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A Quadrature Finite Element Method for Semilinear Second-Order Hyperbolic Problems*

K. Mustapha,[1] **H. Mustapha**[2]
[1] *Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia*

[2] *Reservoir Engineering Research Institute, (RERI), Palo Alto, California 94306*

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In this work we propose and analyze a fully discrete modified Crank–Nicolson finite element (CNFE) method with quadrature for solving semilinear second-order hyperbolic initial-boundary value problems. We prove optimal-order convergence in both time and space for the quadrature-modified CNFE scheme that does not require nonlinear algebraic solvers. Finally, we demonstrate numerically the order of convergence of our scheme for some test problems. Numer Methods Partial Differential Eq 24: 350–367, 2008

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I. INTRODUCTION

Consider the following semilinear second-order hyperbolic initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{x})\frac{\partial u}{\partial x_j}\right) = f(\mathbf{x},t,u), \quad (\mathbf{x},t) \in \Omega \times (0,T], \tag{1.1}$$

$$u(\mathbf{x},0) = g_1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial t}(\mathbf{x},0) = g_2(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.3}$$

$$u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\Omega \times [0,T], \tag{1.4}$$

where $\Omega \subset \mathbb{R}^d$ is a convex bounded polygonal domain with boundary $\partial\Omega$. The given functions a_{ij}, g_1 and g_2 are smooth on $\bar{\Omega}$, with $a_{ij} = a_{ji}$ for $i,\ j = 1,\ldots,d$ and the nonlinear source term f

Correspondence to: K. Mustapha, Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia (email: kassem@kfupm.edu.sa)
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is continuous on $\bar{\Omega} \times [0, T] \times \mathbb{R}$. We assume that a_{ij} satisfies the ellipticity property: there exists a constant $C > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(\mathbf{x})\alpha_i\alpha_j \geq C \sum_{i=1}^{d} \alpha_i^2, \quad \mathbf{x} \in \bar{\Omega}, \ \alpha_1, \ldots, \alpha_n \in \mathbb{R}. \tag{1.5}$$

Consequently, the bilinear form associated with the elliptic part of (1.1)–(1.4), defined by

$$a(v_1, v_2) = \sum_{i,j=1}^{d} \int_{\Omega} a_{ij}(\mathbf{x}) \frac{\partial v_1}{\partial x_i} \frac{\partial v_2}{\partial x_j} \, d\mathbf{x}, \quad v_1, \, v_2 \in H_0^1(\Omega) \tag{1.6}$$

satisfies the coercivity

$$a(v, v) \geq C_1 \|v\|_1^2, \quad v \in H_0^1(\Omega) \tag{1.7}$$

for some positive constant C_1. Throughout the paper, for a nonnegative integer k, the standard norm in the Sobolev space $W_p^k(\Omega)$ is denoted by $\|\cdot\|_{W_p^k(\Omega)}$ where $1 \leq p \leq \infty$. In the special case $p = 2$, we shall write $W_2^k(\Omega) = H^k(\Omega)$ and so $H^0(\Omega) = L^2(\Omega)$. Furthermore, $H_0^1(\Omega)$ denotes the space of all functions $\phi \in H^1(\Omega)$ with $\phi = 0$ on $\partial\Omega$.

A semidiscrete (and fully discrete) finite element analysis has been studied widely for solving partial differential equations but without including the effect of quadrature in practical implementation. In contrast, fully discrete finite element methods with quadrature are frequently ignored. It has been proposed and analyzed by few authors only, see for example [1,2] for elliptic problems, and [3–6] for linear parabolic and hyperbolic problems. However, fully discrete finite element schemes with quadrature for semilinear hyperbolic problems are yet to be analyzed.

In this paper, we are interested in finding the approximate solution of (1.1)–(1.4) by applying a fully discrete quadrature scheme that does not require nonlinear algebraic solvers. In actual practise, we have to evaluate the integrals occurring in the standard Galerkin method without quadrature by a numerical quadrature scheme.

Following [3,5], we construct for $h > 0$, a family of a quasiuniform triangulations T_h of $\bar{\Omega}$ with

$$h = \sup_{K \in T_h} (\mathrm{diam}(K)).$$

With T_h, we associate $S_h \subset H_0^1(\Omega)$ is a subspace of all continuous piecewise polynomials of degree at most r defined on T_h. By the isoparametric finite elements of Cialert and Raviart [7], the following approximation property is satisfied:

$$\inf_{v \in S_h} \|q - v\|_{m,T_h} \leq Ch^{s-m}\|q\|_s, \quad q \in H^s(\Omega) \cap H_0^1(\Omega), \ 0 \leq m \leq s \leq r+1, \tag{1.8}$$

where

$$\|.\|_{m,T_h}^2 = \sum_{K \in T_h} \|.\|_{H^m(K)}^2.$$

For a chosen parameter N^t, throughout the paper $\Pi^t = \{t_n\}_{n=0}^{N^t}$ denotes the uniform partition of the time interval $[0, T]$ with $t_n = n\tau$ and $\tau = T/N^t$. We use the following notations: for a function ϕ defined on Π^t,

$$\phi^n = \phi(t_n), \quad \bar{\phi}^{n+\frac{1}{2}} = \frac{\phi^{n+1} + \phi^{n-1}}{2}, \quad \partial_t \phi^n = \frac{\phi^{n+1} - \phi^n}{\tau}, \tag{1.9}$$

and

$$\tilde{\partial}_t \phi^n = \frac{\phi^{n+1} - \phi^{n-1}}{2\tau}, \quad \partial_t^2 \phi^n = \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{\tau^2}. \tag{1.10}$$

A two-time level quadrature Crank–Nicolson finite element (CNFE) scheme for a linear hyperbolic problem was first proposed by Baker and Dougalis [3], following the work of Raviart [5] on the effect of quadrature on finite element solutions for parabolic problems using triangular finite elements. Generalized results in [3] for rectangular elements using quadrature Crank–Nicolson (CN) and alternating direction implicit (ADI) schemes were recently studied by Ganesh and the first author [4]. The schemes in [4] were obtained by applying the quadrature aspect to the CN and ADI schemes of [8] and [9] respectively. A related, but three-time level, quadrature CNFE scheme for linear hyperbolic problems yielding second-order convergence in both time and space was recently analyzed by Sinha [6].

