)\|_{L_q} \leqslant Ct^{\alpha-1} \end{aligned}\quad (2.21)$$

and

$$\begin{aligned}\|D_{tt}G(t,u(t))\|_{L_q} &= \|G_{tt}(t,u(t)) + 2f_{ut}(t,u(t))u'(t) + f_{uu}(t,u(t))u'(t)^2 + f_u(t,u(t))u''(s)\|_{L_q}\\ &\leqslant C\left(\|G_{tt}(t,u(t))\|_{L_q} + \|f_{ut}(t,u(t))\|_{L_{2q}}\|u'(t)\|_{L_{2q}} + \|u'(t)\|_{L_{2q}}^2 + \|v''(t)\|_{L_q} + \|w''(t)\|_{L_q}\right)\\ &\leqslant C\left(t^{\alpha-2} + \|w''(t)\|_{L_q}\right). \end{aligned}\quad (2.22)$$

Hence (2.18)–(2.22) gives $\|Aw_3\|_{L_q} \leqslant Ct^{\alpha-1} + C\int_{t/2}^{t} \|w''(s)\|_{L_q}\,\mathrm{d}s$. Using this and the estimated results of $\|Aw_1\|_{L_q}$ and $\|Aw_2\|_{L_q}$ in (2.17), we get

$$\|Aw'(t)\|_{L_q} \leqslant Ct^{\alpha-1} + C\int_{t/2}^{t} \|w''(s)\|_{L_q}\,\mathrm{d}s. \tag{2.23}$$

To bound the second term on the right-hand side of (2.16) we use $u(t) = w(t)+v(t)$, (2.23) and Lemma 2.3 to obtain

$$\begin{aligned}
\left\|\int_0^t s^{\alpha-1} AD_t u(t-s)\mathrm{d}s\right\|_{L_q} &\leqslant \int_0^t (t-s)^{\alpha-1}\|Aw'(s)\|_{L_q}\,\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}\|Av'(s)\|_{L_q}\,\mathrm{d}s \\
&\leqslant C\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\,\mathrm{d}s + C\int_0^t (t-s)^{\alpha-1}\int_0^s \|w''(\sigma)\|_{L_q}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\leqslant C\int_0^{t/2} s^{\alpha-1}(t-s)^{\alpha-1}\,\mathrm{d}s + C\int_{t/2}^{t} s^{\alpha-1}(t-s)^{\alpha-1}\,\mathrm{d}s \\
&\quad + Ct^{\alpha}\int_0^t \|w''(s)\|_{L_q}\,\mathrm{d}s \\
&\leqslant Ct^{\alpha}\left(t^{\alpha-1} + \int_0^t \|w''(s)\|_{L_q}\,\mathrm{d}s\right).
\end{aligned}$$

Therefore using (2.21) and (2.23) in (2.16) yields $\|w''(t)\|_{L_q} \leqslant Ct^{\alpha-1} + C\int_0^t \|w''(s)\|_{L_q}\,\mathrm{d}s$. Finally, an application of Gronwall's lemma to this inequality follows by substituting the obtained bound of $\|w''(t)\|_{L_q}$ into (2.23), giving the desired bound of $\|Aw'(t)\|$. Hence the proof of (2.15) is complete. □

In addition to the regularity results that we have shown, we need to prove that $\|Au''(t)\|_{L_q} \leqslant Ct^{\alpha-2}$. This can be done by following the procedure that was used to show that $\|Au'(t)\|_{L_q} \leqslant Ct^{\alpha-1}$ with some modifications and under suitable assumptions.

3. Stability of the numerical solution

In this section we study the stability of the generalized extrapolated Crank–Nicolson scheme (1.3)–(1.5). In Lemma 3.2 we derive some important properties of the approximate solutions U^n and V^1. A key ingredient is the following discrete version of the positivity property $\int_0^T v(t)\int_0^t \beta(t-s)v(s)\mathrm{d}s$ for all $T > 0$ and any continuous function $v\colon [0, T] \to \mathbb{R}$. For the proof of the next lemma see McLean *et al.* (1996).

LEMMA 3.1 If $v^1, \ldots, v^n$ belong to $H_0^1(\Omega)$ then $(v^1, \mathscr{F}_\alpha^{1/2}Bv)k_1 + \sum_{j=2}^n (v^{j-1/2}, \mathscr{F}_\alpha^{j-1/2}Bv)k_j \geqslant 0$.

The next result will be used to prove the stability and the convergence of the approximate solution U^n for $n \geqslant 1$.

LEMMA 3.2 If U^n and V^1 are defined by (1.3)–(1.5) with $U^0 = V^0$, then

$$\|V^1\| \leqslant \|U^0\| + 2k_1\|f(t_{1/2}, U^0)\|$$

and

$$\|U^M\| \leqslant \|U^0\| + 2\sum_{n=2}^{M} k_n \| f(t_{n-1/2}, \widehat{U}^{n-1/2})\| + 2k_1 \| f(t_{1/2}, V^{1/2})\| \quad \text{for } M = 1, \ldots, N.$$

Proof. Taking the inner product of both sides of (1.4) with $2V^1 k_1$, we obtain

$$2\|V^1\|^2 - 2(V^1, V^0) + 2k_1(AV^1, V^1) + 2k_1(\mathscr{F}_\alpha^{1/2} BV, V^1) = 2k_1(f(t_{1/2}, V^0), V^1).$$

Since $\|V^1 - V^0\|^2 = \|V^1\|^2 - 2(V^1, V^0) + \|V^0\|^2$, it follows that

$$\|V^1\|^2 - \|V^0\|^2 + \|V^1 - V^0\|^2 + 2k_1(AV^1, V^1) + 2k_1(\mathscr{F}_\alpha^{1/2} BV, V^1) = 2k_1(f(t_{1/2}, V^0), V^1). \quad (3.1)$$

Using this, $\|V^1 - V^0\| \geqslant 0$, $(AV^1, V^1) \geqslant 0$, $(\mathscr{F}_\alpha^{1/2} BV, V^1) \geqslant 0$ by Lemma 3.1 and the Cauchy–Schwarz inequality, we find that

$$(\|V^1\| + \|V^0\|)(\|V^1\| - \|V^0\|) = \|V^1\|^2 - \|V^0\|^2 \leqslant 2k_1(f(t_{1/2}, V^0), V^1)$$
$$\leqslant 2k_1\|V^1\|\|f(t_{1/2}, V^0)\| \leqslant 2k_1(\|V^1\| + \|V^0\|)\|f(t_{1/2}, V^0)\|.$$

So $\|V^1\| - \|V^0\| \leqslant 2k_1\|(f(t, V^0)\|$ and the first desired result is obtained.

Taking the inner product of both sides of (1.5) with $2U^1 k_1$ and following a similar procedure as before, we obtain

$$\|U^1\|^2 - \|U^0\|^2 + \|U^1 - U^0\|^2 + 2k_1(AU^1, U^1) + 2k_1(\mathscr{F}_\alpha^{1/2} BU, U^1) = 2k_1(f(t_{1/2}, V^{1/2}), U^1) \quad (3.2)$$

and $\|U^1\| - \|U^0\| \leqslant 2k_1\|f(t_{1/2}, V^{1/2})\|$. So the second required result is proved for $M = 1$.

