IMA Journal of Numerical Analysis (2004) **24**, 157–177

A Petrov–Galerkin method with quadrature for elliptic boundary value problems

B. BIALECKI†
Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401, USA

M. GANESH‡
School of Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

AND

K. MUSTAPHA§
School of Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

[Received on 4 July 2002; revised on 12 May 2003]

We propose and analyse a fully discrete Petrov–Galerkin method with quadrature, for solving second-order, variable coefficient, elliptic boundary value problems on rectangular domains. In our scheme, the trial space consists of C^2 splines of degree $r \geqslant 3$, the test space consists of C^0 splines of degree $r - 2$, and we use composite $(r - 1)$-point Gauss quadrature. We show existence and uniqueness of the approximate solution and establish optimal order error bounds in H^2, H^1 and L^2 norms.

Keywords: elliptic boundary value problems; Petrov–Galerkin method; splines; Gauss quadrature.

1. Introduction

We consider the elliptic boundary value problem

$$Lu = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega = (0, 1) \times (0, 1)$,

$$Lu = -a_1(x)u_{xx} - a_2(y)u_{yy} + b_1(x, y)u_x + b_2(x, y)u_y + c(x, y)u, \tag{1.2}$$

f, b_1, b_2, c are defined on Ω, and a_1, a_2 defined on $(0, 1)$ are such that

$$a_1(x),\ a_2(y) \geqslant a_* > 0, \quad x, y \in (0, 1).$$

In this paper, we solve (1.1) using a fully discrete Petrov–Galerkin method with quadrature. The trial space for our scheme consists of C^2 splines of degree $r \geqslant 3$, the

†Email: bbialeck@mines.edu
‡Email: ganesh@maths.unsw.edu.au
§Email: kassim@maths.unsw.edu.au

test space consists of C^0 splines of degree $r-2$, and we use composite $(r-1)$-point Gauss quadrature. Our work is motivated by the fully discrete schemes: C^2 *nodal spline collocation* (NSC) and C^1 *orthogonal spline collocation* (OSC). OSC has been analysed in many papers over the last three decades, see for example the recent OSC survey paper Bialecki & Fairweather (2001) and references therein. In recent years, NSC has been proposed (see Bottcher & Strayer, 1990, 1993; Hadjidimos *et al.*, 1999; Houstis *et al.*, 1988; Kegley *et al.*, 1996; Tsompanopoulou & Vavalis, 1998; Umar, 1990; Umar *et al.*, 1991) as an alternative to OSC. However, in contrast to the quadrature Petrov–Galerkin scheme of this paper, current analysis and applicability of NSC are restricted to uniform partitions and splines of degree 3.

As in OSC with splines of arbitrary degree $r \geqslant 3$, the quadrature Petrov–Galerkin scheme requires evaluation of given functions at Gauss points. However, C^2 smoothness of trial functions in our scheme leads to marked reduction in the dimension of the resulting linear system. For example, in the cubic spline case, the dimension of the quadrature Petrov–Galerkin linear system is about *one-fourth* of that for OSC. (For further relations between these methods, we refer to Sloan *et al.*, 1993.) Similarly, in comparison with quadrature C^0 finite-element Galerkin methods, C^2 smoothness of our approximate solutions leads to significantly smaller linear systems. While the quadrature finite-element Galerkin method with the same trial space as ours leads to the linear system of the same dimension, the matrix arising in our approach has fewer non-zero coefficients and the computation of each of these coefficients requires fewer arithmetic operations (Fairweather, 1978, p. 57)

Our scheme is based on a quadrature version of the semi-discrete Petrov–Galerkin method of Douglas *et al.* (1977). (In Douglas *et al.*, 1977, the Petrov–Galerkin method is called the adjoint local H^{-1} Galerkin procedure.) Like OSC, NSC and the Petrov–Galerkin method of Douglas *et al.* (1977), our scheme for two-dimensional problems is restricted, at this point, to rectangular domains. For one-dimensional problems and arbitrary partitions, the quadrature Petrov–Galerkin method was studied in Sloan *et al.* (1993) and Grigorieff & Sloan (1996). The scheme in Sloan *et al.* (1993) and Grigorieff & Sloan (1996) is obtained by applying quadrature to the Petrov–Galerkin method of de Boor (1966) for solving one-dimensional problems.

To describe our scheme, for an integer $N_x \geqslant 1$, let $\Pi_x = \{x_k\}_{k=0}^{N_x}$ be a partition of $[0, 1]$ in the x-direction such that

$$0 = x_0 < x_1 < \cdots < x_{N_x} = 1.$$

Let $\Pi_y = \{y_l\}_{l=0}^{N_y}$ be the corresponding partition of $[0, 1]$ in the y-direction, and let

$$h_x^k = x_k - x_{k-1}, \quad k = 1, \dots, N_x, \quad h_y^l = y_l - y_{l-1}, \quad l = 1, \dots, N_y.$$

For $r \geqslant 3$, let

$$S_x = \{v \in C^2[0, 1] : v|_{[x_{k-1}, x_k]} \in P_r, k = 1, \dots, N_x, v(0) = v(1) = 0\},$$

$$T_x = \{v \in C[0, 1] : v|_{[x_{k-1}, x_k]} \in P_{r-2}, k = 1, \dots, N_x\},$$

where P_n is the space of all polynomials of degree at most n. Let S_y and T_y be the corresponding spline spaces in the y-direction, and let

$$S = S_x \otimes S_y, \quad T = T_x \otimes T_y.$$

Let $\{w_j\}_{j=1}^{r-1}$ and $\{\xi_j\}_{j=1}^{r-1}$ be respectively the weights and nodes of the $(r-1)$-point Gauss quadrature on [0, 1]. For v and z defined on Ω, we introduce

$$(v, z)_h = \sum_{k=1}^{N_x} \sum_{l=1}^{N_y} h_x^k h_y^l \sum_{m=1}^{r-1} \sum_{n=1}^{r-1} w_m w_n (vz)(x_{k,m}, y_{l,n}), \tag{1.3}$$

where $x_{k,m} = x_{k-1} + h_x^k \xi_m$ and $y_{l,n} = y_{l-1} + h_y^l \xi_n$. Clearly, (1.3) is an approximation to the standard $L^2(\Omega)$ inner product (v, z) obtained using composite $(r-1)$-point Gauss quadrature in the x- and y-direction.

The quadrature Petrov–Galerkin method of this paper for solving (1.1) is: find $U \in S$ such that

$$(LU, v)_h = (f, v)_h, \quad v \in T. \tag{1.4}$$

The Petrov–Galerkin method of Douglas *et al.* (1977) is defined by (1.4) but with the $L^2(\Omega)$ inner product $(\cdot, \cdot)$ in place of $(\cdot, \cdot)_h$. In Douglas *et al.* (1977), the operator L is given by

$$Lu = -\nabla \cdot (a(x, y)\nabla u) + b(x, y) \cdot \nabla u + c(x, y)u, \tag{1.5}$$

where $a(x, y)$ is positive on $\overline{\Omega}$, and $b = (b_1, b_2)$. Assuming that Π_x is a quasi-uniform partition of [0, 1] and that $\Pi_y = \Pi_x$, optimal error bounds in Sobolev norms are derived in Douglas *et al.* (1977) for the Petrov–Galerkin method using analysis of the local H^{-1} Galerkin scheme. Although (1.4) is well defined for L of (1.2) with $a_1(x)$ and $a_2(y)$ replaced by $a_1(x, y)$ and $a_2(x, y)$, our analysis (see, in particular, Lemma 3.1) is not applicable to this more general case. However, our analysis is applicable to L of (1.5), after first dividing (1.1) through by $a(x, y)$, which gives an equivalent equation with $a_1(x) = a_2(y) = 1$.

The purpose of the present paper is to establish optimal-order Sobolev norm error bounds for the quadrature Petrov–Galerkin method. More precisely, assuming that the solution u of (1.1) is in $H^{r+3}(\Omega)$ and that (1.4) is solved on a collection of quasi-uniform partitions $\Pi_x \times \Pi_y$, we show that for $k = 0, 1, 2$, the H^k norm of the error $u - U$ is of order $r + 1 - k$. In the case of the corresponding quadrature finite-element Galerkin method, the same optimal convergence orders are obtained for u in $H^{r+1}(\Omega)$ and for a regular collection of partitions. However, as we demonstrate using numerical experiments, it seems that to obtain optimal convergence orders for the quadrature Petrov–Galerkin method it is sufficient to assume that $u \in H^s(\Omega)$ for some s such that $r + 1 \leqslant s < r + 2$. In general, solutions to elliptic problems on a rectangular domain may only be in $H^s(\Omega)$, $2 \leqslant s < 3$. Even in such cases, numerical experiments demonstrate that our method can be used to compute approximate solutions with the convergence order 2 in the L^2 norm. To prove such convergence results, perhaps we may require a different type of analysis than the one used in this work.

The outline of this paper is as follows. In the next section, we give properties of a comparison function required for our analysis and prove a lemma for bounding quadrature errors. In Section 3, we prove optimal order convergence of the quadrature Petrov–Galerkin solutions in the H^2 norm. The L^2 and H^1 convergence analyses are given in Section 4. Numerical experiments in Section 5 confirm theoretical results and demonstrate the applicability of our scheme for solving (1.1).

2. Preliminaries

For a non-negative integer k, the standard norm in the Sobolev space $H^k(\Omega)$ is denoted by $\|\cdot\|_k$. (Note that $H^0(\Omega) = L^2(\Omega)$ and that $\|\cdot\|_0 = \|\cdot\|_{L^2(\Omega)}$.) In what follows, $\|v\|_h = (v, v)_h^{1/2}$. Let Q be the collection of the rectangles $\rho = (x_{k-1}, x_k) \times (y_{l-1}, y_l)$, $k = 1, \ldots, N_x$, $l = 1, \ldots, N_y$. We define the 'broken' L^2-norm $\|v\|_Q$ by

$$\|v\|_Q^2 = \sum_{\rho \in Q} \|v\|_{0,\rho}^2 ,$$

where $\|\cdot\|_{k,\rho} = \|\cdot\|_{H^k(\rho)}$.