The main contribution of this paper is to propose and analyze a quadrature fully discrete method for solving a class of semilinear hyperbolic problems defined by (1.1)–(1.4) using a modified CN finite difference discretization in time. In the time-dependent problem (1.1)–(1.4), the source term f is a nonlinear function of the unknown solution. Applying the standard CN three-time level schemes lead to solve a nonlinear algebraic system at each time step. In this work, we modify the standard CN method applying to (1.1)–(1.4). This modification lead to solve only a linear algebraic system at each time step.

In [3], second-order convergence in time and optimal order convergence in space was proved for a fully discrete scheme using ℓ-point quadrature rule on $K \in T_h$ with positive weights $w_{j,K}$ and nodes $\sigma_{j,K} \in K,\ j = 1, \dots, \ell$:

$$\int_K v(\mathbf{x})\, d\mathbf{x} \approx \sum_{j=1}^{\ell} w_{j,K} v(\sigma_{j,K}), \qquad (v, z)_h = \sum_{K \in T_h} \sum_{j=1}^{\ell} w_{j,K} (vz)(\sigma_{j,K}). \tag{1.11}$$

Our fully discrete modified CNFE scheme with quadrature for (1.1)–(1.4) involves finding $U : \Pi^t \to S_h$ such that

$$\left(\partial_t^2 U^n, v\right)_h + a_h\left(\bar{U}^{n+\frac{1}{2}}, v\right) = (\mathcal{F}(t_n)U^n, v)_h, \quad v \in S_h,\ n = 1, \dots, N^t - 1, \tag{1.12}$$

where the discrete bilinear form $a_h(\cdot, \cdot)$ is defined by

$$a_h(v_1, v_2) = \sum_{K \in T_h} \sum_{j=1}^{\ell} w_{j,K} \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial v_1}{\partial x_i} \frac{\partial v_2}{\partial x_j} \right] (\sigma_{j,K}),$$

with v_1 and v_2 being differentiable on the interior of each cell K in T_h. For each $t \in [0, T]$, the Nemytskii operator $\mathcal{F}(t)$ is defined by

$$[\mathcal{F}(t)\psi](\mathbf{x}) = f(\mathbf{x}, t, \psi(\mathbf{x})), \quad \mathbf{x} \in \bar{\Omega}.$$

Practically the above scheme is applicable for a wide class of quasilinear and nonlinear hyperbolic problems. For example, if the functions a_{ij} $(i,\ j = 1, \dots, d)$ in (1.1) depend on the unknown solution, i.e., $a_{ij} = a_{ij}(\mathbf{x}, t, u)$ then the second term on the left hand side of (1.12) has to be replaced with

$$a_h\left(U^n, \bar{U}^{n+\frac{1}{2}}, v\right) = \sum_{K \in T_h} \sum_{j=1}^{\ell} w_{j,K} \left[\sum_{i,j=1}^{d} a_{ij}(., (t^n + t^{n+1})/\tau, U^n) \frac{\partial \bar{U}^{n+\frac{1}{2}}}{\partial x_i} \frac{\partial v}{\partial x_j} \right] (\sigma_{j,K}).$$

Theoretically the affect of quadrature will increase the level of complexity especially in the case of quasilinear and nonlinear problems.

The quadrature CNFE scheme (1.12) requires solutions of only *linear* systems and a selection of U^0 and $U^1 \in S_h$. We select U^0 and U^1 using the data g_1 and g_2 in (1.2) and (1.3) respectively:

$$a_h(U^0, v) = a_h(g_1, v), \quad a_h(U^1, v) = a_h(g_2^*, v), \qquad v \in S_h, \tag{1.13}$$

where

$$g_2^*(\mathbf{x}) = g_1(\mathbf{x}) + \tau g_2(\mathbf{x}) + \frac{\tau^2}{2} u_{tt}(\mathbf{x}, 0). \tag{1.14}$$

The purpose of the present paper is to prove second-order convergence in time and optimal order H^k norms convergence in space for the modified CNFE scheme (1.12) with $k = 0, 1$. Moreover, we demonstrate numerically our theoretical convergence results.

The outline of this paper is as follows. In the next section, we recall from [3,5] some suitable assumptions and useful results required in our convergence analysis. Our convergence analysis is based on analyzing error in two stages using a discrete type of an elliptic projection of the exact solution in S_h as a comparison function. In Section III we study the convergence rate of the comparison function to the exact solution of a continuous time problem. We prove optimal-order convergence in both time and space of our implementable solutions in Section IV. The results of numerical computations and their discussion are given in Section V.

II. PRELIMINARIES

Let $L^2(0, T; H^s(\Omega))$, $s \geq 0$ be the space of all functions $v(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, $t \in (0, T)$, such that for each fixed $t \in (0, T)$, $v(\cdot, t) \in H^s(\Omega)$ and $\|v\|_s \in L^2(0, T)$.

Throughout the paper, for each fixed $t \in [0, T]$, we assume that $f(\cdot, \cdot, t, \cdot) \in \mathcal{C}(\bar{\Omega} \times \mathbb{R})$, and locally Lipschitz, that is

$$|f(x, t, z_1) - f(x, t, z_2)| \leq C|z_1 - z_2|, \quad \mathbf{x} \in \bar{\Omega}, \ t \in [0, T], \ |z_1|, |z_2| \leq D(u), \tag{2.1}$$

where $D(u)$ is a positive constant quantity that depends on u, and C denotes a generic positive constant which may depend on r, but which is independent of h and τ.

In order to prove optimal-order convergence results of our quadrature CNFE solution we assume that $\mathcal{F}(t)u \in L^2(0, T; H^{r+1}(\Omega))$,

$$u, \, u_t, \, u_{tt} \in L^2(0, T; H^{r+2}(\Omega)), \quad u_{ttt}, \, u_{tttt} \in L^2(0, T; H^{r+1}(\Omega)),$$

and $a_{ij} \in W_\infty^{r+1}(\Omega)$ for $i, \ j = 1, \ldots, d$. We require such regularity on the exact solution mainly due to the technical details involved in the analysis of the quadrature CNFE method and also due to the nonlinearity of the forcing term f. Slightly different assumptions on u were assumed by Baker and Dougalis in their convergence analysis where f in (1.1) was independent of u.