For $M \geqslant 2$ we take the inner product of both sides of (1.3) with $k_n(U^n + U^{n-1})$ and get

$$\|U^n\|^2 - \|U^{n-1}\|^2 + 2k_n(AU^{n-1/2} + \mathscr{F}_\alpha^{n-1/2} BU, U^{n-1/2}) = 2k_n(f(t_{n-1/2}, \widehat{U}^{n-1/2}), U^{n-1/2}).$$

Summing from $n = 2$ to $n = M$ gives

$$\|U^M\|^2 - \|U^1\|^2 + 2\sum_{n=2}^{M} k_n(AU^{n-1/2}, U^{n-1/2}) + 2\sum_{n=2}^{M} k_n(\mathscr{F}_\alpha^{n-1/2} BU, U^{n-1/2})$$
$$= 2\sum_{n=2}^{M} k_n(f(t_{n-1/2}, \widehat{U}^{n-1/2}), U^{n-1/2}). \quad (3.3)$$

Using (3.2), (3.3), $(AU^{n-1/2}, U^{n-1/2}) \geqslant 0$, $(AU^1, U^1) \geqslant 0$ and $k_1(\mathscr{F}_\alpha^{1/2} BU, U^1) + \sum_{n=2}^{M} k_n (\mathscr{F}_\alpha^{n-1/2} BU, U^{n-1/2}) \geqslant 0$ from Lemma 3.1, we have

$$\|U^M\|^2 - \|U^0\|^2 \leqslant 2\sum_{n=2}^{M} k_n(f(t_{n-1/2}, \widehat{U}^{n-1/2}), U^{n-1/2}) + 2k_1(f(t_{1/2}, V^{1/2}), U^1), \quad (3.4)$$

and hence the Cauchy–Schwarz inequality gives

$$\|U^M\|^2 \leqslant \|U^0\|^2 + 2\sum_{n=2}^{M} k_n \|f(t_{n-1/2}, \widehat{U}^{n-1/2})\|\|U^{n-1/2}\| + 2k_1\|f(t_{1/2}, V^{1/2})\|\|U^1\|. \tag{3.5}$$

Let M_0 be such that $\|U^{M_0}\| = \max_{2\leqslant n\leqslant M} \|U^n\|$. Then (3.5) and the triangle inequality yield

$$\|U^{M_0}\|^2 \leqslant \|U^0\|^2 + 2\left(\sum_{n=2}^{M_0} k_n\|f(t_{n-1/2}, \widehat{U}^{n-1/2})\| + k_1\|f(t_{1/2}, V^{1/2})\|\right)\|U^{M_0}\|.$$

Thus for $M = 2, \dots, N$ we have

$$\|U^M\| \leqslant \|U^{M_0}\| \leqslant \|U^0\| + 2\sum_{n=2}^{M_0} k_n\|f(t_{n-1/2}, \widehat{U}^{n-1/2})\| + 2k_1\|f(t_{1/2}, V^{1/2})\|. \qquad \square$$

LEMMA 3.3 The approximate solution U^n defined by (1.3)–(1.5) satisfies

$$\|U^n\| \leqslant C_{\gamma,L}\|U^0\| + C_{\gamma,L}\sum_{j=1}^{n} k_j\|f(t_{j-1/2}, U^0)\| \quad \text{for } n = 1, \dots, N.$$

Proof. Using Lemma 3.2, (2.1) and our graded mesh assumption (1.7), we obtain

$$\begin{aligned}
\|U^n\| &\leqslant \|U^0\| + 2\sum_{j=2}^{n} k_j\|f(t_{j-1/2}, \widehat{U}^{j-1/2})\| + 2k_1\|f(t_{1/2}, V^{1/2})\| \\
&\leqslant \|U^0\| + 2\sum_{j=2}^{n} k_j\|f(t_{j-1/2}, \widehat{U}^{j-1/2}) - f(t_{j-1/2}, U^0)\| + 2k_1\|f(t_{1/2}, V^{1/2}) - f(t_{1/2}, U^0)\| \\
&\quad + 2\sum_{j=1}^{n} k_j\|f(t_{j-1/2}, U^0)\| \\
&\leqslant \|U^0\| + 2L\sum_{j=2}^{n} k_j\|\widehat{U}^{j-1/2} - U^0\| + 2Lk_1\|V^{1/2} - U^0\| + 2\sum_{j=1}^{n} k_j\|f(t_{j-1/2}, U^0)\| \\
&\leqslant \|U^0\| + L\sum_{j=2}^{n} k_j\|3U^{j-1} - U^{j-2}\| + 2L\|U^0\| + Lk_1(\|V^1\| + \|U^0\|) + 2\sum_{j=1}^{n} k_j\|f(t_{j-1/2}, U^0)\| \\
&\leqslant (1 + 2L + 2Lk_1 + Lk_2)\|U^0\| + 2(Lk_1 + 1)2\sum_{j=1}^{n} k_j\|f(t_{j-1/2}, U^0)\| + C_\gamma L\sum_{j=1}^{n-1} k_j\|U^j\|.
\end{aligned}$$

Therefore an application of the standard discrete analogue of Gronwall's inequality yields the stability result. $\square$

4. Error from the time discretization

In this section we estimate the error $e^n = U^n - u^n$ ($u^n := u(t_n)$), where the approximate solution U^n is obtained using the extrapolated Crank–Nicolson time discretization defined by (1.3)–(1.5) or equivalently by

$$U^n - U^{n-1} + k_n AU^{n-1/2} + \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s)B\overline{U}(s)\mathrm{d}s\,\mathrm{d}t = k_n f(t_{n-1/2}, \widehat{U}^{n-1/2}) \quad \text{for } n \geqslant 2, \tag{4.1}$$

and for $n = 1$ by

$$U^1 - U^0 + k_1 AU^1 + \int_0^{t_1} \int_0^t \beta(t-s)BU^1\,\mathrm{d}s\,\mathrm{d}t = k_1 f(t_{n-1/2}, V^{1/2}), \tag{4.2}$$

where $V^0 = U^0 \approx u(t_0)$ and V^1 is defined by (1.4).

For comparison, integrating (1.1) from $t = t_{n-1}$ to $t = t_n$ shows that the exact solution satisfies

$$u(t_n) - u(t_{n-1}) + \int_{t_{n-1}}^{t_n} \left(Au(t) + \int_0^t \beta(t-s)Bu(s)\mathrm{d}s \right) \mathrm{d}t = \int_{t_{n-1}}^{t_n} f(t, u(t))\mathrm{d}t \quad \text{for } n \geqslant 1. \tag{4.3}$$