In our convergence analysis, we assume that (1.4) is solved on a quasi-uniform collection of partitions $\Pi_x \times \Pi_y$ corresponding to a sequence of values (N_x, N_y). Hence, throughout the paper, C is a generic positive constant which may depend on r, but which is independent of h, where

$$h = \max(h_x, h_y), \quad h_x = \max_{1 \leqslant k \leqslant N_x} h_x^k, \quad h_y = \max_{1 \leqslant l \leqslant N_y} h_y^l.$$

Throughout the paper we make the following two additional assumptions about the operator L of (1.2). To bound quadrature errors, we assume that a_1, a_2 and that b_1, b_2, c are in $C^{r+1}[0,1]$ and $C^{r+1}(\overline{\Omega})$, respectively. Moreover, with L^* being the formal adjoint of L, we assume that for any $v \in L^2(\Omega)$, there exists $\phi \in H^2(\Omega)$ such that

$$L^*\phi = v \text{ in } \Omega, \quad \phi = 0 \text{ on } \Omega, \tag{2.1}$$

$$\|\phi\|_2 \leqslant C\,\|v\|_0 . \tag{2.2}$$

Since $\phi \in H^2(\Omega)$, it follows from Fairweather (1978, Theorem 3.3) that there exists a bilinear spline $\tilde{\phi}$ such that

$$\|\phi - \tilde{\phi}\|_0 \leqslant Ch^2\,\|\phi\|_2 , \quad \|\phi - \tilde{\phi}\|_1 \leqslant Ch\,\|\phi\|_2 . \tag{2.3}$$

The triangle inequality, (2.3) and (2.2) give

$$\|\tilde{\phi}\|_1 \leqslant \|\phi - \tilde{\phi}\|_1 + \|\phi\|_1 \leqslant C\|\phi\|_2 \leqslant C\,\|v\|_0 . \tag{2.4}$$

The following lemma follows easily from Fairweather (1978, Theorem 3.3).

LEMMA 2.1 If $u \in H^{r+2}(\Omega)$, then there exists a spline $z \in H^{r-1}(\Omega)$, of degree at most $r-1$ in each variable, such that

$$\|u_{xx} - z\|_k \leqslant Ch^{r-k}\,\|u\|_{r+2} , \quad k = 0, \ldots, r-1. \tag{2.5}$$

Next, we list properties of a function W which plays a role of the comparison function in our convergence analysis.

LEMMA 2.2 If $u \in H^{r+2}(\Omega)$ and $u = 0$ on $\partial\Omega$, then there exists $W \in S$ such that

$$\left\| \frac{\partial^s}{\partial y^s}(u - W)_x \right\|_Q + \left\| \frac{\partial^s}{\partial x^s}(u - W)_y \right\|_Q \leqslant Ch^{r+1-s} \, \|u\|_{r+2} , \quad 1 \leqslant s \leqslant r, \tag{2.6}$$

$$\left\| \frac{\partial^s}{\partial y^s}(u - W) \right\|_Q + \left\| \frac{\partial^s}{\partial x^s}(u - W) \right\|_Q \leqslant Ch^{r+1-s} \, \|u\|_{r+2} , \quad 1 \leqslant s \leqslant r, \tag{2.7}$$

$$\left\| \frac{\partial^r}{\partial y^r} W_{xx} \right\|_Q + \left\| \frac{\partial^r}{\partial x^r} W_{yy} \right\|_Q \leqslant C \, \|u\|_{r+2} . \tag{2.8}$$

Proof. It follows in Bramble *et al.* (1989, Lemma 2.2) or from Douglas *et al.* (1977, Theorem 3) that there exists $W \in S$ such that

$$\|(u - W)_{xy}\|_0 \leqslant Ch^r \|u\|_{r+2}. \tag{2.9}$$

Fairweather (1978, Theorem 3.3) implies the existence of a spline $z \in H^{r+1}(\Omega)$ such that

$$\|u - z\|_k \leqslant Ch^{r+2-k} \, \|u\|_{r+2} , \quad k = 0, \ldots , r + 1. \tag{2.10}$$

Using the inverse and triangle inequalities, (2.10), and (2.9), we have, for $1 \leqslant s \leqslant r$,

$$\left\| \frac{\partial^s}{\partial y^s}(z - W)_x \right\|_Q \leqslant Ch^{1-s} \, \left\|(z - W)_{xy}\right\|_0$$

$$\leqslant Ch^{1-s} \left\{ \left\|(u - z)_{xy}\right\|_0 + \left\|(u - W)_{xy}\right\|_0 \right\} \leqslant Ch^{r+1-s} \|u\|_{r+2}. \tag{2.11}$$

The triangle inequality, (2.10) and (2.11) give

$$\left\| \frac{\partial^s}{\partial y^s}(u - W)_x \right\|_Q \leqslant \left\| \frac{\partial^s}{\partial y^s}(u - z)_x \right\|_0 + \left\| \frac{\partial^s}{\partial y^s}(z - W)_x \right\|_Q \leqslant Ch^{r+1-s} \|u\|_{r+2}.$$

This and a similar inequality for $\left\| \frac{\partial^s}{\partial x^s}(u - W)_y \right\|_Q$ imply (2.6).

Using $(u - W)_y(\alpha, y) = 0$, $y \in [0, 1]$, $\alpha = 0, 1$, and the Poincaré inequality, we have

$$\|(u - W)_y\|_0^2 = \int_0^1 \int_0^1 (u - W)_y^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant C \int_0^1 \int_0^1 (u - W)_{xy}^2 \, \mathrm{d}x \, \mathrm{d}y = C \|(u - W)_{xy}\|_0^2,$$

which along with (2.9) gives

$$\|(u - W)_y\|_0 \leqslant Ch^r \|u\|_{r+2}. \tag{2.12}$$

Using the inverse and triangle inequalities, (2.10) and (2.12), we have, for $1 \leqslant s \leqslant r$,

$$\left\| \frac{\partial^s}{\partial y^s}(z - W) \right\|_Q \leqslant Ch^{1-s} \, \left\|(z - W)_y\right\|_0$$

$$\leqslant Ch^{1-s} \left\{ \left\|(u - z)_y\right\|_0 + \left\|(u - W)_y\right\|_0 \right\} \leqslant Ch^{r+1-s} \|u\|_{r+2}. \tag{2.13}$$

The triangle inequality, (2.10) and (2.13) give

$$\left\|\frac{\partial^s}{\partial y^s}(u-W)\right\|_Q \leqslant \left\|\frac{\partial^s}{\partial y^s}(u-z)\right\|_0 + \left\|\frac{\partial^s}{\partial y^s}(z-W)\right\|_Q \leqslant Ch^{r+1-s}\|u\|_{r+2}.$$

This and a similar inequality for $\left\|\frac{\partial^s}{\partial x^s}(u-W)\right\|_Q$ imply (2.7).

Using the inverse and triangle inequalities, (2.10) and (2.9), we have

$$\begin{aligned} \|(z-W)_{xxy}\|_0 &\leqslant Ch^{-1}\|(z-W)_{xy}\|_0 \\ &\leqslant Ch^{-1}\left\{\|(u-z)_{xy}\|_0 + \|(u-W)_{xy}\|_0\right\} \leqslant Ch^{r-1}\|u\|_{r+2}. \end{aligned} \tag{2.14}$$

The triangle inequality, (2.10) and (2.14) give

$$\|(u-W)_{xxy}\|_0 \leqslant \|(u-z)_{xxy}\|_0 + \|(z-W)_{xxy}\|_0 \leqslant Ch^{r-1}\|u\|_{r+2}. \tag{2.15}$$

Let, in the remainder of this proof, z be a spline of Lemma 2.1. Using the fact that z is of degree at most $r-1$ in each variable, the inverse and triangle inequalities, (2.5) and (2.15), we have

$$\begin{aligned} \left\|\frac{\partial^r}{\partial y^r}W_{xx}\right\|_Q &= \left\|\frac{\partial^r}{\partial y^r}(z-W_{xx})\right\|_Q \leqslant Ch^{1-r}\left\|(z-W_{xx})_y\right\|_0 \\ &\leqslant Ch^{1-r}\left\{\left\|(u_{xx}-z)_y\right\|_0 + \left\|(u-W)_{xxy}\right\|_0\right\} \leqslant C\|u\|_{r+2}. \end{aligned} \tag{2.16}$$

This and a similar inequality for $\left\|\frac{\partial^r}{\partial x^r}W_{yy}\right\|_Q$ imply (2.8). □

LEMMA 2.3 We have

$$\|v\|_2 \leqslant C(\left\|v_{xxy}\right\|_0 + \left\|v_{xyy}\right\|_0), \quad v \in H^3(\Omega), \quad v|_{\partial\Omega} = 0.$$

Proof. Since $v_{xx}(x,\alpha) = 0$, $x \in [0,1]$, $\alpha = 0, 1$, the Poincaré inequality gives

$$\|v_{xx}\|_0^2 = \int_0^1\int_0^1 v_{xx}^2(x,y)\,\mathrm{d}y\,\mathrm{d}x \leqslant C\int_0^1\int_0^1 v_{xxy}^2(x,y)\,\mathrm{d}y\,\mathrm{d}x = C\|v_{xxy}\|_0^2. \tag{2.17}$$

Using $v(\alpha,y) = 0$, $\alpha = 0, 1$, $y \in [0,1]$, integration by parts and the Cauchy–Schwarz inequality, we have

$$\|v_x\|_0^2 = \int_0^1\int_0^1 (v_x v_x)(x,y)\,\mathrm{d}x\,\mathrm{d}y = -(v_{xx}, v) \leqslant \|v_{xx}\|_0\,\|v\|_0\,, \tag{2.18}$$

which along with $\|v\|_0 \leqslant C\|v_x\|_0$ gives

$$\|v_x\|_0 \leqslant \|v_{xx}\|_0\,. \tag{2.19}$$

Using $v_x(x,\alpha) = 0$, $x \in [0,1]$, $\alpha = 0, 1$, integration by parts and the Cauchy–Schwarz inequality, we have

$$\left\|v_{xy}\right\|_0^2 = \int_0^1\int_0^1 (v_{xy}v_{xy})(x,y)\,\mathrm{d}y\,\mathrm{d}x = -(v_{xyy}, v_x) \leqslant \left\|v_{xyy}\right\|_0\|v_x\|_0\,, \tag{2.20}$$

which along with (2.19) and (2.17) gives

$$\|v_{xy}\|_0^2 \leqslant C\|v_{xyy}\|_0\|v_{xxy}\|_0 \leqslant C(\|v_{xxy}\|_0 + \|v_{xyy}\|_0)^2. \tag{2.21}$$

The desired inequality follows from $\|v\|_0 \leqslant C\|v_x\|_0$, (2.19) and a similar bound on $\|v_y\|_0$, (2.17) and a similar bound on $\|v_{yy}\|_0^2$, and (2.21). □

LEMMA 2.4 We have $\|v\|_h \leqslant C\|v\|_0$ for any spline v.