Throughout this paper, we follow [5] in our assumption on the quadrature formula (1.11). We assume that the quadrature formula (1.11) satisfies the following accuracy requirement. Let $\psi \in W_\infty^{r+1}(\Omega)$. For each $K \in T_h$, let $g \in H^{r+1}(K)$ and v be a polynomial of degree r on K, then for $r + 1 - \frac{d}{2} > 0$,

$$|(\psi g, v) - (\psi g, v)_h| \leq Ch^{r+1} \|\psi\|_{W_\infty^{r+1}(\Omega)} \|g\|_{r+1, T_h} \|v\|_{0, T_h}, \tag{2.2}$$

and for $v \in S_h$ and $r - \frac{d}{2} > 0$,

$$|a(g,v) - a_h(g,v)| \leq Ch^r \max_{1\leq i,j\leq d} \|a_{ij}\|_{W^r_\infty(\Omega)} \|g\|_{r+1,T_h} \|v\|_{1,T_h}. \tag{2.3}$$

Since the forcing term f in (1.1) depends on the unknown solution u, our convergence analysis requires us to assume that the quadrature formula satisfying the following additional error estimate: for $g \in W_1^{r+1}(K)$ and $r + 1 - d > 0$,

$$\left| \int_K g(\mathbf{x})\, d\mathbf{x} - \sum_{j=1}^{\ell} w_{j,K}\, g(\sigma_{j,K}) \right| \leq Ch^{r+1} \|g\|_{W_1^{r+1}(K)}. \tag{2.4}$$

(The proof of (2.3) and (2.4) can be found in [10]). We also assume that $(\cdot,\cdot)_h$ is an inner product on $S_h \times S_h$ and thus induces a norm on S_h defined by

$$\|v\|_h = (v,v)_h^{1/2}, \quad v \in S_h.$$

We assume that the norms $\|.\|_0$ and $\|.\|_h$ are equivalent on S_h: there exist positive constants C_1 and C_2 such that

$$C_1\|v\|_h \leq \|v\|_0 \leq C_2\|v\|_h, \quad v \in S_h. \tag{2.5}$$

Furthermore, we assume that $a_h(.,.)$ is coercive on $S_h \times S_h$: there exists a positive constant C_1 such that

$$a_h(v,v) \geq C\|v\|_1^2, \quad v \in S_h. \tag{2.6}$$

The above assumptions on the quadrature formula (1.11) holds under suitable hypothesis on the set of the nodes $\sigma_{j,K} \in K,\ j = 1,\ldots,\ell$ and the degree of precision.

Throughout the paper, our analysis required the use of two different types of inverse inequalities: if v is a polynomial of degree $s > 0$ on $K \in T_h$, then

$$\|v\|_{W^0_\infty(K)} \leq Ch^{-d/2}\|v\|_{L^2(K)} \quad \text{and} \quad \|v\|_{H^j(K)} \leq Ch^{-j}\|v\|_{L^2(K)}, \qquad j \leq s.$$

III. ERROR ESTIMATE OF THE COMPARISON FUNCTION

Following a traditional approach, our convergence analysis is based on dividing the error between the exact solution u and the computable U as $u - U = (u - W) + (W - U)$, where W is a comparison function obtained using an appropriate quadrature elliptic projection of the exact solution u in S_h, for each fixed time.

In this section we analyze the first-stage error of the comparison function, by choosing $W : [0,T] \to S_h$ to be the solution of

$$a_h(W,v) = a_h(u,v), \quad v \in S_h, \ t \in [0,T]. \tag{3.1}$$

In the next theorem we bound the comparison function error $\eta := u - W$ and its derivatives.

Theorem 3.1. *For each $t \in [0,T]$, W satisfying* (3.1) *exists and*

$$\left\| \frac{\partial^i \eta}{\partial t^i} \right\|_m \leq Ch^{r+1-m} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{r+2}, \quad m = 0,1, \ i = 0,1,2. \tag{3.2}$$

Proof. Since the linear system of equations corresponding to (3.1) is square, existence of W follows from uniqueness. To prove the uniqueness, we assume that there exist W and V satisfying (3.1). Hence

$$a_h(W - V, v) = 0, \quad v \in S_h, \ t \in [0, T]. \tag{3.3}$$

From (2.6) and (3.3) with $v = W - V$, we have $\|V - W\|_1^2 \leq a_h(V - W, V - W) = 0$ and hence $V = W$.

Next, we prove (3.2) for $i = 0$ and $m = 1$. By the hypothesis of S_h, there exists a function $u_h : [0, T] \longrightarrow S_h$ such that

$$\|u - u_h\|_m \leq Ch^{r+1-m}\|u\|_{r+1}, \quad m = 0, \ldots, r + 1. \tag{3.4}$$

The triangle inequality and (3.4) yield

$$\|\eta\|_1 \leq \|u - u_h\|_1 + \|u_h - W\|_1 \leq Ch^r\|u\|_{r+1} + \|u_h - W\|_1, \quad t \in [0, T]. \tag{3.5}$$

To bound $\|u_h - W\|_1$, we use (3.1), (2.3), the Cauchy-Schwarz inequality, and (3.4) to obtain for any $v \in S_h$

$$\begin{aligned} a_h(W - u_h, v) &= [a_h(u - u_h, v) - a(u - u_h, v)] + a(u - u_h, v) \\ &\leq Ch^r\|u - u_h\|_{r+1,T_h}\|v\|_1 + \|u - u_h\|_1\|v\|_1 \leq Ch^r\|u\|_{r+1}\|v\|_1, \end{aligned}$$

and hence, from (2.6) with $v = W - u_h$,

$$\|W - u_h\|_1^2 \leq Ca_h(W - u_h, W - u_h) \leq Ch^r\|u\|_{r+1}\|W - u_h\|_1. \tag{3.6}$$

Using (3.5) and (3.6), we obtain (3.2) for $i = 0$ and $m = 1$.