Hence the error satisfies

$$e^n - e^{n-1} + k_n Ae^{n-1/2} + \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s)B\overline{e}(s)\mathrm{d}s\,\mathrm{d}t = \eta^{n-1/2} \quad \text{for } n \geqslant 2, \tag{4.4}$$

and for $n = 1$ we have

$$e^1 - e^0 + k_1 Ae^1 + \int_0^{t_1} \int_0^t \beta(t-s)B\overline{e}(s)\mathrm{d}s\,\mathrm{d}t = \eta^{1/2}, \tag{4.5}$$

where $\eta^{n-1/2} = \eta_1^{n-1/2} + \eta_2^{n-1/2} + \eta_3^{n-1/2}$ with

$$\begin{aligned}
\eta_1^{1/2} &= \int_{t_{n-1}}^{t_n} [f(t_{1/2}, V^{1/2}) - f(t, u(t))]\mathrm{d}t, \\
\eta_1^{n-1/2} &= \int_{t_{n-1}}^{t_n} [f(t_{n-1/2}, \widehat{U}^{n-1/2}) - f(t, u(t))]\mathrm{d}t \quad \text{for } n \geqslant 2, \\
\eta_2^{n-1/2} &= \int_{t_{n-1}}^{t_n} [Au(t) - Au(t_n)]\mathrm{d}t \quad \text{for } n \geqslant 1, \\
\eta_3^{n-1/2} &= \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s)[Bu(s) - B\overline{u}(s)]\mathrm{d}s\,\mathrm{d}t \quad \text{for } n \geqslant 1.
\end{aligned} \tag{4.6}$$

Since (4.4) and (4.5) have the same form as (4.1) and (4.2), respectively, Lemma 3.2 implies that

$$\|e^n\| \leqslant \|e^0\| + 2\sum_{j=1}^{n} \|\eta^{j-1/2}\| \quad \text{for } n \geqslant 1. \tag{4.7}$$

Thus our task is reduced to estimating the sum on the right-hand side of this inequality. In our process we allow for the possibility that $\|Au''(t)\|$, $\|Bu''(t)\|$ and $\|u''(t)\|$ are not integrable on the time step size $(0, t_{1/2})$.

Lemma 3.2 and Theorem 3.3 in the recent work of McLean & Mustapha (2007) provide the following estimate of $\|\eta_3^{j-1/2}\|$:

$$\sum_{j=1}^{n} \|\eta_3^{j-1/2}\| \leqslant C_{\alpha,T} \left(\int_0^{t_1} s\|Bu'(s)\|\mathrm{d}s + k_2 \int_{t_1}^{t_2} \|Bu'(s)\|\mathrm{d}s + \sum_{j=2}^{n} k_j^2 \int_{t_{j-1}}^{t_j} \|Bu''(s)\|\mathrm{d}s \right)$$
$$+ C_{\alpha,T} \sum_{j=2}^{n-1} \left(k_j^2 \int_{t_j}^{t_j^*} \|Bu'(s)\|\mathrm{d}s + (k_{j+1} - k_j) \int_{t_{j-1}}^{t_{j+1}} \|Bu'(s)\|\mathrm{d}s \right), \tag{4.8}$$

where

$$t_j^* = \frac{t_{j+1} + t_{j-1}}{2} \quad \text{for } j \geqslant 1.$$

LEMMA 4.1 For $\eta_1^{j-1/2}$ defined as in (4.6) for $j \geqslant 1$, we have

$$\sum_{j=1}^{n} \|\eta_1^{j-1/2}\| \leqslant C_L \left(\sum_{j=2}^{n} k_j \|\widehat{e}^{j-1/2}\| + k_1(\|V^1 - u^1\| + \|e^0\|) \right)$$
$$+ \sum_{j=2}^{n} k_j \left(C_L k_j \int_{t_{j-1}}^{t_j} (\|u''(t)\| + \|D_{tt} f(t, u(t))\|)\mathrm{d}t + C_L \int_{t_{j-1}}^{t_{j-1}^*} \|u'(t)\|\mathrm{d}t \right)$$
$$+ \int_0^{t_1^*} [C_L k_1 t\|u''(t)\| + t^2\|D_{tt} f(t, u(t))\|]\mathrm{d}t$$
$$+ k_1^2 \int_{t_{1/2}}^{t_1} [C_L \|u''(t)\| + \|D_{tt} f(t, u(t))\|]\mathrm{d}t. \tag{4.9}$$

Proof. To show the stated result we make the splitting $\eta_1^{j-1/2} = \eta_{11}^{j-1/2} + \eta_{12}^{j-1/2}$, where

$$\eta_{11}^{1/2} = k_1[f(t_{1/2}, V^{1/2}) - f(t_{1/2}, u(t_{1/2}))],$$
$$\eta_{11}^{j-1/2} = k_j[f(t_{j-1/2}, \widehat{U}^{j-1/2}) - f(t_{j-1/2}, u(t_{j-1/2}))] \quad \text{for } j \geqslant 2,$$
$$\eta_{12}^{j-1/2} = -\int_{t_{j-1}}^{t_j} [f(t, u(t)) - f(t_{j-1/2}, u(t_{j-1/2}))]\mathrm{d}t \quad \text{for } j \geqslant 1.$$

To bound $\|\eta_{11}^{j-1/2}\|$ for $j \geqslant 2$ we use (2.1), and by adding and subtracting $\widehat{u}^{j-1/2} := \frac{3u(t_{j-1})-u(t_{j-2})}{2}$ we get

$$\|\eta_{11}^{j-1/2}\| \leqslant C_L k_j \|\widehat{U}^{j-1/2} - u(t_{j-1/2})\| \leqslant C_L k_j (\|\widehat{e}^{j-1/2}\| + \|\widehat{u}^{j-1/2} - u(t_{j-1/2})\|).$$

Adding and subtracting $u(t^*_{j-1})$, we get

$$\widehat{u}^{j-1/2} - u(t_{j-1/2}) = [u^{j-1/2} - u(t_{j-1/2})] - \left[\frac{u^j + u^{j-2}}{2} - u(t^*_{j-1})\right] + [u^{j-1} - u(t^*_{j-1})],$$

$$u^{j-1/2} := \frac{u(t_{j-1}) + u(t_j)}{2}.$$

For $\mu_1 < \mu_2$ and $\mu_{12} = \frac{\mu_1+\mu_2}{2}$ we observe that

$$\frac{u(\mu_1) + u(\mu_2)}{2} - u(\mu_{12}) = \frac{1}{2}\left[\int_{\mu_1}^{\mu_{12}} (s - \mu_1)u''(s)\mathrm{d}s - \int_{\mu_{12}}^{\mu_2} (s - \mu_2)u''(s)\mathrm{d}s\right]. \tag{4.10}$$

Since $k_{j-1} \leqslant k_j$, we have $t_{j-1} \leqslant t^*_{j-1} \leqslant t_j$, and hence for $j \geqslant 2$ it follows that

$$\|\widehat{u}^{j-1/2} - u(t_{j-1/2})\| \leqslant k_j \int_{t_{j-1}}^{t_j} \|u''(s)\|\mathrm{d}s + \frac{1}{2}\int_{t_{j-2}}^{t^*_{j-1}} (s - t_{j-2})\|u''(s)\|\mathrm{d}s + \|u^{j-1} - u(t^*_{j-1})\|.$$

Thus for $j \geqslant 2$ we have

$$\|\eta_{11}^{j-1/2}\| \leqslant C_L k_j \left(\|\widehat{e}^{j-1/2}\| + k_j \int_{t_{j-1}}^{t_j} \|u''(s)\|\mathrm{d}s + \int_{t_{j-2}}^{t^*_{j-1}} (s - t_{j-2})\|u''(s)\|\mathrm{d}s + \int_{t_{j-1}}^{t^*_{j-1}} \|u'(s)\|\mathrm{d}s\right).$$