Proof. Using (1.3) and the inverse inequality $\|v\|_{L^\infty(\rho)} \leqslant Ch^{-1}\|v\|_{0,\rho}$, we have

$$\|v\|_h^2 \leqslant Ch^2 \sum_{\rho\in Q} \|v\|_{L^\infty(\rho)}^2 \leqslant Ch^2 \sum_{\rho\in Q} h^{-2}\|v\|_{0,\rho}^2 = C\|v\|_0^2.$$ □

LEMMA 2.5 If $u \in H^{r+2}(\Omega)$, $u = 0$ on $\partial\Omega$, and W is that of Lemma 2.2, then

$$\|L(u - W)\|_h \leqslant Ch^{r-1}\|u\|_{r+2}.$$

Proof. It follows from Bialecki (1998, Lemmas 2.3 and 2.5) that there exists a spline $z \in H^2(\Omega)$ such that

$$\left\|\frac{\partial^{i+j}}{\partial x^i \partial y^j}(u - z)\right\|_h \leqslant Ch^{r+1-i-j}\|u\|_{r+1}, \quad 0 \leqslant i + j \leqslant 2, \tag{2.22}$$

and

$$\|u - z\|_2 \leqslant Ch^{r-1}\|u\|_{r+1}. \tag{2.23}$$

Using Lemma 2.4, the triangle inequality, (2.23), Lemma 2.3, and (2.6), we obtain

$$\begin{aligned}\sum_{0\leqslant i+j\leqslant 2} \left\|\frac{\partial^{i+j}}{\partial x^i \partial y^j}(z - W)\right\|_h &\leqslant C\|z - W\|_2 \leqslant C(\|u - z\|_2 + \|u - W\|_2)\\ &\leqslant Ch^{r-1}\|u\|_{r+2}.\end{aligned} \tag{2.24}$$

Hence (1.2), the triangle inequality, (2.22) and (2.24) give

$$\begin{aligned}\|L(u - W)\|_h &\leqslant C \sum_{0\leqslant i+j\leqslant 2} \left\|\frac{\partial^{i+j}}{\partial x^i \partial y^j}(u - W)\right\|_h\\ &\leqslant C \sum_{0\leqslant i+j\leqslant 2} \left\|\frac{\partial^{i+j}}{\partial x^i \partial y^j}(u - z)\right\|_h + C \sum_{0\leqslant i+j\leqslant 2} \left\|\frac{\partial^{i+j}}{\partial x^i \partial y^j}(z - W)\right\|_h \leqslant Ch^{r-1}\|u\|_{r+2}.\end{aligned}$$ □

The next lemma will be used for bounding quadrature errors.

LEMMA 2.6 Let integers s and q be such that $s \geqslant 2$, $q \geqslant 0$ and $s+q \leqslant 2r-2$. Then there exists a positive constant C independent of $\rho = (x_{k-1}, x_k) \times (y_{l-1}, y_l)$, $k = 1, \dots, N_x$, $l = 1, \dots, N_y$, such that for any $g \in H^s(\rho)$ and $v \in P_q \otimes P_q$,

$$\begin{aligned}&\left|\int_\rho gv \,\mathrm{d}x\,\mathrm{d}y - h_x^k h_y^l \sum_{m=1}^{r-1}\sum_{n=1}^{r-1} w_m w_n (gv)(x_{k,m}, y_{l,m})\right|\\ &\qquad \leqslant Ch^s \left(\left\|\frac{\partial^s g}{\partial x^s}\right\|_{0,\rho} + \left\|\frac{\partial^s g}{\partial y^s}\right\|_{0,\rho}\right)\|v\|_{0,\rho}.\end{aligned}$$

Proof. For $\hat{x}, \hat{y} \in \Omega$, let

$$\hat{g}(\hat{x}, \hat{y}) = g(x_{k-1} + h_x^k \hat{x}, y_{l-1} + h_y^l \hat{y}), \qquad \hat{v}(\hat{x}, \hat{y}) = v(x_{k-1} + h_x^k \hat{x}, y_{l-1} + h_y^l \hat{y}).$$

Then

$$\int_\rho gv \, \mathrm{d}x \, \mathrm{d}y - h_x^k h_y^l \sum_{m=1}^{r-1} \sum_{n=1}^{r-1} w_m w_n (gv)(x_{k,m}, y_{l,m}) = h_x^k h_y^l F_{\hat{v}}(\hat{g}), \tag{2.25}$$

where

$$F_{\hat{v}}(\hat{g}) = \int_\Omega \hat{g}\hat{v} \, \mathrm{d}\hat{x} \, \mathrm{d}\hat{y} - \sum_{m=1}^{r-1} \sum_{n=1}^{r-1} w_m w_n (\hat{g}\hat{v})(\xi_m, \xi_m).$$

Using the Sobolev embedding theorem and equivalence of norms in a finite-dimensional space, we have

$$|F_{\hat{v}}(\hat{g})| \leqslant C \|\hat{g}\|_{C(\overline{\Omega})} \|\hat{v}\|_{C(\overline{\Omega})} \leqslant C \|\hat{g}\|_s \|\hat{v}\|_0.$$

Hence, for a given $\hat{v} \in P_q \times P_q$, $F_{\hat{v}}$ is a bounded linear functional on $H^s(\rho)$ with the norm $\leqslant C\|\hat{v}\|_0$. Moreover, $F_{\hat{v}}$ vanishes on the space $P_{s-1} \otimes P_{s-1}$. Thus, by the Bramble–Hilbert lemma (see Bramble & Hilbert, 1971, Theorem 2),

$$|F_{\hat{v}}(\hat{g})| \leqslant C \left(\left\| \frac{\partial^s \hat{g}}{\partial \hat{x}^s} \right\|_0 + \left\| \frac{\partial^s \hat{g}}{\partial \hat{y}^s} \right\|_0 \right) \|\hat{v}\|_0. \tag{2.26}$$

Since

$$\left\| \frac{\partial^s \hat{g}}{\partial \hat{x}^s} \right\|_0 = (h_x^k)^s (h_x^k h_y^l)^{-1/2} \left\| \frac{\partial^s g}{\partial x^s} \right\|_{0,\rho}, \qquad \left\| \frac{\partial^s \hat{g}}{\partial \hat{y}^s} \right\|_0 = (h_y^l)^s (h_x^k h_y^l)^{-1/2} \left\| \frac{\partial^s g}{\partial y^s} \right\|_{0,\rho},$$

and $\|\hat{v}\|_0 = (h_x^k h_y^l)^{-1/2} \|v\|_{0,\rho}$, the desired inequality follows from (2.25) and (2.26). □

3. H^2 convergence analysis

In this and the remaining sections of the paper, u and U are the solutions of (1.1) and (1.4), respectively, and W is that of Lemma 2.2. We prove the main result of this section in Theorem 3.4, after establishing stability-type results in Lemmas 3.1–3.3.

LEMMA 3.1 If h is sufficiently small, then

$$\|v_{xxy}\|_0^2 + \|v_{xyy}\|_0^2 \leqslant C \left\{ (Lv, v_{xxyy})_h + \|v\|_0^2 \right\}, \quad v \in S.$$

Proof. It follows from (1.3), Douglas & Dupont (1974, Lemma 3.3) and the exactness property of the quadrature that

$$
\begin{aligned}
-(a_1 v_{xx}, v_{xxyy})_h &= \sum_{k=1}^{N_x} h_x^k \sum_{m=1}^{r-1} w_m a_1(x_{k,m}) \sum_{l=1}^{N_y} h_y^l \sum_{n=1}^{r-1} w_n (-v_{xx} v_{xxyy})(x_{k,m}, y_{l,n}) \\
&\geqslant \sum_{k=1}^{N_x} h_x^k \sum_{m=1}^{r-1} w_m a_1(x_{k,m}) \int_0^1 v_{xxy}^2(x_{k,m}, y)\,\mathrm{d}y \\
&\geqslant C \int_0^1 \sum_{k=1}^{N_x} h_x^k \sum_{m=1}^{r-1} w_m v_{xxy}^2(x_{k,m}, y)\,\mathrm{d}y = C\|v_{xxy}\|_0^2.
\end{aligned}
$$

This and a similar inequality for $-(a_2 v_{yy}, v_{xxyy})_h$ give

$$
-(a_1 v_{xx}, v_{xxyy})_h - (a_2 v_{yy}, v_{xxyy})_h \geqslant C\left(\left\|v_{xxy}\right\|_0^2 + \left\|v_{xyy}\right\|_0^2\right). \tag{3.1}
$$