Next we prove (3.2) for $i, m = 0$. For each fixed $t \in [0, T]$, let $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique solution of

$$a(\phi, v) = (\eta, v), \quad v \in H_0^1(\Omega), \tag{3.7}$$

satisfying the regularity property

$$\|\phi\|_2 \leq C\|\eta\|_0. \tag{3.8}$$

It is well known from the approximation theory that there exists a piecewise linear polynomial $\tilde{\phi}$ defined on T_h such that

$$\|\phi - \tilde{\phi}\|_1 \leq Ch\|\phi\|_2, \tag{3.9}$$

and hence (3.8) and the triangle inequality give

$$\|\phi - \tilde{\phi}\|_1 \leq Ch\|\eta\|_0, \quad \|\tilde{\phi}\|_1 \leq C\|\phi\|_2 \leq C\|\eta\|_0. \tag{3.10}$$

Further, (3.7) and (3.1) yield

$$\|\eta\|_0^2 = (\eta, \eta) = a(\phi, \eta) = a(\phi - \tilde{\phi}, \eta) + [a(\tilde{\phi}, \eta) - a_h(\tilde{\phi}, \eta)]. \tag{3.11}$$

The Cauchy–Schwarz inequality, (3.10), and (3.2) (for $m = 1, i = 0$),

$$|a(\phi - \tilde{\phi}, \eta)| \leq C\|\phi - \tilde{\phi}\|_1\|\eta\|_1 \leq Ch^{r+1}\|u\|_{r+2}\|\eta\|_0. \tag{3.12}$$

Using $\eta = u - W$, the triangle inequality, (2.2), (3.10), and (3.2) (for $m = 1, i = 0$), yield

$$\begin{aligned}|a(\eta,\tilde{\phi}) - a_h(\eta,\tilde{\phi})| &\le |a(u,\tilde{\phi}) - a_h(u,\tilde{\phi})| + |a(W,\tilde{\phi}) - a_h(W,\tilde{\phi})|\\ &\le Ch^{r+1}\|u\|_{r+2}\|\tilde{\phi}\|_1 + Ch^{r+1}\|\tilde{\phi}\|_1\|W\|_1\\ &\le Ch^{r+1}(\|u\|_{r+2} + \|W\|_1)\|\eta\|_0\\ &\le Ch^{r+1}(\|u\|_{r+2} + \|\eta\|_1 + \|u\|_1)\|\eta\|_0 \le Ch^{r+1}\|u\|_{r+2}\|\eta\|_0. \end{aligned} \tag{3.13}$$

Now, (3.2) for $i = m = 0$ follows from (3.11)–(3.13) and hence (3.2) is proved for $i = 0$ and $m = 0, 1$. In the above arguments, if we replace u and W by $\frac{\partial^i u}{\partial t^i}$ and $\frac{\partial^i W}{\partial t^i}$ respectively, we get (3.2) for $i = 1, 2$. ∎

In the next lemma we prove that the comparison function W is uniformly bounded. This result will be used later to prove that the approximate solution U^n for $n = 1, \dots, N^t$ is also uniformly bounded. The main aim behind proving such results is to avoid using a global Lipschitz assumption on f (see (2.1)).

Lemma 3.2. $\|W\|_\infty \le C\|u\|_{r+2}$ *for each* $t \in [0, T]$.

Proof. For a fixed $t \in [0, T]$, let z_K be a linear polynomial interpolate u at the corners of K in T_h, see [10]. Then $\|z_K\|_{W^0_\infty(K)} \le \|u\|_{W^0_\infty(K)} \le \|u\|_\infty \le C\|u\|_{r+1}$ where in the last inequality we used the Sobolev embedding theorem.

Using this and the triangle inequality we get

$$\begin{aligned}\|W\|_\infty &= \max_{K\in T_h}\|W\|_{W^0_\infty(K)} \le \max_{K\in T_h}\big(\|z_K\|_{W^0_\infty(K)} + \|W - z_K\|_{W^0_\infty(K)}\big)\\ &\le C\|u\|_{r+1} + \max_{K\in T_h}\|W - z_K\|_{W^0_\infty(K)}.\end{aligned}$$

Now, the inverse inequality ($W - z_K$ is a polynomial on K), $z_K - W = \eta - (u - z_K)$ on K, the triangle inequality, (3.2), and $\|z\|_{W^0_\infty(K)} \le \|u\|_{W^0_\infty(K)} \le C\|u\|_{r+1}$, yield

$$\begin{aligned}\|W - z_K\|_{W^0_\infty(K)} &\le Ch^{-d/2}\|W - z_K\|_{L^2(K)}\\ &\le Ch^{-d/2}[\|\eta\|_{L^2(K)} + \|u\|_{L^2(K)} + \|z_K\|_{L^2(K)}]\\ &\le Ch^{-d/2}\big[\|\eta\|_0 + h^{d/2}\|u\|_{W^0_\infty(K)} + h^{d/2}\|z_K\|_{W^0_\infty(K)}\big]\\ &\le C(\|u\|_{r+2} + \|u\|_\infty) \le C\|u\|_{r+2},\end{aligned}$$

and hence the proof is complete. ∎

IV. CONVERGENCE ANALYSIS

In this section, we prove optimal-order convergence of our scheme by analyzing the error $u^n - U^n$, for $n = 0, \dots, N^t$ in the L^2 and H^1 norms, where U^0, U^1 and $\{U^n\}_{n=2}^{N^t}$ are defined by (1.13) and (1.12) respectively. In the remainder of the paper, we use the following notation

$$\xi^n = U^n - W^n, \quad n = 0, \dots, N^t. \tag{4.1}$$

It is clear that $\xi^0 = 0$. In the next Lemma, we derive the bound ξ^1. We aim to bound partially the second stage error bound ξ^n for $n = 2, \ldots, N^t$ in Theorem 4.3 after proving some needed results in Lemma 4.2.

Lemma 4.1. *If U^1 is given by* (1.13) *and $\tau \le Ch$, then*

$$\tau^{-1}\|\xi^1\|_1 \le Ch^{r+1}\int_0^{t_1}\|u_{ttt}\|_{r+1}\,ds + \tau^2\max_{0\le t\le t_1}\|u_{ttt}\|_1$$

Proof. Using (2.6), (4.1), (1.13), (3.1), (2.3), and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\|\xi^1\|_1^2 &\le Ca_h(\xi^1,\xi^1) = Ca_h(U^1 - W^1,\xi^1) = Ca_h\big(g_2^* - u^1,\xi^1\big)\\ &\le C|a_h\big(g_2^* - u^1,\xi^1\big) - a\big(g_2^* - u^1,\xi^1\big)| + C|a\big(g_2^* - u^1,\xi^1\big)|\\ &\le Ch^r\big\|g_2^* - u^1\big\|_{r+1}\|\xi^1\|_1 + C\big\|g_2^* - u^1\big\|_1\|\xi^1\|_1. \end{aligned} \tag{4.2}$$

Using the Taylor expansion, (1.2), (1.3), (1.14), we obtain for $\mathbf{x} \in \Omega$,

$$\begin{aligned}u^1(\mathbf{x}) = u(\mathbf{x},\tau) &= u(\mathbf{x},0) + \tau u_t(\mathbf{x},0) + \frac{\tau^2}{2}u_{tt}(\mathbf{x},0) + \frac{1}{2}\int_0^{t_1}(t_1 - s)^2u_{ttt}\,ds\\ &= g_1(\mathbf{x}) + \tau g_2(\mathbf{x}) + \frac{\tau^2}{2}u_{tt}(\mathbf{x},0) + \frac{1}{2}\int_0^{t_1}(t_1 - s)^2u_{ttt}\,ds\\ &= g_2^*(\mathbf{x}) + \frac{1}{2}\int_0^{t_1}(t_1 - s)^2u_{ttt}\,ds.\end{aligned}$$