Similarly, for $j = 1$ we have

$$\begin{aligned}\|\eta_{11}^{1/2}\| &\leqslant C_L k_1 \|\widehat{V}^{1/2} - u(t_{1/2})\| \leqslant C_L k_1 (\|V^{1/2} - u^{1/2}\| + \|u^{1/2} - u(t_{1/2})\|) \\ &\leqslant C_L k_1 \left(\|V^1 - u^1\| + \|e^0\| + \int_0^{t_{1/2}} s\|u''(s)\|\mathrm{d}s + k_1 \int_{t_{1/2}}^{t_1} \|u''(s)\|\mathrm{d}s\right) \\ &\leqslant C_L k_1 \left(\|V^1 - u^1\| + \|e^0\| + \int_0^{t^*_1} s\|u''(s)\|\mathrm{d}s + k_1 \int_{t_{1/2}}^{t_1} \|u''(s)\|\mathrm{d}s\right).\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{j=1}^n \|\eta_{11}^{j-1/2}\| \leqslant C_L \sum_{j=2}^n k_j &\left(\|\widehat{e}^{j-1/2}\| + k_j \int_{t_{j-1}}^{t_j} \|u''(t)\|\mathrm{d}t + \int_{t_{j-1}}^{t^*_{j-1}} \|u'(t)\|\mathrm{d}t\right) \\ &+ C_L k_1 \left(\int_0^{t^*_1} t\|u''(t)\|\mathrm{d}t + k_1 \int_{t_{1/2}}^{t_1} \|u''(t)\|\mathrm{d}t + \|V^1 - u^1\| + \|e^0\|\right).\end{aligned} \tag{4.11}$$

For the estimation of $\eta_{12}^{j-1/2}$, using Taylor's expansion for integrals, splitting, changing the order of integrals and integrating, we get

$$
\begin{aligned}
-\eta_{12}^{j-1/2} &= \int_{t_{j-1}}^{t_j} \left((t - t_{j-1/2}) D_t f(t_{j-1/2}, u(t_{j-1/2})) + \int_{t_{j-1/2}}^{t} (t-s) D_{ss} f(s, u(s)) \mathrm{d}s \right) \mathrm{d}t \\
&= \int_{t_{j-1}}^{t_j} \int_{t_{j-1/2}}^{t} (t-s) D_{ss} f(s, u(s)) \mathrm{d}s \, \mathrm{d}t \\
&= \int_{t_{j-1}}^{t_{j-1/2}} \int_{t}^{t_{j-1/2}} (s-t) D_{ss} f(s, u(s)) \mathrm{d}s \, \mathrm{d}t + \int_{t_{j-1/2}}^{t_j} \int_{t_{j-1/2}}^{t} (t-s) D_{ss} f(s, u(s)) \mathrm{d}s \, \mathrm{d}t \\
&= \int_{t_{j-1}}^{t_{j-1/2}} \int_{t_{j-1}}^{s} (s-t) D_{ss} f(s, u(s)) \mathrm{d}t \, \mathrm{d}s + \int_{t_{j-1/2}}^{t_j} \int_{s}^{t_j} (t-s) D_{ss} f(s, u(s)) \mathrm{d}t \, \mathrm{d}s \\
&= \frac{1}{2} \int_{t_{j-1}}^{t_{j-1/2}} (s - t_{j-1})^2 D_{ss} f(s, u(s)) \mathrm{d}s + \frac{1}{2} \int_{t_{j-1/2}}^{t_j} (t_j - s)^2 D_{ss} f(s, u(s)) \mathrm{d}s.
\end{aligned}
$$

So

$$
\sum_{j=1}^{n} \|\eta_{12}^{j-1/2}\| \leqslant \sum_{j=2}^{n} k_j^2 \int_{t_{j-1}}^{t_j} \|D_{tt} f(t, u(t))\| \mathrm{d}t + \frac{1}{2} \left(\int_0^{t_{1/2}} t^2 \|D_{tt} f(t, u(t))\| \mathrm{d}t + k_1^2 \int_{t_{1/2}}^{t_1} \|D_{tt} f(t, u(t))\| \mathrm{d}t \right).
$$

Now, using the estimated bound of $\|\eta_{11}^{j-1/2}\|$ and $\|\eta_{12}^{j-1/2}\|$, we complete the proof. □

At this stage $\eta_2^{j-1/2}$ remains to be estimated. Using

$$
u(t) - u(t_{j-1/2}) = (t - t_{j-1/2}) u'(t_{j-1/2}) + \int_{t_{j-1/2}}^{t} (t-s) u''(s) \mathrm{d}s
$$

and (4.10), we get for $j \geqslant 2$ that

$$
\begin{aligned}
\eta_2^{j-1/2} &= \int_{t_{j-1}}^{t_j} A[u(t) - u(t_{j-1/2})] \mathrm{d}t + \int_{t_{j-1}}^{t_j} A[u(t_{j-1/2}) - u^{j-1/2}] \mathrm{d}t \\
&= \int_{t_{j-1}}^{t_j} \int_{t_{j-1/2}}^{t} (t-s) A u''(s) \mathrm{d}s \, \mathrm{d}t - \frac{k_j}{2} \left(\int_{t_{j-1}}^{t_{j-1/2}} (s - t_{j-1}) A u''(s) \mathrm{d}s - \int_{t_{j-1/2}}^{t_j} (s - t_j) A u''(s) \mathrm{d}s \right).
\end{aligned}
$$

For $j = 1$, changing the order of integrals, we have

$$
\eta_2^{1/2} = -\int_0^{t_1} \int_t^{t_1} A u'(s) \mathrm{d}s \, \mathrm{d}t = -\int_0^{t_1} \int_0^{s} A u'(s) \mathrm{d}t \, \mathrm{d}s = -\int_0^{t_1} s A u'(s) \mathrm{d}s.
$$

Hence

$$
\sum_{j=1}^{n} \|\eta_2^{j-1/2}\| \leqslant \int_0^{t_1} t \|A u'(t)\| \mathrm{d}t + \sum_{j=2}^{n} k_j^2 \int_{t_{j-1}}^{t_j} \|A u''(t)\| \mathrm{d}t. \tag{4.12}
$$

To state the error bound from the time discretization we need to estimate the term $\|V^1 - u^1\|$ that appears in the estimation of $\|\eta_1^{n-1/2}\|$ in Lemma 4.1.