On the other hand, using (1.2), we have

$$
\begin{aligned}
&- (a_1 v_{xx}, v_{xxyy})_h - (a_2 v_{yy}, v_{xxyy})_h \\
&= (Lv, v_{xxyy})_h - (b_1 v_x, v_{xxyy})_h - (b_2 v_y, v_{xxyy})_h - (cv, v_{xxyy})_h \\
&= (Lv, v_{xxyy})_h - (b_1 v_x, v_{xxyy}) - (b_2 v_y, v_{xxyy}) - (cv, v_{xxyy}) + I_1 + I_2 + I_3,
\end{aligned} \tag{3.2}
$$

where

$$
I_1 = (b_1 v_x, v_{xxyy}) - (b_1 v_x, v_{xxyy})_h, \quad I_2 = (b_2 v_y, v_{xxyy}) - (b_2 v_y, v_{xxyy})_h,
$$

$$
I_3 = (cv, v_{xxyy}) - (cv, v_{xxyy})_h.
$$

Using integration by parts, the Cauchy–Schwarz, triangle and ϵ inequalities, we have

$$
\begin{aligned}
|(b_1 v_x, v_{xxyy})| &= |((b_1 v_x)_y, v_{xxy})| \leqslant C(\|v_x\|_0 + \left\|v_{xy}\right\|_0) \left\|v_{xxy}\right\|_0 \\
&\leqslant \epsilon_1 \left\|v_{xxy}\right\|_0^2 + C(\epsilon_1)(\|v_x\|_0^2 + \left\|v_{xy}\right\|_0^2),
\end{aligned} \tag{3.3}
$$

which, by symmetry with respect to x and y, gives also

$$
|(b_2 v_y, v_{xxyy})| \leqslant \epsilon_2 \left\|v_{xyy}\right\|_0^2 + C(\epsilon_2)(\left\|v_y\right\|_0^2 + \left\|v_{xy}\right\|_0^2). \tag{3.4}
$$

In a similar way, using $\|v\|_0 \leqslant C\|v_y\|_0$, we obtain

$$
|(cv, v_{xxyy})| = |((cv)_y, v_{xxy})| \leqslant C(\|v\|_0 + \left\|v_y\right\|_0) \left\|v_{xxy}\right\|_0 \leqslant \epsilon_3 \left\|v_{xxy}\right\|_0^2 + C(\epsilon_3) \left\|v_y\right\|_0^2. \tag{3.5}
$$

Hence, for sufficiently small ϵ_1, ϵ_2, and ϵ_3, (3.1)–(3.5) give

$$
\left\|v_{xxy}\right\|_0^2 + \left\|v_{xyy}\right\|_0^2 \leqslant C\left\{(Lv, v_{xxyy})_h + \|v_x\|_0^2 + \|v_y\|_0^2 + \|v_{xy}\|_0^2 + I_1 + I_2 + I_3\right\}. \tag{3.6}
$$

It follows from (2.20), (2.18), (2.17) and the ϵ inequality that

$$\left\| v_{xy} \right\|_0^2 \leqslant \epsilon_1 \left\| v_{xyy} \right\|_0^2 + C(\epsilon_1) \left\| v_x \right\|_0^2, \quad \left\| v_x \right\|_0^2 \leqslant \epsilon_2 \left\| v_{xxy} \right\|_0^2 + C(\epsilon_2) \left\| v \right\|_0^2,$$

and hence, by symmetry with respect to x and y, we also have

$$\left\| v_y \right\|_0^2 \leqslant \epsilon_3 \left\| v_{xyy} \right\|_0^2 + C(\epsilon_3) \left\| v \right\|_0^2.$$

Therefore, (3.6) reduces further to

$$\left\| v_{xxy} \right\|_0^2 + \left\| v_{xyy} \right\|_0^2 \leqslant C \left\{ (Lv, v_{xxyy})_h + \left\| v \right\|_0^2 + I_1 + I_2 + I_3 \right\}. \tag{3.7}$$

Since I_1 and I_2 are symmetric with respect to x and y, it remains to bound I_1 and I_3.

Using Lemma 2.6, we have

$$I_1 \leqslant Ch^r \sum_{\rho \in Q} \left\{ \left\| \frac{\partial^r}{\partial x^r}(b_1 v_x) \right\|_{0,\rho} + \left\| \frac{\partial^r}{\partial y^r}(b_1 v_x) \right\|_{0,\rho} \right\} \left\| v_{xxyy} \right\|_{0,\rho}. \tag{3.8}$$

Using Leibnitz's rule, the triangle inequality, the fact that the spline v is of degree at most r in each variable and the inverse inequality, we obtain

$$\left\| \frac{\partial^r}{\partial x^r}(b_1 v_x) \right\|_{0,\rho} \left\| v_{xxyy} \right\|_{0,\rho} \leqslant C \sum_{j=1}^{r} \left\| \frac{\partial^j}{\partial x^j} v \right\|_{0,\rho} \left\| v_{xxyy} \right\|_{0,\rho} \leqslant C h^{1-r} \left\| v \right\|_{2,\rho} \left\| v_{xyy} \right\|_{0,\rho}. \tag{3.9}$$

In a similar way, we have

$$\begin{aligned} &\left\| \frac{\partial^r}{\partial y^r}(b_1 v_x) \right\|_{0,\rho} \left\| v_{xxyy} \right\|_{0,\rho} \leqslant C \sum_{j=0}^{r} \left\| \frac{\partial^j}{\partial y^j} v_x \right\|_{0,\rho} \left\| v_{xxyy} \right\|_{0,\rho} \\ &\leqslant C h^{-1} \left(\left\| v \right\|_{2,\rho} + h^{2-r} \left\| v_{xyy} \right\|_{0,\rho} \right) \left\| v_{xxy} \right\|_{0,\rho} \leqslant C h^{1-r} (\left\| v \right\|_{2,\rho} + \left\| v_{xyy} \right\|_{0,\rho}) \left\| v_{xxy} \right\|_{0,\rho}. \end{aligned} \tag{3.10}$$

Hence (3.8)–(3.10), the Cauchy–Schwarz inequality, and Lemma 2.3 give

$$I_1 \leqslant Ch(\left\| v \right\|_2 + \left\| v_{xyy} \right\|_0)(\left\| v_{xyy} \right\|_0 + \left\| v_{xxy} \right\|_0) \leqslant Ch(\left\| v_{xxy} \right\|_0^2 + \left\| v_{xyy} \right\|_0^2). \tag{3.11}$$

Following the procedure for bounding I_1, we have

$$\begin{aligned} I_3 &\leqslant Ch^r \sum_{\rho \in Q} \left\{ \left\| \frac{\partial^r}{\partial x^r}(cv) \right\|_{0,\rho} + \left\| \frac{\partial^r}{\partial y^r}(cv) \right\|_{0,\rho} \right\} \left\| v_{xxyy} \right\|_{0,\rho} \\ &\leqslant Ch^{r-1} \sum_{\rho \in Q} \left\{ \sum_{j=0}^{r} \left\| \frac{\partial^j}{\partial x^j} v \right\|_{0,\rho} + \sum_{j=0}^{r} \left\| \frac{\partial^j}{\partial y^j} v \right\|_{0,\rho} \right\} \left\| v_{xyy} \right\|_{0,\rho} \\ &\leqslant Ch \left\| v \right\|_{2,\rho} \left\| v_{xyy} \right\|_{0,\rho} \leqslant Ch(\left\| v_{xxy} \right\|_0^2 + \left\| v_{xyy} \right\|_0^2). \end{aligned} \tag{3.12}$$

Finally, for h sufficiently small, the desired result follows from (3.7), (3.11) and a similar bound for I_2, and (3.12). □

LEMMA 3.2 If $u \in H^{r+2}(\Omega)$ and $v = U - W$, then

$$(Lv, v_{xxyy})_h \leqslant Ch^{r-1} \|u\|_{r+2} (\|v_{xxy}\|_0 + \|v_{xyy}\|_0).$$

Proof. Since $v_{xxyy} \in T$, it follows from (1.4) and (1.1) that

$$(Lv, v_{xxyy})_h = (L(U-W), v_{xxyy})_h = (L(u-W), v_{xxyy})_h = (L(u-W), v_{xxyy}) + \{(L(u-W), v_{xxyy})_h - (L(u-W), v_{xxyy})\} \equiv I_1 + I_2. \tag{3.13}$$

Using (1.2), integration by parts, the fact that a_1 and a_2 are respectively functions of x and y only, the Cauchy–Schwarz inequality, $\|u-W\|_0 \leqslant C\|(u-W)_y\|_0$, (2.6) and (2.7), we obtain

$$\begin{aligned} I_1 &= (-a_1(u-W)_{xx} - a_2(u-W)_{yy} + b_1(u-W)_x + b_2(u-W)_y + c(u-W), v_{xxyy}) \\ &= (a_1(u-W)_{xxy}, v_{xxy}) + (a_2(u-W)_{yyx}, v_{xyy}) - ([b_1(u-W)_x]_y, v_{xxy}) \\ &- ([b_2(u-W)_y]_x, v_{xyy}) - ([c(u-W)]_y, v_{xxy}) \leqslant Ch^{r-1}\|u\|_{r+2}(\|v_{xxy}\|_0 + \|v_{xyy}\|_0). \end{aligned} \tag{3.14}$$

On the other hand, using (1.2) and the triangle inequality, we have

$$\begin{aligned} I_2 \leqslant\ & |(a_1(u-W)_{xx}, v_{xxyy}) - (a_1(u-W)_{xx}, v_{xxyy})_h| \\ &+ |(a_2(u-W)_{yy}, v_{xxyy}) - (a_2(u-W)_{yy}, v_{xxyy})_h| \\ &+ |(b_1(u-W)_x, v_{xxyy}) - (b_1(u-W)_x, v_{xxyy})_h| \\ &+ |(b_2(u-W)_y, v_{xxyy}) - (b_2(u-W)_y, v_{xxyy})_h| \\ &+ |(c(u-W), v_{xxyy}) - (c(u-W), v_{xxyy})_h| \equiv \sum_{i=1}^{5} J_i. \end{aligned} \tag{3.15}$$

Since J_1 and J_2, and J_3 and J_4 are symmetric with respect to x and y, it is sufficient to bound J_1, J_3, and J_5 only.