Hence, (4.2) and $\tau \le Ch$ yield

$$\begin{aligned}\|\xi^1\|_1^2 &\le C\left(h^r\left\|\frac{1}{2}\int_0^{t_1}(t_1 - s)^2u_{ttt}\right\|_{r+1} + \left\|\frac{1}{2}\int_0^{t_1}(t_1 - s)^2u_{ttt}\right\|_1\right)\|\xi^1\|_1\\ &\le C\tau^2\left(h^r\int_0^{t_1}\|u_{ttt}\|_{r+1}\,ds + \int_0^{t_1}\|u_{ttt}\|_1\,ds\right)\|\xi^1\|_1\\ &\le C\tau\left(h^{r+1}\int_0^{t_1}\|u_{ttt}\|_{r+1}\,ds + \tau^2\max_{0\le t\le t_1}\|u_{ttt}\|_1\right)\|\xi^1\|_1,\end{aligned}$$

and thus the proof is complete. ∎

The next lemma is needed to derive bounds for some terms that appear in Theorem 4.3.

Lemma 4.2. *If $v^n \in S_h$, then for $4 \le j \le N^t$*

$$\tau\sum_{n=1}^{j-1}|a(u^n,\tilde{\partial}_t v^n) - a_h(u^n,\tilde{\partial}_t v^n)| \le \frac{C(u)}{\epsilon}h^{2r+2} + \left(C\tau\sum_{n=2}^{j-2}\|v^n\|_1^2 + \epsilon\sum_{i=j-1}^{j}\|v^i\|_1^2 + C\sum_{i=0}^{1}\|v^i\|_1^2\right),$$

where in the remainder of the paper, $\epsilon > 0$ and $C(u)$ denotes a generic positive constant which may depend on r and u, but which is independent of h and τ.

Proof. Using (1.10), we have

$$2\tau \sum_{n=1}^{j-1} a(u^n, \tilde{\partial}_t v^n) = \sum_{n=1}^{j-1} a(u^n, v^{n+1}) - \sum_{n=1}^{j-1} a(u^n, v^{n-1}) = \sum_{n=2}^{j} a(u^{n-1}, v^n) - \sum_{n=0}^{j-2} a(u^{n+1}, v^n)$$

$$= 2\tau \sum_{n=2}^{j-2} a(\tilde{\partial}_t u^n, v^n) + \sum_{i=j-1}^{j} a(u^{i-1}, v^i) - \sum_{i=0}^{1} a(u^{i+1}, v^i). \quad (4.3)$$

Similarly,

$$2\tau \sum_{n=1}^{j-1} a_h(u^n, \tilde{\partial}_t v^n) = 2\tau \sum_{n=2}^{j-2} a_h(\tilde{\partial}_t u^n, v^n) + \sum_{i=j-1}^{j} a_h(u^{i-1}, v^i) - \sum_{i=0}^{1} a_h(u^{i+1}, v^i). \quad (4.4)$$

Using (4.3), (4.4), the triangle inequality, and (2.2), we obtain

$$\tau \sum_{n=1}^{j-1} |a(u^n, \tilde{\partial}_t v^n) - a_h(u^n, \tilde{\partial}_t v^n)|$$

$$\leq \tau \sum_{n=2}^{j-2} |a(\tilde{\partial}_t u^n, v^n) - a_h(\tilde{\partial}_t u^n, v^n)|$$

$$+ \sum_{i=j-1}^{j} |a(u^{i-1}, v^i) - a_h(u^{i-1}, v^i)| + \sum_{i=0}^{1} |a(u^{i+1}, v^i) - a_h(u^{i+1}, v^i)|.$$

$$\leq Ch^{r+1} \left(\tau \sum_{n=2}^{j-2} \|\tilde{\partial}_t u^n\|_{r+2} \|v^n\|_1 + \sum_{i=j-1}^{j} \|u^{i-1}\|_{r+2} \|v^i\|_1 + \sum_{i=0}^{1} \|u^{i+1}\|_{r+2} \|v^i\|_1 \right). \quad (4.5)$$

Using $2\tau \tilde{\partial}_t u^n = \int_{t_{n-1}}^{t_{n+1}} u_t \, ds$, the Cauchy–Schwarz and Cauchy's inequalities, we get

$$\tau h^{r+1} \|\tilde{\partial}_t u^n\|_{r+2} \|v^n\|_1 \leq Ch^{r+1} \int_{t_{n-1}}^{t_{n+1}} \|u_t\|_{r+2} \, ds \, \|v^n\|_1$$

$$\leq Ch^{r+1} \left(\int_{t_{n-1}}^{t_{n+1}} \|u_t\|_{r+2}^2 \, ds \right)^{(1/2)} \tau^{1/2} \|v^n\|_1$$

$$\leq Ch^{2r+2} \int_{t_{n-1}}^{t_{n+1}} \|u_t\|_{r+2}^2 \, ds + \tau \|v^n\|_1^2,$$

and so

$$\tau h^{r+1} \sum_{n=2}^{j-2} \|\tilde{\partial}_t u^n\|_{r+2} \|v^n\|_1 \leq Ch^{2r+2} \int_{t_1}^{t_{j-1}} \|u_t\|_{r+2}^2 \, ds + \tau \sum_{n=2}^{j-2} \|v^n\|_1^2$$

$$\leq C(u) h^{2r+2} + \tau \sum_{n=2}^{j-2} \|v^n\|_1^2. \quad (4.6)$$

For bounding the second term in (4.5), we use the ϵ inequality to get

$$Ch^{r+1}\sum_{i=j-1}^{j}\|u^{i-1}\|_{r+2}\|v^i\|_1 \le \sum_{i=j-1}^{j}\left(\frac{C}{\epsilon}h^{2r+2}\|u^i\|_{r+2}^2+\epsilon\|v^i\|_1^2\right)$$

$$\le \frac{C(u)}{\epsilon}h^{2r+2}+\epsilon\sum_{i=j-1}^{j}\|v^i\|_1^2. \tag{4.7}$$

To bound the last term in (4.5), we use the Cauchy's inequality to obtain

$$h^{r+1}\sum_{i=0}^{1}\|u^{i+1}\|_{r+2}\|v^i\|_1 \le \sum_{i=0}^{1}\left(h^{2r+2}\|u^i\|_{r+2}^2+\|v^i\|_1^2\right) \le C(u)h^{2r+2}+\sum_{i=0}^{1}\|v^i\|_1^2. \tag{4.8}$$

Therefore desired result follows from (4.5)–(4.8). ∎

In the next theorem, we partially derive the second-stage error bound ξ^n for $n = 1, \dots, N^t$.