LEMMA 4.2 Let u be the exact solution of (1.1) and V^1 be defined by (1.4) with $V^0 = U^0$. Then

$$\|V^1 - u^1\| \leqslant C_L \|e^0\| + C_{\alpha,T} \int_0^{t_1} t(\|u'(t)\| + \|Au'(t)\| + \|Bu'(t)\|)\mathrm{d}t.$$

Proof. From (1.4) and by using $V^0 = U^0$ we observe that

$$V^1 - U^0 + k_1 AV^1 + \int_0^{t_1} \int_0^t \beta(t-s)B\overline{V}(s)\mathrm{d}s\,\mathrm{d}t = k_1 f(t_{1/2}, U^0). \tag{4.13}$$

Comparing this with (4.3) for $n = 1$, we find that

$$(V^1 - u^1) - e^0 + k_1 A(V^1 - u^1) + \int_0^{t_1} \int_0^t \beta(t-s)[B\overline{V}(s) - B\overline{u}(s)]\mathrm{d}s\,\mathrm{d}t = \eta^1 + \eta_2^{1/2} + \eta_3^{1/2} \tag{4.14}$$

for the same $\eta_2^{1/2}$ and $\eta_3^{1/2}$ as in (4.6) and where

$$\eta^1 = \int_0^{t_1} [f(t_{1/2}, U^0) - f(t, u(t))]\mathrm{d}t. \tag{4.15}$$

Since (4.14) has the same form as (4.13), the stability of $V^1 - u^1$ is obtained from the first result of Lemma 3.2, i.e.,

$$\|V^1 - u^1\| \leqslant \|e^0\| + 2\|\eta^1\| + 2\|\eta_2^{1/2}\| + 2\|\eta_3^{1/2}\|.$$

Using (2.1) and changing the order of integrals yields

$$\begin{aligned} \|\eta^1\| &\leqslant L\int_0^{t_1} \|U^0 - u(t)\|\mathrm{d}t \leqslant L\int_0^{t_1} \|U^0 - u_0\|\mathrm{d}t + L\int_0^{t_1} \|u_0 - u(t)\|\mathrm{d}t \\ &\leqslant Lk_1 e^0 + \int_0^{t_1} t\|u'(t)\|\mathrm{d}t. \end{aligned} \tag{4.16}$$

Using this and the obtained bounds of $\|\eta_2^{1/2}\|$ and $\|\eta_3^{1/2}\|$ in (4.12) and (4.8), we complete the proof. □

Now we are at the stage of showing the convergence for the time discretization scheme (1.3)–(1.5).

THEOREM 4.3 Assume that the t_n satisfy (1.6)–(1.8) and let U^n be the solution of the discrete-time scheme (1.3)–(1.5). If the exact solution u of (1.1) and the source term satisfy

$$\|Au'(t)\| + \|Bu'(t)\| + t\|Au''(t)\| + t\|Bu''(t)\| \leqslant Mt^{\alpha-1} \quad \text{and} \quad t\|D_{tt} f(t, u(t))\| \leqslant Mt^{\alpha-1}$$

for $t > 0$, then, we have for $0 \leqslant t_n \leqslant T$, we have

$$\|U^n - u(t_n)\| \leqslant C\|U^0 - u_0\| + C \times \begin{cases} k^{\gamma(1+\alpha)} & \text{if } 1 \leqslant \gamma < 2/(1+\alpha), \\ k^2 \log(t_n/t_1) & \text{if } \gamma = 2/(1+\alpha), \\ k^2 & \text{if } \gamma > 2/(1+\alpha), \end{cases}$$

where C is a generic positive constant that is independent of k_n but may depend on L, α, γ, T and M.

Proof. We plan to bound the quantities on the right-hand side of (4.9), (4.12) and (4.8) using the properties of the graded mesh and the given regularity assumptions on u and f. Then, by substituting these bounds into (4.7) we obtain the desired result.

Since $t^*_{j-1} - t_{j-1} = \frac{1}{2}(k_j - k_{j-1})$, (4.9), (1.7) and (1.8) imply that

$$\sum_{j=1}^{n} \|\eta_1^{j-1/2}\| \leqslant C\left(\sum_{j=2}^{n} k_j \|\widehat{e}^{j-1/2}\| + k_1(\|V^1 - u^1\| + \|e^0\|)\right)$$
$$+ C\sum_{j=2}^{n}(t_j^{\alpha-2}k_j^3 + k_j(t^*_{j-1} - t_{j-1})t_j^{\alpha-1}) + CMt_1^{*1+\alpha} + Ck_1^2 t_1^{\alpha-1}$$
$$\leqslant C\sum_{j=2}^{n} k_j\|\widehat{e}^{j-1/2}\| + Ck_1(\|V^1 - u^1\| + \|e^0\|) + Ck^2\sum_{j=2}^{n} t_j^{\alpha-2/\gamma}k_j + Ck^{\gamma(1+\alpha)}.$$

From (4.12) and the mesh assumption (1.7) we get

$$\sum_{j=1}^{n} \|\eta_2^{j-1/2}\| \leqslant C\int_0^{t_1} t^\alpha \, \mathrm{d}t + C\sum_{j=2}^{n} k_j^3 t_j^{\alpha-2} \leqslant Ck^2\sum_{j=2}^{n} t_j^{\alpha-2/\gamma}k_j + Ck^{\gamma(1+\alpha)}. \tag{4.17}$$

Using (4.8), $t_1^{-1}k_2 \leqslant Ct_1^{-1}kt_2^{1-1/\gamma} \leqslant Ckt_1^{-1/\gamma} \leqslant C_\gamma$, $t_j^* - t_j = \frac{1}{2}(k_{j+1} - k_j)$, (1.7) and (1.8), we have

$$\sum_{j=1}^{n} \|\eta_3^{j-1/2}\| \leqslant C(t_1^{\alpha+1} + t_1^\alpha k_2) + C\sum_{j=2}^{n}(k_j^2 t_j^{\alpha-2}k_j + k_j(t_j^* - t_j)t_j^{\alpha-1} + (k_{j+1} - k_j)t_j^{\alpha-1}(k_j + k_{j+1}))$$
$$\leqslant Ck^{\gamma(1+\alpha)} + Ck^2\sum_{j=2}^{n} t_j^{\alpha-2/\gamma}k_j.$$

From (4.7), the above estimations,

Lemma 4.2 and $\int_0^{t_1} t(\|u'(t)\| + \|Au'(t)\| + \|Bu'(t)\|)\,\mathrm{d}t \leqslant Ct_1^{\alpha+1} \leqslant Ck^{\gamma(1+\alpha)}$, we find that

$$\|U^n - u^n\| \leqslant C\|U^0 - u_0\| + C\sum_{j=1}^{n-1} k_j\|U^j - u^j\|$$
$$+Ck^2\sum_{j=2}^{n} t_j^{\alpha-2/\gamma}k_j + Ck^{\gamma(1+\alpha)} \quad \text{for } n = 1, \ldots, N, \tag{4.18}$$

and hence an application of the standard discrete analogue of Gronwall's inequality yields

$$\|U^n - u^n\| \leqslant C\|U^0 - u_0\| + C\left(k^2\sum_{j=2}^{n} t_j^{\alpha-2/\gamma}k_j + Ck^{1+\alpha}\right).$$

Estimating the sum, we obtain

$$\sum_{j=2}^{n} t_j^{\alpha-2/\gamma} k_j \leqslant C \int_{t_1}^{t_n} t^{\alpha-2/\gamma}\, \mathrm{d}t \leqslant C \times \begin{cases} t_1^{\alpha-2/\gamma+1}/(2/\gamma-\alpha-1) & \text{if } \gamma < 2/(\alpha+1), \\ \log(t_n/t_1) & \text{if } \gamma = 2/(\alpha+1), \\ t_n^{\alpha-2/\gamma+1}/(\alpha-2/\gamma+1) & \text{if } \gamma > 2/(\alpha+1), \end{cases}$$

and we now complete the proof by noting that $t_1^{\alpha-2/\gamma+1} = (k_1)^{-(2/\gamma-\alpha-1)} \leqslant (c_\gamma k^\gamma)^{-(2/\gamma-\alpha-1)} = Ck^{\gamma(1+\alpha)-2}$. □

5. Discretization in time and space

Throughout this section we carry out our analysis assuming that Ω is a bounded, convex or $C^{1,1}$ domain and that the differential operators A and B in (1.1) are equal.