First we bound J_1. Let z be a spline of Lemma 2.1. The triangle inequality gives

$$\begin{aligned} J_1 \leqslant\ & |(a_1(u_{xx} - z), v_{xxyy}) - (a_1(u_{xx} - z), v_{xxyy})_h| \\ &+ |(a_1(W_{xx} - z), v_{xxyy}) - (a_1(W_{xx} - z), v_{xxyy})_h| \equiv K_1 + K_2. \end{aligned} \tag{3.16}$$

Using Lemma 2.6, we have

$$K_1 \leqslant Ch^r \sum_{\rho \in Q} \left\{ \left\| \frac{\partial^r}{\partial x^r}[a_1(u_{xx} - z)] \right\|_{0,\rho} + \left\| \frac{\partial^r}{\partial y^r}[a_1(u_{xx} - z)] \right\|_{0,\rho} \right\} \|v_{xxyy}\|_{0,\rho}. \tag{3.17}$$

Using Leibnitz's rule, the triangle and inverse inequalities, the fact that the spline z is of degree at most $r-1$ in each variable, the Cauchy–Schwarz inequality and (2.5), we obtain

$$\begin{aligned} \sum_{\rho \in Q} \left\| \frac{\partial^r}{\partial x^r}[a_1(u_{xx} - z)] \right\|_{0,\rho} \|v_{xxyy}\|_{0,\rho} &\leqslant Ch^{-1} \sum_{\rho \in Q} \sum_{j=0}^{r} \left\| \frac{\partial^j}{\partial x^j}(u_{xx} - z) \right\|_{0,\rho} \|v_{xyy}\|_{0,\rho} \\ \leqslant Ch^{-1} \left\{ \sum_{j=0}^{r-1} \left\| \frac{\partial^j}{\partial x^j}(u_{xx} - z) \right\|_0 + \|u\|_{r+2} \right\} \|v_{xyy}\|_0 &\leqslant Ch^{-1}\|u\|_{r+2}\|v_{xyy}\|_0. \end{aligned} \tag{3.18}$$

In a similar way, using the fact that a_1 is a function of x only, we obtain

$$\sum_{\rho\in Q}\left\|\frac{\partial^r}{\partial y^r}[a_1(u_{xx}-z)]\right\|_{0,\rho}\|v_{xxyy}\|_{0,\rho}\leqslant Ch^{-1}\sum_{\rho\in Q}\left\|\frac{\partial^r}{\partial y^r}u_{xx}\right\|_{0,\rho}\|v_{xxy}\|_{0,\rho}$$
$$\leqslant Ch^{-1}\|u\|_{r+2}\|v_{xxy}\|_0. \tag{3.19}$$

It follows from (3.17)–(3.19) that

$$K_1\leqslant Ch^{r-1}\|u\|_{r+2}\left(\|v_{xxy}\|_0+\|v_{xyy}\|_0\right). \tag{3.20}$$

To bound K_2, we first use Lemma 2.6 to obtain

$$K_2\leqslant Ch^r\sum_{\rho\in Q}\left\{\left\|\frac{\partial^r}{\partial x^r}[a_1(W_{xx}-z)]\right\|_{0,\rho}+\left\|\frac{\partial^r}{\partial y^r}[a_1(W_{xx}-z)]\right\|_{0,\rho}\right\}\|v_{xxyy}\|_{0,\rho}. \tag{3.21}$$

Using Leibnitz's rule, the triangle and inverse inequalities, the fact that the splines W and z are respectively of degrees at most r and $r-1$ in each variable, the Cauchy–Schwarz and triangle inequalities, (2.7) and (2.5), we have

$$\sum_{\rho\in Q}\left\|\frac{\partial^r}{\partial x^r}[a_1(W_{xx}-z)]\right\|_{0,\rho}\|v_{xxyy}\|_{0,\rho}\leqslant Ch^{-1}\sum_{\rho\in Q}\sum_{j=0}^{r}\left\|\frac{\partial^j}{\partial x^j}(W_{xx}-z)\right\|_{0,\rho}\|v_{xyy}\|_{0,\rho}$$
$$\leqslant Ch^{-1}\left\{\sum_{j=0}^{r-2}\left\|\frac{\partial^j}{\partial x^j}(W_{xx}-z)\right\|_Q+\left\|\frac{\partial^{r-1}}{\partial x^{r-1}}z\right\|_0\right\}\|v_{xyy}\|_0$$
$$\leqslant Ch^{-1}\left\{\sum_{j=2}^{r}\left\|\frac{\partial^j}{\partial x^j}(u-W)\right\|_Q+\sum_{j=0}^{r-1}\left\|\frac{\partial^j}{\partial x^j}(u_{xx}-z)\right\|_0+\|u\|_{r+1}\right\}\|v_{xyy}\|_0$$
$$\leqslant Ch^{-1}\left\{h\|u\|_{r+2}+\|u\|_{r+1}\right\}\|v_{xyy}\|_0\leqslant Ch^{-1}\|u\|_{r+2}\|v_{xyy}\|_0. \tag{3.22}$$

In a similar way, using the fact that a_1 is a function of x only, by (2.8), we have

$$\sum_{\rho\in Q}\left\|\frac{\partial^r}{\partial y^r}[a_1(W_{xx}-z)]\right\|_{0,\rho}\|v_{xxyy}\|_{0,\rho}\leqslant Ch^{-1}\sum_{\rho\in Q}\left\|\frac{\partial^r}{\partial y^r}W_{xx}\right\|_{0,\rho}\|v_{xxy}\|_{0,\rho}$$
$$\leqslant Ch^{-1}\left\|\frac{\partial^r}{\partial y^r}W_{xx}\right\|_Q\|v_{xxy}\|_0\leqslant Ch^{-1}\|u\|_{r+2}\|v_{xxy}\|_0. \tag{3.23}$$

Therefore, it follows from (3.21)–(3.23) that

$$K_2\leqslant Ch^{r-1}\|u\|_{r+2}\left(\|v_{xxy}\|_0+\|v_{xyy}\|_0\right). \tag{3.24}$$

Combining (3.16), (3.20) and (3.24), we obtain

$$J_1\leqslant Ch^{r-1}\|u\|_{r+2}(\|v_{xxy}\|_0+\|v_{xyy}\|_0). \tag{3.25}$$

Next we bound J_3. Using again Lemma 2.6, we have

$$J_3 \leqslant Ch^r \sum_{\rho\in Q} \left\{ \left\| \frac{\partial^r}{\partial x^r}[b_1(u-W)_x] \right\|_{0,\rho} + \left\| \frac{\partial^r}{\partial y^r}[b_1(u-W)_x] \right\|_{0,\rho} \right\} \left\| v_{xxyy} \right\|_{0,\rho} . \tag{3.26}$$

Proceeding as before and using (2.7), we obtain

$$\begin{aligned} \sum_{\rho\in Q} \left\| \frac{\partial^r}{\partial x^r}[b_1(u-W)_x] \right\|_{0,\rho} \left\| v_{xxyy} \right\|_{0,\rho} &\leqslant Ch^{-1} \sum_{\rho\in Q} \sum_{j=1}^{r+1} \left\| \frac{\partial^j}{\partial x^j}(u-W) \right\|_{0,\rho} \|v_{xyy}\|_{0,\rho} \\ \leqslant Ch^{-1} \left\{ \sum_{j=1}^{r} \left\| \frac{\partial^j}{\partial x^j}(u-W) \right\|_Q + \|u\|_{r+1} \right\} \|v_{xyy}\|_0 &\leqslant Ch^{-1} \|u\|_{r+2} \left\| v_{xyy} \right\|_0 . \end{aligned} \tag{3.27}$$

Moreover, by (2.6) and (2.7),

$$\begin{aligned} \sum_{\rho\in Q} \left\| \frac{\partial^r}{\partial y^r}[b_1(u-W)_x] \right\|_{0,\rho} \|v_{xxyy}\|_{0,\rho} &\leqslant Ch^{-1} \sum_{\rho\in Q} \sum_{j=0}^{r} \left\| \frac{\partial^j}{\partial y^j}(u-W)_x \right\|_{0,\rho} \|v_{xxy}\|_{0,\rho} \\ \leqslant Ch^{-1} \sum_{j=0}^{r} \left\| \frac{\partial^j}{\partial y^j}(u-W)_x \right\|_Q \|v_{xxy}\|_0 &\leqslant C \|u\|_{r+2} \left\| v_{xxy} \right\|_0 . \end{aligned} \tag{3.28}$$

Therefore, (3.26)–(3.28) give

$$J_3 \leqslant Ch^{r-1} \|u\|_{r+2} \left(\left\| v_{xxy} \right\|_0 + \left\| v_{xyy} \right\|_0 \right) . \tag{3.29}$$

Following the procedure for bounding J_3, we have

$$J_5 \leqslant Ch^r \sum_{\rho\in T} \left\{ \left\| \frac{\partial^r}{\partial x^r}[c(u-W)] \right\|_{0,\rho} + \left\| \frac{\partial^r}{\partial y^r}[c(u-W)] \right\|_{0,\rho} \right\} \left\| v_{xxyy} \right\|_{0,\rho} , \tag{3.30}$$

and using $\|u-W\|_0 \leqslant C\|(u-W)_x\|_0$ and (2.7), we obtain

$$\begin{aligned} \sum_{\rho\in Q} \left\| \frac{\partial^r}{\partial x^r}[c(u-W)] \right\|_{0,\rho} \|v_{xxyy}\|_{0,\rho} &\leqslant Ch^{-1} \sum_{\rho\in Q} \sum_{j=0}^{r} \left\| \frac{\partial^j}{\partial x^j}(u-W) \right\|_{0,\rho} \|v_{xyy}\|_{0,\rho} \\ \leqslant Ch^{-1} \sum_{j=1}^{r} \left\| \frac{\partial^j}{\partial x^j}(u-W) \right\|_Q \left\| v_{xyy} \right\|_0 &\leqslant C \|u\|_{r+2} \left\| v_{xyy} \right\|_0 . \end{aligned} \tag{3.31}$$