Theorem 4.3. *If τ is sufficiently small, then*

$$\|\xi^{n-1}\|_1^2+\|\xi^n\|_1^2 \le C\tau\sum_{i=1}^{n-1}\|\mathcal{F}(t_i)U^i-\mathcal{F}(t_i)W^i\|_h^2+C(u)(\tau^4+h^{2r+2}), \quad n=1,\dots,N^t.$$

Proof. Setting $Lu = -\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}(a_{ij}(\mathbf{x})\frac{\partial u}{\partial x_j})$. Using (1.12), $\eta = u - W$, (3.1), and the fact that $(Lu, \tilde{\partial}_t\xi^n) = a(u, \tilde{\partial}_t\xi^n)$, we obtain

$$\begin{aligned}
&\left(\partial_t^2\xi^n, \tilde{\partial}_t\xi^n\right)_h + a_h(\bar{\xi}^{n+\frac{1}{2}}, \tilde{\partial}_t\xi^n)\\
&= \left(\partial_t^2U^n, \tilde{\partial}_t\xi^n\right)_h + a_h(\bar{U}^{n+\frac{1}{2}}, \tilde{\partial}_t\xi^n) - \left(\partial_t^2W^n, \tilde{\partial}_t\xi^n\right)_h - a_h(\bar{W}^{n+\frac{1}{2}}, \tilde{\partial}_t\xi^n)\\
&= (\mathcal{F}(t_n)U^n, \tilde{\partial}_t\xi^n)_h + \left(\partial_t^2\eta^n - \partial_t^2u^n, \tilde{\partial}_t\xi^n\right)_h - a_h(\bar{u}^{n+\frac{1}{2}}, \tilde{\partial}_t\xi^n)\\
&= (\mathcal{F}(t_n)U^n - \mathcal{F}(t_n)u^n, \tilde{\partial}_t\xi^n)_h + \left(\partial_t^2\eta^n - \partial_t^2u^n, \tilde{\partial}_t\xi^n\right)_h + (\mathcal{F}(t_n)u^n - Lu^n, \tilde{\partial}_t\xi^n)_h\\
&\quad + [(Lu^n, \tilde{\partial}_t\xi^n)_h - (Lu^n, \tilde{\partial}_t\xi^n)] + [a(u^n, \tilde{\partial}_t\xi^n) - a_h(u^n, \tilde{\partial}_t\xi^n)] + a_h(u^n - \bar{u}^{n+\frac{1}{2}}, \tilde{\partial}_t\xi^n),
\end{aligned}$$

and hence from (1.1), we get

$$\left(\partial_t^2\xi^n, \tilde{\partial}_t\xi^n\right)_h + a_h(\bar{\xi}^{n+\frac{1}{2}}, \tilde{\partial}_t\xi^n) = \sum_{i=1}^{6} I_i^n, \quad n = 1, \dots, N^t - 1, \tag{4.9}$$

where

$$I_1^n = \left(u_{tt}^n - \partial_t^2u^n, \tilde{\partial}_t\xi^n\right)_h, \tag{4.10}$$

$$I_2^n = \left(\partial_t^2\eta^n, \tilde{\partial}_t\xi^n\right)_h, \tag{4.11}$$

$$I_3^n = (Lu^n, \tilde{\partial}_t\xi^n)_h - (Lu^n, \tilde{\partial}_t\xi^n), \tag{4.12}$$

$$I_4^n = a(u^n, \tilde{\partial}_t \xi^n) - a_h(u^n, \tilde{\partial}_t \xi^n), \tag{4.13}$$

$$I_5^n = a_h(u^n - \bar{u}^{n+\frac{1}{2}}, \tilde{\partial}_t \xi^n) = -\frac{1}{2} a_h\big(\tau^2 \partial_t^2 u^n, \tilde{\partial}_t \xi^n\big), \tag{4.14}$$

$$I_6^n = (\mathcal{F}(t_n)U^n - \mathcal{F}(t_n)u^n, \tilde{\partial}_t \xi^n)_h. \tag{4.15}$$

For the first term on the left-hand side of (4.9), we use (1.9) and (1.10) to obtain

$$\big(\partial_t^2 \xi^n, \tilde{\partial}_t \xi^n\big)_h = \frac{1}{2\tau}[(\partial_t \xi^n - \partial_t \xi^{n-1}, \partial_t \xi^n + \partial_t \xi^{n-1})_h] = \frac{1}{2\tau}\big[\|\partial_t \xi^n\|_h^2 - \|\partial_t \xi^{n-1}\|_h^2\big]. \tag{4.16}$$

Similarly, we get

$$\begin{aligned} a_h(\bar{\xi}^{n+\frac{1}{2}}, \tilde{\partial}_t \xi^n) &= \frac{1}{4\tau}[a_h(\xi^{n+1} + \xi^{n-1}, \xi^{n+1} - \xi^{n-1})] \\ &= \frac{1}{4\tau}[a_h(\xi^{n+1}, \xi^{n+1}) - a_h(\xi^{n-1}, \xi^{n-1})]. \end{aligned} \tag{4.17}$$

Next we bound terms on the right-hand side of (4.9). Using (4.10), the triangle inequality, (2.2), and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |I_1^n| &\le \big|\big(u_{tt}^n - \partial_t^2 u^n, \tilde{\partial}_t \xi^n\big) - \big(u_{tt}^n - \partial_t^2 u^n, \tilde{\partial}_t \xi^n\big)_h\big| + \big|\big(u_{tt}^n - \partial_t^2 u^n, \tilde{\partial}_t \xi^n\big)\big| \\ &\le Ch^{r+1}\big\|u_{tt}^n - \partial_t^2 u^n\big\|_{r+1}\|\tilde{\partial}_t \xi^n\|_0 + \big\|u_{tt}^n - \partial_t^2 u^n\big\|_0 \|\tilde{\partial}_t \xi^n\|_0. \end{aligned} \tag{4.18}$$