We choose a finite-dimensional subspace $S_h \subseteq H_0^1(\Omega)$ formed of continuous, piecewise-linear finite-element spaces constructed on a triangulation of Ω for which h is the maximum element diameter such that

$$\min_{\chi \in S_h} (\|v-\chi\| + h\|\nabla(v-\chi)\|) \leqslant Ch^2 \|v\|_{H^2(\Omega)} \quad \text{for } v \in H_0^1 \cap H^2(\Omega).$$

Our fully discrete scheme for (1.1) is based on the extrapolated Crank–Nicolson method and the standard Galerkin method for the time and space discretizations, respectively. So we describe our fully discrete scheme by applying spatial Galerkin finite elements to the time discretization scheme (1.3)–(1.5) in which $U_h^n, V_h^1 \in S_h$ satisfy

$$\left(\frac{U_h^n - U_h^{n-1}}{k_n}, \chi\right) + A(U_h^{n-1/2}, \chi) + \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t-s) A(\overline{U}_h(s), \chi)\mathrm{d}s\, \mathrm{d}t$$

$$= (f(t_{n-1/2}, \widehat{U}_h^{n-1}), \chi) \quad \text{for } n \geqslant 2, \tag{5.1}$$

$$\left(\frac{V_h^1 - V_h^0}{k_1}, \chi\right) + A(V_h^1, \chi) + \frac{1}{k_1}\int_0^{t_1}\int_0^t \beta(t-s) B(\overline{V}_h(s), \chi)\mathrm{d}s\, \mathrm{d}t = (f(t_{1/2}, V_h^0), \chi), \tag{5.2}$$

$$\left(\frac{U_h^1 - U_h^0}{k_1}, \chi\right) + A(U_h^1, \chi) + \frac{1}{k_1}\int_0^{t_1}\int_0^t \beta(t-s) B(\overline{U}_h(s), \chi)\mathrm{d}s\, \mathrm{d}t = (f(t_{1/2}, V_h^{1/2}), \chi), \tag{5.3}$$

for all $\chi \in S_h$, with suitable approximations $U_h^0 = V_h^0 \approx u_0$. Here $A(\cdot,\cdot)$ is the bilinear form associated with A.

To state our error bound for the time–space discretizations we let $R_h\colon H_0^1(\Omega) \to S_h$ denote the Ritz projector for the operator A. Thus

$$A(R_h v, \chi) = A(v, \chi) \quad \text{for } \chi \in S_h,$$

and we recall that

$$\|v - R_h v\| \leqslant Ch^2 \|u(t)\|_2. \tag{5.4}$$

The error bound in the next theorem allows for the possibility that $\|u'(t)\|_2$ is not integrable on the first subinterval $(0, t_1)$.

THEOREM 5.1 Let $U_h^n \in S_h$ be the solution of the discrete-time, discrete-space scheme (5.1)–(5.3). In addition to the assumptions of Theorem 4.3, if the exact solution u of (1.1) satisfies $\|u(t)\|_2 + t\|u'(t)\|_2 \leqslant Mt^\nu$ for $t > 0$, with $\nu \geqslant 0$, then, for $0 \leqslant t_n \leqslant T$, we have

$$\|U_h^n - u(t_n)\| \leqslant C\|U_h^0 - R_h u_0\| + C \times \begin{cases} k^{\gamma(1+\alpha)} & \text{if } 1 \leqslant \gamma < 2/(1+\alpha), \\ k^2 \log(t_n/t_1) & \text{if } \gamma = 2/(1+\alpha), \\ k^2 & \text{if } \gamma > 2/(1+\alpha) \end{cases}$$

$$+ Ch^2 \times \begin{cases} 1 + \log(t_n/t_1) & \text{if } \nu = 0, \\ 1 & \text{if } \nu > 0, \end{cases}$$

where C is a generic positive constant that is independent of k_n and h but may depend on L, α, γ, M, T and ν.

Proof. We split the error into two terms: $U_h^n - u(t_n) = \theta^n + \rho(t_n)$, where

$$\theta^n = U_h^n - R_h u(t_n) \in S_h \quad \text{and} \quad \rho(t) = R_h u(t) - u(t).$$

Integrating (1.1) from $t = t_{n-1}$ to $t = t_n$ and then taking the inner product with $\chi \in S_h$, we have

$$(u(t_n) - u(t_{n-1}), \chi) + \int_{t_{n-1}}^{t_n} \left(A(u(t), \chi) + \int_0^t \beta(t-s) B(u(s), \chi) \mathrm{d}s \right) \mathrm{d}t = \int_{t_{n-1}}^{t_n} (f(t, u(t)), \chi) \mathrm{d}t. \tag{5.5}$$

Since

$$(U_h^n - U_h^{n-1}) - (u(t_n) - u(t_{n-1})) = (\theta^n - \theta^{n-1}) + (\rho(t_n) - \rho(t_{n-1}))$$

and for $t_{n-1} \leqslant t \leqslant t_n$ we have

$$A(U_h^n, \chi) - A(u(t), \chi) + \int_0^t \beta(t-s) A(\overline{U}_h(s), \chi) \mathrm{d}s - \int_0^t \beta(t-s) A(u(s), \chi) \mathrm{d}s$$

$$= A(\theta^n, \chi) + A(u(t_n) - u(t), \chi) + \int_0^t \beta(t-s) A(\overline{\theta}(s), \chi) \mathrm{d}s + \int_0^t \beta(t-s) A(\overline{u}(s) - u(s), \chi) \mathrm{d}s,$$

we see that

$$(\theta^n - \theta^{n-1}, \chi) + k_n A(\theta^n, \chi) + \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s) B(\overline{\theta}(s), \chi) \mathrm{d}s \, \mathrm{d}t = (\eta^{n-1/2}, \chi),$$

where $\eta^{n-1/2} = \eta_1^{n-1/2} + \eta_2^{n-1/2} + \eta_3^{n-1/2} + \eta_4^{n-1/2}$, for the same $\eta_1^{n-1/2}$, $\eta_2^{n-1/2}$ and $\eta_3^{n-1/2}$ as in (4.6) and for

$$\eta_4^{n-1/2} = -[\rho(t_n) - \rho(t_{n-1})]. \tag{5.6}$$

Following the derivation used in Lemma 3.2 and using $\theta^0 = U_h^0 - R_h u_0$, we have for $n = 1, \ldots, N$ that

$$\|\theta^n\| \leqslant \|U_h^0 - R_h u_0\| + 2\sum_{j=1}^{n} \|\eta^{j-1/2}\|.$$

Using the bounds of $\|\eta_1^{n-1/2}\|$, $\|\eta_2^{n-1/2}\|$ and $\|\eta_3^{n-1/2}\|$ that were already dealt with in Theorem 4.3, we have

$$\sum_{j=1}^{n}(\|\eta_1^{n-1/2}\|+\|\eta_2^{n-1/2}\|+\|\eta_3^{n-1/2}\|)$$
$$\leqslant C\sum_{j=1}^{n-1}k_j\|U_h^j-u^j\|+Ck_1(\|V_h^1-u(t_1)\|+\|U_h^0-u_0\|)+C\left(k^2\sum_{j=2}^{n}t_j^{\alpha-2/\gamma}k_j+k^{\gamma(1+\alpha)}\right). \tag{5.7}$$

Now it suffices to estimate $\|\rho(t_n)\|$, $\|V_h^1-u(t_1)\|$ and the contribution from $\eta_4^{n-1/2}$.