By symmetry with respect to x and y, we also have

$$\sum_{\rho\in Q} \left\| \frac{\partial^r}{\partial y^r}[c(u-W)] \right\|_{0,\rho} \|v_{xxyy}\|_{0,\rho} \leqslant C \|u\|_{r+2} \left\| v_{xxy} \right\|_0 . \tag{3.32}$$

Therefore, (3.30)–(3.32) give

$$J_5 \leqslant Ch^r \|u\|_{r+2} \left(\left\| v_{xxy} \right\|_0 + \left\| v_{xyy} \right\|_0 \right) . \tag{3.33}$$

Finally, the desired result follows from (3.13)–(3.15), (3.25) and a similar bound for J_2, (3.29) and a similar bound for J_4, and (3.33). □

LEMMA 3.3 If $u \in H^{r+2}(\Omega)$ and $v = U - W$, then

$$\|v\|_0 \leqslant C\left\{h\left(\|v_{xxy}\|_0 + \|v_{xyy}\|_0\right) + h^{r-1}\|u\|_{r+2}\right\}.$$

Proof. With $v = U - W$, let $\phi \in H^2(\Omega)$ be that of (2.1) and (2.2), and let a bilinear spline $\tilde{\phi}$ be that of (2.3) and (2.4). Using (2.1), we have

$$\begin{aligned}\|v\|_0^2 &= (v, v) = (L^*\phi, v) = (\phi, Lv)\\ &= (\phi - \tilde{\phi}, Lv) + \left\{(\tilde{\phi}, Lv) - (\tilde{\phi}, Lv)_h\right\} + (\tilde{\phi}, Lv)_h \equiv I_1 + I_2 + I_3.\end{aligned} \tag{3.34}$$

Using the Cauchy–Schwarz inequality, (2.3), (1.2), (2.2) and Lemma 2.3, we have

$$I_1 \leqslant \|\phi - \tilde{\phi}\|_0\|Lv\|_0 \leqslant Ch^2\|\phi\|_2\|v\|_2 \leqslant Ch^2\|v\|_0\left(\|v_{xxy}\|_0 + \|v_{xyy}\|_0\right). \tag{3.35}$$

Using Lemma 2.6, we have

$$I_2 \leqslant Ch^{r-1}\sum_{\rho\in Q}\left\{\left\|\frac{\partial^{r-1}}{\partial x^{r-1}}(Lv)\right\|_{0,\rho} + \left\|\frac{\partial^{r-1}}{\partial y^{r-1}}(Lv)\right\|_{0,\rho}\right\}\left\|\tilde{\phi}\right\|_{0,\rho}. \tag{3.36}$$

Using (1.2), the triangle inequality, Leibnitz's rule, the fact that the spline v is of degree at most r in each variable, the fact that a_2 is a function of y only and the inverse inequality, we obtain

$$\begin{aligned}\left\|\frac{\partial^{r-1}}{\partial x^{r-1}}(Lv)\right\|_{0,\rho} &\leqslant C\left\{\left\|\frac{\partial^{r-1}}{\partial x^{r-1}}(a_1v_{xx})\right\|_{0,\rho} + \left\|\frac{\partial^{r-1}}{\partial x^{r-1}}(a_2v_{yy})\right\|_{0,\rho}\right.\\ &\quad\left.+\left\|\frac{\partial^{r-1}}{\partial x^{r-1}}(b_1v_x)\right\|_{0,\rho} + \left\|\frac{\partial^{r-1}}{\partial x^{r-1}}(b_2v_y)\right\|_{0,\rho} + \left\|\frac{\partial^{r-1}}{\partial x^{r-1}}(cv)\right\|_{0,\rho}\right\}\\ &\leqslant C\left\{\sum_{j=0}^{r}\left\|\frac{\partial^j}{\partial x^j}v\right\|_{0,\rho} + \left\|\frac{\partial^{r-1}}{\partial x^{r-1}}v_{yy}\right\|_{0,\rho} + \sum_{j=0}^{r-1}\left\|\frac{\partial^j}{\partial x^j}v_y\right\|_{0,\rho}\right\}\\ &\leqslant Ch^{2-r}\left\{\|v\|_{2,\rho} + \|v_{xyy}\|_{0,\rho}\right\}.\end{aligned} \tag{3.37}$$

Hence, using (3.37), the Cauchy–Schwarz inequality, (2.4) and Lemma 2.3, we have

$$\begin{aligned}\sum_{\rho\in Q}\left\|\frac{\partial^{r-1}}{\partial x^{r-1}}(Lv)\right\|_{0,\rho}\left\|\tilde{\phi}\right\|_{0,\rho} &\leqslant Ch^{2-r}\|\tilde{\phi}\|_0\left\{\|v\|_2 + \|v_{xyy}\|_0\right\}\\ &\leqslant Ch^{2-r}\|v\|_0\left\{\|v_{xxy}\|_0 + \|v_{xyy}\|_0\right\}.\end{aligned} \tag{3.38}$$

By symmetry with respect to x and y, we also have

$$\sum_{\rho\in Q}\left\|\frac{\partial^{r-1}}{\partial y^{r-1}}(Lv)\right\|_{0,\rho}\left\|\tilde{\phi}\right\|_{0,\rho} \leqslant Ch^{2-r}\|v\|_0\left\{\|v_{xxy}\|_0 + \|v_{xyy}\|_0\right\}. \tag{3.39}$$

Therefore, (3.36), (3.38), and (3.39) give

$$I_2 \leqslant Ch \|v\|_0 \left\{ \|v_{xxy}\|_0 + \|v_{xyy}\|_0 \right\}. \tag{3.40}$$

Since $\tilde{\phi} \in T$, using (1.4), (1.1), the Cauchy–Schwarz inequality, Lemma 2.4, (2.4) and Lemma 2.5, we obtain

$$I_3 = (\tilde{\phi}, L(U - W))_h = (\tilde{\phi}, L(u - W))_h \leqslant C\|\tilde{\phi}\|_h \|L(u - W)\|_h \leqslant Ch^{r-1} \|v\|_0 \|u\|_{r+2}. \tag{3.41}$$

The desired inequality follows now from (3.34), (3.35), (3.40) and (3.41). □

THEOREM 3.4 Assume that h is sufficiently small. Then (1.4) has a unique solution. Moreover, if $u \in H^{r+2}(\Omega)$, then

$$\|(u - U)_{xxy}\|_0 + \|(u - U)_{xyy}\|_0 \leqslant Ch^{r-1} \|u\|_{r+2}.$$

Proof. We assume for the moment that (1.4) has a solution. Then, with $v = U - W$, Lemmas 3.1–3.3 and the ϵ inequality give

$$\begin{aligned}
&\|v_{xxy}\|_0^2 + \|v_{xyy}\|_0^2 \\
&\leqslant C \left\{ h^{r-1} \|u\|_{r+2} \left(\|v_{xxy}\|_0 + \|v_{xyy}\|_0 \right) + h^2 \left(\|v_{xxy}\|_0^2 + \|v_{xyy}\|_0^2 \right) + h^{2r-2} \|u\|_{r+2}^2 \right\} \\
&\leqslant C \left\{ (\epsilon + h^2) \left(\|v_{xxy}\|_0^2 + \|v_{xyy}\|_0^2 \right) + C(\epsilon) h^{2r-2} \|u\|_{r+2}^2 + h^{2r-2} \|u\|_{r+2}^2 \right\}.
\end{aligned}$$

Hence, for sufficiently small h and for a particular sufficiently small ϵ, we obtain

$$\|v_{xxy}\|_0 + \|v_{xyy}\|_0 \leqslant Ch^{r-1} \|u\|_{r+2}. \tag{3.42}$$

The desired inequality follows from $u - U = (u - W) - v$, the triangle inequality, (2.6) and (3.42).

Since the linear system corresponding to (1.4) is square, it is sufficient to prove uniqueness of the solution to (1.4) in order to show its existence. If U_1 and U_2 are two solutions of (1.4), then

$$(L(U_1 - U_2), v)_h = 0, \quad v \in T.$$

Hence, $U_1 - U_2$ is the quadrature Petrov–Galerkin solution to (1.1) with f replaced by 0. The proved convergence result, with u and U replaced respectively by 0 and $U_1 - U_2$, and Lemma 2.3 imply $U_1 - U_2 = 0$. □

COROLLARY 3.5 If $u \in H^{r+2}(\Omega)$ and h is sufficiently small, then

$$\|u - U\|_2 \leqslant Ch^{r-1} \|u\|_{r+2}.$$

Proof. The desired result follows from Lemma 2.3 and Theorem 3.4. □

4. L^2 and H^1 convergence analyses

THEOREM 4.1 If $u \in H^{r+3}(\Omega)$ and h is sufficiently small, then

$$\|u - U\|_0 \leqslant Ch^{r+1}\, \|u\|_{r+3}\,, \quad \|u - U\|_1 \leqslant Ch^{r}\, \|u\|_{r+3}\,.$$

Proof. With $v = u - U$, let $\phi \in H^2(\Omega)$ be that of (2.1) and (2.2), and let a bilinear spline $\tilde{\phi}$ be that of (2.3) and (2.4). Since $\tilde{\phi} \in T$, (1.1) and (1.4) give

$$(Lv, \tilde{\phi})_h = (L(u - U), \tilde{\phi})_h = (f, \tilde{\phi})_h - (f, \tilde{\phi})_h = 0,$$

and hence, by (2.1), we obtain

$$\|v\|_0^2 = (v, v) = (L^*\phi, v) = (\phi, Lv) = (\phi - \tilde{\phi}, Lv) + \{(\tilde{\phi}, Lv) - (\tilde{\phi}, Lv)_h\} \equiv I_1 + I_2. \tag{4.1}$$