It follows from the Peano-Kernel Theorem (see, e.g., Theorem 1.5 in [11]) that

$$u_{tt}^n - \partial_t^2 u^n = \int_{t_{n-1}}^{t_{n+1}} G(s) u_{tttt}\, ds, \quad |G(s)| \le C\tau, \;\; s \in [t_{n-1}, t_{n+1}],$$

hence

$$\big\|u_{tt}^n - \partial_t^2 u^n\big\|_{r+1} \le C\tau \int_{t_{n-1}}^{t_{n+1}} \|u_{tttt}\|_{r+1}\, ds. \tag{4.19}$$

Using (2.5), (1.10), and the triangle inequality, we have

$$\|\tilde{\partial}_t \xi^n\|_h \le C\|\tilde{\partial}_t \xi^n\|_0 \le C \sum_{i=n-1}^{n} \|\partial_t \xi^i\|_0. \tag{4.20}$$

Now, (4.18)–(4.20), and the Cauchy–Schwarz and Cauchy's inequalities yield

$$\big|I_1^n\big| \le C\tau \int_{t_{n-1}}^{t_{n+1}} \|u_{tttt}\|_{r+1}\, ds \sum_{i=n-1}^{n} \|\partial_t \xi^i\|_0 \le C\tau^3 \int_{t_{n-1}}^{t_{n+1}} \|u_{tttt}\|_{r+1}^2\, ds + \sum_{i=n-1}^{n} \|\partial_t \xi^i\|_0^2. \tag{4.21}$$

Next we bound I_2^n in (4.11). By the hypothesis of S_h, for each $n = 1, \dots, N^t - 1$, there exists V^n in S_h such that

$$\big\|\partial_t^2 u^n - V^n\big\|_{m,T_h} \le Ch^{r+1-m}\big\|\partial_t^2 u^n\big\|_{r+1}, \quad m = 0, \dots, r+1,$$

and thus, using

$$\tau^2 \partial_t^2 u^n = \int_{t_{n-1}}^{t_{n+1}} (\tau - |s - t_n|) u_{tt}\, ds, \tag{4.22}$$

yields

$$\left\| \partial_t^2 u^n - V^n \right\|_{m,T_h} \le C\tau^{-1} h^{r+1-m} \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|_{r+1}\, ds, \quad m = 0, \ldots, r+1. \tag{4.23}$$

Using (4.11), $\eta = u - W$, the triangle inequality, (2.2), the Cauchy–Schwarz inequality, and (4.23) we obtain

$$\begin{aligned} \left|I_2^n\right| &\le \left|\left(\partial_t^2 u^n - V^n, \tilde{\partial}_t \xi^n\right)_h - \left(\partial_t^2 u^n - V^n, \tilde{\partial}_t \xi^n\right)\right| + \left|\left(\partial_t^2 u^n - V^n, \tilde{\partial}_t \xi^n\right)\right| + \left|\left(\partial_t^2 W^n - V^n, \tilde{\partial}_t \xi^n\right)_h\right| \\ &\le Ch^{r+1} \left\|\partial_t^2 u^n - V^n\right\|_{r+1,T_h} \|\tilde{\partial}_t \xi^n\|_0 + C\left\|\partial_t^2 u^n - V^n\right\|_0 \|\tilde{\partial}_t \xi^n\|_0 + \left|\left(\partial_t^2 W^n - V^n, \tilde{\partial}_t \xi^n\right)_h\right| \\ &\le C\tau^{-1} h^{r+1} \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|_{r+1}\, ds \|\tilde{\partial}_t \xi^n\|_0 + \left|\left(\partial_t^2 W^n - V^n, \tilde{\partial}_t \xi^n\right)_h\right|. \end{aligned} \tag{4.24}$$

To bound $|(\partial_t^2 W^n - V^n, \tilde{\partial}_t \xi^n)_h|$ we use the Cauchy–Schwarz inequality, (2.5), $W = u - \eta$, the triangle inequality, (4.22) with η in place of u, and (3.2) to obtain

$$\begin{aligned} \left|\left(\partial_t^2 W^n - V^n, \tilde{\partial}_t \xi^n\right)_h\right| &\le C\left\|\partial_t^2 W^n - V^n\right\|_h \|\tilde{\partial}_t \xi^n\|_h \le C\left\|\partial_t^2 W^n - V^n\right\|_0 \|\tilde{\partial}_t \xi^n\|_0 \\ &\le C\left(\left\|\partial_t^2 u^n - V^n\right\|_0 + \left\|\partial_t^2 \eta^n\right\|_0\right) \|\tilde{\partial}_t \xi^n\|_0 \\ &\le C\tau^{-1} \int_{t_{n-1}}^{t_{n+1}} (h^{r+1} \|u_{tt}\|_{r+1} + \|\eta_{tt}\|_0)\, ds\, \|\tilde{\partial}_t \xi^n\|_0 \\ &\le C\tau^{-1} h^{r+1} \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|_{r+2}\, ds\, \|\tilde{\partial}_t \xi^n\|_0. \end{aligned} \tag{4.25}$$

Hence, (4.20), (4.24), (4.25) the Cauchy–Schwarz and Cauchy's inequalities give

$$\begin{aligned} \left|I_2^n\right| &\le C\tau^{-1} h^{r+1} \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|_{r+2}\, ds \|\tilde{\partial}_t \xi^n\|_0 \\ &\le C\tau^{-1} h^{r+1} \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|_{r+2}\, ds \sum_{i=n-1}^{n} \|\partial_t \xi^i\|_0 \\ &\le C\tau^{-1/2} h^{r+1} \left(\int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|_{r+2}^2\, ds\right)^{1/2} \sum_{i=n-1}^{n} \|\partial_t \xi^i\|_0 \\ &\le C\tau^{-1} h^{2r+2} \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|_{r+2}^2\, ds + \sum_{i=n-1}^{n} \|\partial_t \xi^i\|_0^2. \end{aligned} \tag{4.26}$$