We see from (5.4) that $\|\rho(t)\|\leqslant Ch^2\|u(t)\|_2$, so

$$\|\rho(t_n)\|\leqslant Ch^2\left(\|u(t_1)\|_2+\int_{t_1}^{t_n}\|u'(t)\|_2\,\mathrm{d}t\right)\quad \text{for } n\geqslant 1. \tag{5.8}$$

To deal with $\eta_4^{j-1/2}$ we note that

$$\sum_{j=1}^{n}\|\eta_4^{j-1/2}\|=\|\rho(t_1)-\rho(t_0)\|+\sum_{j=2}^{n}\left\|\int_{t_{j-1}}^{t_j}\rho'(t)\mathrm{d}t\right\|$$
$$\leqslant\|(R_h-I)[u(t_1)-u_0]\|+\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\|\rho'(t)\|\mathrm{d}t\leqslant Ch^2\|u(t_1)-u_0\|_2+\int_{t_1}^{t_n}\|\rho'(t)\|\mathrm{d}t,$$

and by (5.4) we observe that $\|\rho'(t)\|=\|(R_h-I)u'(t)\|\leqslant Ch^2\|u'(t)\|_2$. Thus

$$\sum_{j=1}^{n}\|\eta_4^{j-1/2}\|\leqslant Ch^2\left(\|u(t_1)-u_0\|_2+\int_{t_1}^{t_n}\|u'(t)\|_2\,\mathrm{d}t\right). \tag{5.9}$$

To estimate $\|V_h^1-u(t_1)\|$ we split it as $V_h^1-u(t_1)=\widetilde{\theta}+\rho(t_1)$ with $\widetilde{\theta}=V_h^1-R_hu(t_1)$. Since

$$(V_h^1-U_h^0)-(u(t_1)-u_0)=(\widetilde{\theta}-\theta^0)+(\rho(t_1)-\rho(t_0))$$

and for $0\leqslant t\leqslant t_1$ we have

$$A(V_h^1,\chi)-A(u(t),\chi)+\int_0^t\beta(t-s)A(\overline{V}_h(s),\chi)\mathrm{d}s-\int_0^t\beta(t-s)A(u(s),\chi)\mathrm{d}s$$
$$=A(\widetilde{\theta},\chi)+A(u(t_1)-u(t),\chi)+\int_0^t\beta(t-s)A(\widetilde{\theta},\chi)\mathrm{d}s+\int_0^t\beta(t-s)A(u(t_1)-u(s),\chi)\mathrm{d}s,$$

we see that

$$(\widetilde{\theta}-\theta^0,\chi)+k_1A(\widetilde{\theta},\chi)+\int_0^{t_1}\int_0^t\beta(t-s)B(\widetilde{\theta},\chi)\mathrm{d}s\,\mathrm{d}t=(\widetilde{\eta}^{1/2},\chi),$$

where $\widetilde{\eta}^{1/2} = \eta^1 + \eta_2^{1/2} + \eta_3^{1/2} + \eta_4^{1/2}$, for the same η^1, $\eta_2^{1/2}$ and $\eta_3^{1/2}$ as in (4.15) and (4.6) and for $\eta_4^{1/2}$ as in (5.6). Following the derivation used in Lemma 3.2 to show the stability of V^1, we have

$$\|\widetilde{\theta}\| \leqslant \|\theta^0\| + 2\sum_{j=1}^{n} \|\widetilde{\eta}^{1/2}\|.$$

Using the obtained estimations of $\|\eta^1\|$, $\|\eta_2^{1/2}\|$, $\|\eta_3^{1/2}\|$ and $\|\eta_4^{1/2}\|$ from (4.16), (4.12), (4.8) and (5.9), respectively, we have

$$\|\widetilde{\theta}\| \leqslant \|\theta^0\| + Lk_1\|U_h^0 - u_0\| + Ch^2\|u(t_1) - u_0\|_2 + C\int_0^{t_1} t(\|u'(t)\| + \|Au'(t)\|)\mathrm{d}t,$$

and hence

$$\begin{aligned}\|V_h^1 - u(t_1)\| &\leqslant \|\widetilde{\theta}\| + \|\rho(t_1)\| \\ &\leqslant \|\theta^0\| + Lk_1\|U_h^0 - u_0\| + Ch^2(\|u(t_1)\|_2 + \|u_0\|_2) + C\int_0^{t_1} t\|Au'(t)\|\mathrm{d}t. \qquad (5.10)\end{aligned}$$

Using the stability of θ^n, combining (5.7)–(5.10), we find that

$$\begin{aligned}\|U_h^n - u(t_n)\| &\leqslant \|\theta^n\| + \|\rho(t_n)\| \\ &\leqslant (1 + Ck_1)\|\theta^0\| + C\left(\sum_{j=1}^{n-1} k_j\|U_h^j - u^j\| + k^2\sum_{j=2}^{n} t_j^{\alpha-2/\gamma}k_j + k^{\gamma(1+\alpha)}\right) \\ &\quad + Ch^2\left(\|u(t_1)\| + \|u_0\|_2 + \int_{t_1}^{t_n} \|u'(t)\|_2\,\mathrm{d}t\right) + C\int_0^{t_1} t\|Au'(t)\|\mathrm{d}t.\end{aligned}$$

TABLE 1 *Errors in* $\|U_h - u\|_{0,\infty}$ *and apparent convergence rates for three different mesh gradings when* $\alpha = 0.2$

N	$\gamma = 1$		$\gamma = 5/3$		$\gamma = 2$	
60	2.13×10^{-3}		7.94×10^{-4}		8.18×10^{-4}	
120	8.78×10^{-4}	1.28	1.98×10^{-4}	1.99	2.05×10^{-4}	1.99
240	3.92×10^{-4}	1.16	4.97×10^{-5}	1.99	5.13×10^{-5}	1.99
480	1.79×10^{-4}	1.13	1.24×10^{-5}	2.00	1.28×10^{-5}	2.00
960	8.12×10^{-5}	1.13	2.89×10^{-6}	2.10	3.00×10^{-6}	2.09

TABLE 2 *Errors in* $\|U_h - u\|_{0,\infty}$ *and apparent convergence rates for three different mesh gradings when* $\alpha = 0.6$

N	$\gamma = 1$		$\gamma = 5/4$		$\gamma = 2$	
60	7.94×10^{-4}		8.22×10^{-4}		9.54×10^{-4}	
120	1.99×10^{-4}	1.99	2.05×10^{-4}	1.99	2.38×10^{-4}	1.99
240	4.99×10^{-5}	1.99	5.14×10^{-5}	1.99	5.97×10^{-5}	1.99
480	1.66×10^{-5}	1.58	1.28×10^{-5}	2.00	1.49×10^{-5}	2.00
960	5.89×10^{-6}	1.49	2.99×10^{-6}	2.09	3.55×10^{-6}	2.06

Therefore, noting that $\theta^0 = U_h^0 - R_h u_0$ and following the steps after (4.18) in Theorem 4.3 yields the desired error bound. □