Using the Cauchy–Schwarz inequality, (2.3), (1.2), (2.2) and Corollary 3.5, we have

$$I_1 \leqslant C\|\phi - \tilde{\phi}\|_0\, \|Lv\|_0 \leqslant Ch^2\, \|\phi\|_2\, \|v\|_2 \leqslant Ch^{r+1}\, \|v\|_0\, \|u\|_{r+2}\,. \tag{4.2}$$

Using the Bramble–Hilbert lemma for the quadrature formula (see Bramble & Hilbert, 1971, Theorem 2), we have

$$I_2 \leqslant Ch^{r+1} \sum_{\rho \in Q} \left\{ \left\| \frac{\partial^{r+1}}{\partial x^{r+1}}(\tilde{\phi} Lv) \right\|_{L^1(\rho)} + \left\| \frac{\partial^{r+1}}{\partial y^{r+1}}(\tilde{\phi} Lv) \right\|_{L^1(\rho)} \right\}. \tag{4.3}$$

Using Leibnitz's rule, the fact that the spline $\tilde{\phi}$ is of degree at most 1 in each variable, and the triangle and Cauchy–Schwarz inequalities, we obtain

$$\left\| \frac{\partial^{r+1}}{\partial x^{r+1}}(\tilde{\phi} Lv) \right\|_{L^1(\rho)} \leqslant C \sum_{i=r}^{r+1} \left\| \frac{\partial^{r+1-i}}{\partial x^{r+1-i}}\tilde{\phi}\, \frac{\partial^{i}}{\partial x^{i}}(Lv) \right\|_{L^1(\rho)} \leqslant C\|\tilde{\phi}\|_{1,\rho} \sum_{i=r}^{r+1} \left\| \frac{\partial^{i}}{\partial x^{i}}(Lv) \right\|_{0,\rho}. \tag{4.4}$$

For $i = r$ or $r + 1$, using (1.2), the triangle inequality, Leibnitz's rule, the fact that a_2 is a function of y only and the fact that the spline U is of degree at most r in each variable, we obtain

$$\begin{aligned}
\left\| \frac{\partial^{i}}{\partial x^{i}}(Lv) \right\|_{0,\rho} &\leqslant C \left\{ \left\| \frac{\partial^{i}}{\partial x^{i}}(a_1 v_{xx}) \right\|_{0,\rho} + \left\| \frac{\partial^{i}}{\partial x^{i}}(a_2 v_{yy}) \right\|_{0,\rho} \right.\\
&\quad \left. + \left\| \frac{\partial^{i}}{\partial x^{i}}(b_1 v_{x}) \right\|_{0,\rho} + \left\| \frac{\partial^{i}}{\partial x^{i}}(b_2 v_{y}) \right\|_{0,\rho} + \left\| \frac{\partial^{i}}{\partial x^{i}}(cv) \right\|_{0,\rho} \right\}\\
&\leqslant C \left\{ \sum_{j=0}^{r+3} \left\| \frac{\partial^{j}}{\partial x^{j}} v \right\|_{0,\rho} + \left\| \frac{\partial^{i}}{\partial x^{i}} v_{yy} \right\|_{0,\rho} + \sum_{j=0}^{r+1} \left\| \frac{\partial^{j}}{\partial x^{j}} v_y \right\|_{0,\rho} \right\}\\
&\leqslant C \left\{ \|u - U\|_{2,\rho} + \|u\|_{r+3,\rho} + I_\rho \right\},
\end{aligned} \tag{4.5}$$

where

$$I_\rho = \sum_{j=3}^{r} \left\| \frac{\partial^j}{\partial x^j}(u-U) \right\|_{0,\rho} + \sum_{j=2}^{r} \left\| \frac{\partial^j}{\partial x^j}(u-U)_y \right\|_{0,\rho} + \left\| \frac{\partial^r}{\partial x^r}(u-U)_{yy} \right\|_{0,\rho}. \quad (4.6)$$

To bound I_ρ, we use Fairweather (1978, Theorem 3.3) to conclude that there exists a spline $z \in H^{r+2}(\Omega)$ such that

$$\|u-z\|_k \leqslant Ch^{r+3-k}\,\|u\|_{r+3}, \quad k = 0, \dots, r+2. \quad (4.7)$$

Using the triangle and inverse inequalities, we have, for $3 \leqslant j \leqslant r$,

$$\begin{aligned} \left\| \frac{\partial^j}{\partial x^j}(u-U) \right\|_{0,\rho} &\leqslant \left\| \frac{\partial^j}{\partial x^j}(u-z) \right\|_{0,\rho} + \left\| \frac{\partial^j}{\partial x^j}(z-U) \right\|_{0,\rho} \\ &\leqslant C\left\{ \|u-z\|_{r,\rho} + h^{2-j}\|z-U\|_{2,\rho} \right\} \\ &\leqslant C\left\{ \|u-z\|_{r,\rho} + h^{2-r}\left(\|u-U\|_{2,\rho} + \|u-z\|_{2,\rho} \right) \right\}. \end{aligned} \quad (4.8)$$

In a similar way, we have, for $2 \leqslant j \leqslant r$,

$$\begin{aligned} \left\| \frac{\partial^j}{\partial x^j}(u-U)_y \right\|_{0,\rho} &\leqslant \left\| \frac{\partial^j}{\partial x^j}(u-z)_y \right\|_{0,\rho} + \left\| \frac{\partial^j}{\partial x^j}(z-U)_y \right\|_{0,\rho} \\ &\leqslant C\left\{ \|u-z\|_{r+1,\rho} + h^{2-j}\|(z-U)_{xxy}\|_{0,\rho} \right\} \\ &\leqslant C\left\{ \|u-z\|_{r+1,\rho} + h^{2-r}\left(\left\|(u-U)_{xxy}\right\|_{0,\rho} + \|u-z\|_{3,\rho} \right) \right\}. \end{aligned} \quad (4.9)$$

Moreover,

$$\begin{aligned} \left\| \frac{\partial^r}{\partial x^r}(u-U)_{yy} \right\|_{0,\rho} &\leqslant \left\| \frac{\partial^r}{\partial x^r}(u-z)_{yy} \right\|_{0,\rho} + \left\| \frac{\partial^r}{\partial x^r}(z-U)_{yy} \right\|_{0,\rho} \\ &\leqslant C\left\{ \|u-z\|_{r+2,\rho} + h^{1-r}\|(z-U)_{xxy}\|_{0,\rho} \right\} \\ &\leqslant C\left\{ \|u-z\|_{r+2,\rho} + h^{1-r}\left(\left\|(u-U)_{xxy}\right\|_{0,\rho} + \|u-z\|_{3,\rho} \right) \right\}. \end{aligned} \quad (4.10)$$

It follows from (4.6) and (4.8)–(4.10) that

$$I_\rho \leqslant C\left\{ \|u-z\|_{r+2,\rho} + h^{1-r}(\|u-U\|_{2,\rho} + \|(u-U)_{xxy}\|_{0,\rho} + \|u-z\|_{3,\rho}) \right\}. \quad (4.11)$$

Hence, (4.4), (4.5), (4.11), the Cauchy–Schwarz inequality, (2.4), (4.7), Corollary 3.5, and Theorem 3.4 give

$$\begin{aligned} &\sum_{\rho \in Q} \left\| \frac{\partial^{r+1}}{\partial x^{r+1}}(\tilde{\phi} L v) \right\|_{L^1(\rho)} \\ &\leqslant C\|\tilde{\phi}\|_1 \left\{ \|u\|_{r+3} + \|u-z\|_{r+2} + h^{1-r}\left(\|u-U\|_2 + \left\|(u-U)_{xxy}\right\|_0 + \|u-z\|_3 \right) \right\} \\ &\leqslant C\,\|v\|_0\,\|u\|_{r+3}. \end{aligned} \quad (4.12)$$

By symmetry with respect to x and y, we also have

$$\sum_{\rho\in Q}\left\|\frac{\partial^{r+1}}{\partial y^{r+1}}(\tilde{\phi}Lv)\right\|_{L^1(\rho)} \leqslant C\,\|v\|_0\,\|u\|_{r+3}\,. \tag{4.13}$$

Therefore, using (4.3), (4.12), and (4.13), we have

$$I_2 \leqslant Ch^{r+1}\,\|v\|_0\,\|u\|_{r+3}\,. \tag{4.14}$$

Hence (4.1), (4.2), and (4.14) give the first desired bound

$$\|v\|_0 \leqslant Ch^{r+1}\,\|u\|_{r+3}\,. \tag{4.15}$$

Finally, using $\|v\|_0 \leqslant C\|v_x\|_0$, integration by parts, the Cauchy–Schwarz inequality, Corollary 3.5 and (4.15), we obtain

$$\|v\|_1^2 \leqslant C(\|v_x\|_0^2 + \|v_y\|_0^2) = -C(\Delta v, v) \leqslant C\,\|v\|_2\,\|v\|_0 \leqslant Ch^{2r}\,\|u\|_{r+3}^2\,. \qquad \square$$

5. Numerical experiments

In this section, we demonstrate applicability of our quadrature Petrov–Galerkin scheme (1.4) even in some cases where the variable coefficients of L in (1.2) are non-smooth or the solution u to (1.1) does not satisfy the smoothness requirements in our analysis. In our numerical experiments, for several values of $N_x = N_y = N$, we used uniform partitions in the x- and y-directions and C^2 cubic splines ($r = 3$). We calculated the errors $\|u - U\|_k$, $k = 0, 1, 2$, using 25 translated Gauss quadrature points on each cell of the 32×32 uniform partition of Ω. The estimated convergence rate ($R(H^k)$) for the H^k norm was computed in the usual way.

EXAMPLE 1 We consider the boundary value problem

$$-u_{xx} - u_{yy} + b_1(x, y)u_x + b_2(x, y)u_y + c(x, y)u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5.1}$$

with

$$b_1(x, y) = \cos(x)\sin^2(y) + g(x, y, \alpha_1), \qquad b_2(x, y) = \exp(x + y) + g(x, y, \alpha_2),$$
$$c(x, y) = \exp(x + y)\sin(x)\cos(y) + g(x, y, \alpha_3),$$

where, for a non-negative parameter α, the function $g(x, y, \alpha)$ is given by

$$g(x, y, \alpha) = \alpha\left[x^{-\alpha}y^{-\alpha} + |x - 0{\cdot}5|^{\alpha}|y - 0{\cdot}5|^{\alpha} + \exp(x^{\alpha} + y^{\alpha})\right].$$

The right-hand side function $f(x, y)$ in (5.1) is chosen in such a way that

$$u(x, y) = \sin(\pi x)\sin(\pi y)$$

is the exact solution of (5.1).

TABLE 1 *Example 1, Case 1 (smooth coefficients and smooth solution)*

N	$\|u-U\|_0$	$R(H^0)$	$\|u-U\|_1$	$R(H^1)$	$\|u-U\|_2$	$R(H^2)$
2	5·3313e−03		5·3190e−02		6·3221e−01	
4	3·7817e−04	3·8174	7·1529e−03	2·8945	1·6854e−01	1·9073
8	2·0095e−05	4·2341	8·0622e−04	3·1493	4·0738e−02	2·0486
16	1·1975e−06	4·0688	9·7752e−05	3·0440	1·0071e−02	2·0163
32	7·3931e−08	4·0177	1·2121e−05	3·0116	2·5096e−03	2·0046

TABLE 2 *Example 1, Case 2 (non-smooth coefficients and smooth solution)*

N	$\|u-U\|_0$	$R(H^0)$	$\|u-U\|_1$	$R(H^1)$	$\|u-U\|_2$	$R(H^2)$
2	4·7960e−03		5·0726e−02		6·3591e−01	
4	3·5739e−04	3·7463	7·1328e−03	2·8302	1·6856e−01	1·9156
8	1·8602e−05	4·2640	8·0564e−04	3·1463	4·0739e−02	2·0488
16	1·1011e−06	4·0784	9·7733e−05	3·0432	1·0070e−02	2·0163
32	6·7864e−08	4·0201	1·2120e−05	3·0114	2·5096e−03	2·0045

CASE 1 First, we choose $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so that the coefficients b_1, b_2 and c are smooth. Our computational results for this case are given in Table 1. As expected (see Corollary 3.5 and Theorem 4.1), we observe the optimal-order $O(h^{4-k})$ convergence in the H^k norm, $k = 0, 1, 2$.

CASE 2 We choose $\alpha_1 = 1/2$, $\alpha_2 = 3/4$ and $\alpha_3 = 3/2$ so that $b_1, b_2, c \notin L^2(\Omega)$. Computational results of Table 2 demonstrate that for some second-order elliptic problems with non-smooth coefficients, our method retains the optimal convergence orders, as long as the exact solution is sufficiently smooth. However, theoretically it is difficult to prove optimal-order convergence results in such cases.

In our next numerical experiment, we demonstrate that our scheme retains the optimal convergence orders when $u \in H^s(\Omega)$ for some s such that $4 \leqslant s < 5$, which is less than required in our convergence analysis, and that, for $\alpha = 2, 3$, the convergence rate in the H^0 norm is α when $u \in H^s(\Omega)$ for some s such that $\alpha \leqslant s < \alpha + 1$.

EXAMPLE 2 For a given positive integer α and a function f_α, we consider the Poisson equation

$$-\Delta u_\alpha = f_\alpha \qquad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega. \tag{5.2}$$

We choose f_α so that

$$u_\alpha(x, y) = \left[x^{(\alpha-0\cdot1)} - x\right]\left[y^{(\alpha-0\cdot1)} - y\right]$$

is the exact solution of (5.2). It can be shown that $u_\alpha \in H^s(\Omega)$ for $s < s_0$, where s_0 is such that $\alpha \leqslant s_0 < \alpha + 1$.

We computed approximate solutions U_α of (5.2) with $\alpha = 2, 3, 4$ and 5. Results in Table 3 demonstrate the applicability of our method in the case of solutions with various

TABLE 3 *Example 2 (solutions with various order of smoothness)*

N	$\|u_2 - U_2\|_0$	$R(H^0)$	$\|u_3 - U_3\|_0$	$R(H^0)$	$\|u_5 - U_5\|_0$	$R(H^0)$
2	8·8965e−05		8·0333e−05		3·5878e−03	
4	1·9713e−05	2·1740	8·9776e−06	3·1616	2·0114e−04	4·1568
8	4·4901e−06	2·1343	1·0164e−06	3·1428	1·2189e−05	4·0445
16	1·0713e−06	2·0674	1·1885e−07	3·0963	7·5516e−07	4·0127
32	2·6684e−07	2·0052	1·4436e−08	3·0414	4·7083e−08	4·0035

TABLE 4 *Example 2 (optimal convergence rates for $u_4 \in H^s(\Omega)$, $s < s_0$, $4 \leqslant s_0 < 5$)*

N	$\|u_4 - U_4\|_0$	$R(H^0)$	$\|u_4 - U_4\|_1$	$R(H^1)$	$\|u_4 - U_4\|_2$	$R(H^2)$
2	9·0940e−04		7·4716e−03		9·2850e−02	
4	5·8455e−05	3·9595	9·0549e−04	3·0447	2·3202e−02	2·0007
8	3·7207e−06	3·9737	1·1253e−04	3·0084	5·8193e−03	1·9953
16	2·3631e−07	3·9768	1·4067e−05	2·9999	1·4581e−03	1·9967
32	1·5004e−08	3·9772	1·7601e−06	2·9987	3·6503e−04	1·9980

orders of smoothness, and those in Table 4 show that our scheme yields optimal-order $O(h^{4-k})$ convergence in the H^k norm, $k = 0, 1, 2$, with solution $u_4 \in H^s(\Omega)$, $s < s_0$, $4 \leqslant s_0 < 5$.

Acknowledgements

The support of Australian Research Council (ARC) is gratefully acknowledged. In particular, BB was supported for six months in 2002 at the University of New South Wales (UNSW) by an ARC Linkage–International grant. KM's (2001–2003) Ph.D. scholarship was supported by the ARC and the School of Mathematics, UNSW. We thank the referees for valuable comments.

REFERENCES

BIALECKI, B. (1998) Convergence analysis of orthogonal spline collocation for elliptic boundary value problems. *SIAM J. Numer. Anal.*, **35**, 617–631.

BIALECKI, B. & FAIRWEATHER, G. (2001) Orthogonal spline collocation methods for partial differential equations. *J. Comput. Appl. Math.*, **128**, 55–85.

BOTTCHER, C. & STRAYER, M. R. (1990) The basis spline method and associated techniques. *Computational Atomic and Nuclear Physics*. (C. Bottcher, M. R. Strayer & J. B. McGrory, eds). Singapore: World Scientific, pp. 217–240.

BOTTCHER, C. & STRAYER, M. R. (1993) Spline methods for conservation equations. *Computational Acoustics—Volume 2*. (D. Lee, A. R. Robinson & R. Vichnevetsky, eds). Amsterdam: Elsevier, pp. 317–338.

BRAMBLE, J. H. & HILBERT, S. R. (1971) Bounds for a class of linear functionals with applications to Hermite interpolation. *Numer. Math.*, **16**, 362–369.

BRAMBLE, J. H., EWING, R. E. & LI, G. (1989) Alternating direction multistep methods for parabolic problems-iterative stabilization. *SIAM J. Numer. Anal.*, **26**, 904–919.

DE BOOR, C. (1966) The method of projections as applied to the numerical solution of the two point boundary value problems using cubic splines. *Ph.D. Thesis*, University of Michigan.

DOUGLAS, J. & DUPONT, T. (1974) *Collocation Method for Parabolic Equations in a Single Space Variable.* Lecture Notes in Mathematics 385. New York: Springer.

DOUGLAS, J., DUPONT, T., RACHFORD, H. H. & WHEELER, M. F. (1977) Local H^{-1} Galerkin and adjoint local H^{-1} Galerkin procedures for elliptic equations. *RAIRO Anal. Numér.*, **11**, 3–12.

GRIGORIEFF, R. D. & SLOAN, I. H. (1996) Spline Petrov–Galerkin methods with quadrature. *Numer. Funct. Anal. Optimiz.*, **17**, 755–784.

HADJIDIMOS, A., HOUSTIS, E. N., RICE, J. R. & VAVALIS, E. (1999) Analysis of iterative line cubic spline collocation methods for elliptic partial differential equations. *SIAM J. Matrix Anal. Appl.*, **21**, 508–521.

HOUSTIS, E. N., VAVALIS, E. A. & RICE, J. R. (1988) Convergence of $O(h^4)$ cubic spline collocation methods for elliptic partial differential equations. *SIAM J. Numer. Anal.*, **25**, 54–74.

KEGLEY, JR, D. R., OBERACKER, V. E., STRAYER, M. R., UMAR, A. S. & WELLS, J. C. (1996) Basis spline collocation method for solving the Schrödinger equation in axillary symmetric systems. *J. Comput. Phys.*, **128**, 197–208.

FAIRWEATHER, G. (1978) *Finite Element Galerkin Methods for Differential Equations*, Lecture Notes in Pure and Applied Mathematics 34. New York: Dekker.

SLOAN, I. H., TRAN, D. & FAIRWEATHER, G. (1993) A fourth-order cubic spline method for linear second-order two-point boundary value problems. *IMA J. Numer. Anal.*, **13**, 591–607.

TSOMPANOPOULOU, P. & VAVALIS, E. (1998) ADI methods for cubic spline collocation discretizations of elliptic PDEs. *SIAM J. Sci. Comput.*, **19**, 341–363.

UMAR, A. S. (1990) Three-dimensional HF and TDHF calculations with the basis-spline collocation technique. *Computational Atomic and Nuclear Physics*. (C. Bottcher, M. R. Strayer & J. B. McGrory, eds). Singapore: World Scientific, pp. 377–390.

UMAR, A. S., WU, J., STRAYER, M. R. & BOTTCHER, C. (1991) Basis-spline collocation method for the lattice solution of boundary value problems. *J. Comput. Phys.*, **93**, 426–448.