6. Numerical experiment

In this section we apply our fully discrete extrapolated Crank–Nicolson scheme (5.1)–(5.3) to a concrete problem of the form (1.1) in one space dimension, taking $T = 1$ and $Au = Bu = -\frac{\partial^2 u}{\partial x^2}$ on the interval $\Omega = (0, 1)$ with homogeneous Dirichlet boundary conditions. We employ a time mesh of the form (1.9) with various choices of $\gamma \geqslant 1$. In each case we use a uniform spatial mesh with N subintervals of length $h = 1/N$. Since N is the same for the space and time meshes, this means $h = k$. For the finite-element space S_h we choose the space of continuous, piecewise-linear functions, and for U_h^0 we choose the elliptic projection of the initial data u_0 onto S_h. The data are chosen so that the exact solution is

$$u(x, t) = \sin(\pi x) - t^{1+\alpha} \sin(2\pi x)$$

and the inhomogeneous term is $f(x, t, u) = t^\alpha u^2(x, t) + f_1$. So we put $u(x, 0) = \sin(\pi x)$ and

$$f_1 = \left(\pi^2 \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)}\right) - t^\alpha \sin(\pi x) + 2t^{1+2\alpha} \sin(2\pi x)\right) \sin(\pi x)$$
$$- t^\alpha \left((1+\alpha) + 4\pi^2 t \left(1 + t^\alpha \frac{\Gamma(2+\alpha)}{\Gamma(2+2\alpha)}\right) + t^{2+2\alpha} \sin(2\pi x)\right) \sin(2\pi x).$$

For $t > 0$ it is clear that

$$\|u(t)\|_2 + t\|u'(t)\|_2 \leqslant M, \quad \|Au'(t)\| + t\|Au''(t)\| \leqslant Mt^\alpha \quad \text{and} \quad t\|D_{tt} f(t, u(t))\| \leqslant Mt^{\alpha-1}.$$

So the assumptions of Theorem 5.1 are satisfied. Tables 1 and 2 show the error $\|U_h - u\|_{0,\infty} = \max_{0\leqslant n\leqslant N} \|U_h^n - u(t_n)\|_{L^2(0,1)}$. The observed convergence rate is almost $\mathcal{O}(k^{1+\alpha} + h^2 |\log k|) = \mathcal{O}(N^{-1-\alpha})$ for uniform time steps, i.e., for $\gamma = 1$. This rate improves to $\mathcal{O}(N^{-2\log N})$ if $\gamma = 2/(\alpha + 1)$. For $\gamma = 2$ the mesh is over-graded: theoretically, we have a slightly better convergence rate $\mathcal{O}(N^{-2})$, but in practice the errors are larger when $\gamma = 2/(\alpha + 1)$.

References

ADOLFSSON, K., ENELUND, M. & LARSSON, S. (2003) Adaptive discretization of an integro-differential equation with a weakly singular convolation kernel. *Comput. Methods Appl. Mech. Eng.*, **192**, 5285–5304.

BRUNNER, H., PEDAS, A. & VAINIKKO, G. (2001) Piecewise polynomial collocation for Volterra integro-differential equations. *SIAM J. Numer. Anal.*, **39**, 957–982.

CANNON, J. R. & LIN, Y. P. (1988) Nonclassical H^1 projection and Galerkin methods for nonlinear parabolic integro-differential equations. *Calcolo*, **25**, 187–201.

CANNON, J. R. & LIN, Y. P. (1990) A priori L^2 error estimates for finite-element methods for nonlinear diffusion equations with memory. *SIAM J. Numer. Anal.*, **27**, 595–607.

CHANDLER, G. A. & GRAHAM, I. G. (1988) Product integration-collocation methods for noncompact integral operator equations. *Math. Comput.*, **50**, 125–138.

CHEN, C., THOME, V. & WAHLBIN, L. (1992) Finite element approximation of a parabolic integro-differential equation with a weakly singular kernel. *Math. Comput.*, **58**, 587–602.

CUESTA, E., LUBICH, C. H. & PALENCIA, C. (2006) Convolution quadrature time discretization of fractional diffusion-wave equations. *Math. Comput.*, **75**, 673–696.

DUPONT, T., FAIRWEATHER, G. & JOHNSON, G. (1974) Three level Galerkin methods for parabolic equations. *SIAM J. Numer. Anal.*, **11**, 392–409.

Marcel Dekker.

FAIRWEATHER, G. (1994) Spline collocation methods for a class of hyperbolic partial integro-differential equations. *SIAM J. Numer. Anal.*, **31**, 444–460.

FRIEDMAN, A. & SHINBROT, M. (1967) Volterra integral equations in Banach space. *Trans. Am. Math. Soc.*, **126**, 131–179.

HEARD, M. L. (1982) An abstract parabolic Volterra integrodifferential equation. *SIAM J. Math. Anal.*, **13**, 81–105.

LARSSON, S., THOḾEE, V. & WAHLBIN, L. (1998) Numerical solution of parabolic integro-differential equations by the discontinuous Galerkin method. *Math. Comput.*, **67**, 45–71.

LIN, Y., THOMÉE, V. & WAHLBIN, L. (1991) Ritz–Volterra projections to finite element spaces and applications to integro-differential and related equations. *SIAM J. Numer. Anal.*, **28**, 1047–1070.

LÓPEZ-FERNÁNDEZ, M. & PALENCIA, C. (2004) On the numerical inversion of the Laplace transform of certain holomorphic functions. *Appl. Numer. Math.*, **51**, 289–303.

LUBICH, C. H. (1988) Convolution quadrature and discretized operational calculus. I, II. *Numer. Math.*, **52**, 129–145.

MCLEAN, W. & MUSTAPHA, K. (2007) A second-order accurate numerical method for a fractional wave equation. *Numer. Math.*, **105**, 481–510.

MCLEAN, W., THOMÉE, V. & WAHLBIN, L. B. (1996) Discretization with variable time steps of an evolution equation with a positive-type memory term. *J. Comput. Appl. Math.*, **69**, 49–69.

PAZY, A. (1983) *Semigroups of Linear Operators and Applications to Partial Differential Equations*. New York: Springer.

RENARDY, M., HRUSA, W. J. & NOBEL, J. A. (1987) Mathematical problems in viscoelasticity. Pitman Monographs and Surveys in Pure and Applied Mathematics. *Longman Science and Technical*, **35**, New York: Wiley. pp. 134–144.

SLOAN, I. H. & THOMÉE, V. (1986) Time discretization of an integro-differential equation of parabolic type. *SIAM. J. Numer. Anal.*, **23**, 1052–1061.

TE RIELE, H. J. J. (1982) Collocation methods for weakly singular second-kind Volterra integral equations with nonsmooth solution. *IMA J. Numer. Anal.*, **2**, 437–449.

THOMÉE, V. & ZHANG, N. Y. (1989) Error estimates for semidiscrete finite element methods for parabolic integro-differential equations. *Math. Comput.*, **53**, 121–139.

YANIK, E. G. & FAIRWEATHER, G. (1988) Finite element methods for parabolic and hyperbolic partial integro-differential equations. *Nonlinear Anal.*, **12**, 785–809.

ZHANG, N. Y. (1993) On fully discrete Galerkin approximations for partial integro-differential equations of parabolic type. *Math. Comput.*, **60**, 133–166.