To bound I_3^n given in (4.12), we use (1.1), (2.2), (4.20), and Cauchy's inequality to obtain

$$\begin{aligned}\left|I_3^n\right| &\le \left|\left(u_{tt}^n, \tilde{\partial}_t\xi^n\right) - \left(u_{tt}^n, \tilde{\partial}_t\xi^n\right)_h\right| + |(\mathcal{F}(t_n)u^n, \tilde{\partial}_t\xi^n) - (\mathcal{F}(t_n)u^n, \tilde{\partial}_t\xi^n)_h| \\ &\le Ch^{r+1}\left(\left\|u_{tt}^n\right\|_{r+1} + \|\mathcal{F}(t_n)u^n\|_{r+1}\right)\|\tilde{\partial}_t\xi^n\|_0 \\ &\le Ch^{2r+2}\left(\left\|u_{tt}^n\right\|_{r+1}^2 + \|\mathcal{F}(t_n)u^n\|_{r+1}^2\right) + \sum_{i=n-1}^{n}\|\partial_t\xi^i\|_0^2. \end{aligned} \tag{4.27}$$

The bound of I_4 will be given later. For bounding I_5^n, we use (4.14), the triangle inequality, (2.3), integration by parts, (4.22), the Cauchy–Schwarz, inverse, and Cauchy's inequalities, and (4.20) to get

$$\begin{aligned}\left|I_5^n\right| &\le \left|a\left(\tau^2\partial_t^2u^n, \tilde{\partial}_t\xi^n\right) - a_h\left(\tau^2\partial_t^2u^n, \tilde{\partial}_t\xi^n\right)\right| + \left|a\left(\tau^2\partial_t^2u^n, \tilde{\partial}_t\xi^n\right)\right| \\ &\le Ch^r\left\|\tau^2\partial_t^2u^n\right\|_{r+1}\|\tilde{\partial}_t\xi^n\|_1 + \left|\left(\tau^2L\partial_t^2u^n, \tilde{\partial}_t\xi^n\right)\right| \\ &\le Ch^{r-1}\tau\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_{r+1}\,ds\,\|\tilde{\partial}_t\xi^n\|_0 + C\tau\int_{t_{n-1}}^{t_{n+1}}\|Lu_{tt}\|_0\,ds\,\|\tilde{\partial}_t\xi^n\|_0 \\ &\le C\tau^{3/2}\left[h^{r-1}\left(\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_{r+1}^2\,ds\right)^{1/2} + \left(\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_2^2\,ds\right)^{1/2}\right]\|\tilde{\partial}_t\xi^n\|_0 \\ &\le C\tau^3\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_{r+1}^2\,ds + \sum_{i=n-1}^{n}\|\partial_t\xi^i\|_0^2. \end{aligned} \tag{4.28}$$

To bound $|I_6^n|$ given in (4.15), we use the triangle and Cauchy–Schwarz inequalities, (2.1), (2.5), (4.20), and Cauchy's inequality, to obtain

$$\begin{aligned}\left|I_6^n\right| &\le |(\mathcal{F}(t_n)u^n - \mathcal{F}(t_n)W^n, \tilde{\partial}_t\xi^n)_h| + |(\mathcal{F}(t_n)U^n - \mathcal{F}(t_n)W^n, \tilde{\partial}_t\xi^n)_h| \\ &\le \|\mathcal{F}(t_n)u^n - \mathcal{F}(t_n)W^n\|_h\|\tilde{\partial}_t\xi^n\|_h + \|\mathcal{F}(t_n)U^n - \mathcal{F}(t_n)W^n\|_h\|\tilde{\partial}_t\xi^n\|_h \\ &\le C(\|\eta^n\|_h + \|\mathcal{F}(t_n)U^n - \mathcal{F}(t_n)W^n\|_h)\sum_{i=n-1}^{n}\|\partial_t\xi^i\|_0 \\ &\le C\left[\|\eta^n\|_h^2 + \|\mathcal{F}(t_n)U^n - \mathcal{F}(t_n)W^n\|_h^2\right] + \sum_{i=n-1}^{n}\|\partial_t\xi^i\|_0^2. \end{aligned} \tag{4.29}$$

Adding and subtracting $\|\eta^n\|_0^2$, (2.4), (3.2), and the Cauchy–Schwarz inequality yield

$$\begin{aligned}\|\eta^n\|_h^2 &\le |\|\eta^n\|_0^2 - \|\eta^n\|_h^2| + \|\eta^n\|_0^2 \\ &\le Ch^{r+1}\sum_{K\in T_h}\|[\eta^n]^2\|_{W_1^{r+1}(K)} + Ch^{2r+2}\|u^n\|_{r+1}^2 \\ &\le Ch^{r+1}\sum_{K\in T_h}\left(\sum_{j=0}^{r+1}\|\eta^n\|_{H^j(K)}\|\eta^n\|_{H^{r+1-j}(K)}\right) + Ch^{2r+2}\|u^n\|_{r+1}^2 \end{aligned}$$

$$\leq Ch^{r+1}\sum_{j=0}^{r+1}\|\eta^n\|_{j,T_h}\|\eta^n\|_{r+1-j,T_h}+Ch^{2r+2}\|u^n\|_{r+2}^2$$
$$\leq Ch^{2r+2}\|u^n\|_{r+2}^2. \tag{4.30}$$

Using the triangle inequality, (3.4), the inverse inequality, (3.2), we obtain

$$\|\eta^n\|_{j,T_h}\leq\left\|u^n-u_h^n\right\|_j+\left\|u_h^n-W^n\right\|_{j,T_h}\leq Ch^{r+1-j}\|u^n\|_{r+1}+Ch^{-j}\left\|u_h^n-W^n\right\|_0$$
$$\leq Ch^{r+1-j}\|u^n\|_{r+1}+Ch^{-j}\left(\left\|u_h^n-u^n\right\|_0+\|\eta^n\|_0\right)\leq Ch^{r+1-j}\|u^n\|_{r+2}. \tag{4.31}$$

Similarly, we get

$$\|\eta^n\|_{j,T_h}\leq Ch^j\|u^n\|_{r+2}. \tag{4.32}$$

Thus (4.29)–(4.32) give

$$\left|I_6^n\right|\leq C\left[\|\mathcal{F}(t_n)U^n-\mathcal{F}(t_n)W^n\|_h^2+h^{2r+2}\|u^n\|_{r+2}^2\right]+\sum_{i=n-1}^{n}\|\partial_t\xi^i\|_0^2. \tag{4.33}$$

Using (4.9), (4.16), (4.17), (4.21), (4.26)–(4.28), (4.33), and multiplying through by 4τ, we obtain for $n=1,\ldots,N^t-1$,

$$2\left[\|\partial_t\xi^n\|_h^2-\|\partial_t\xi^{n-1}\|_h^2\right]+a_h(\xi^{n+1},\xi^{n+1})-a_h(\xi^{n-1},\xi^{n-1})$$
$$\leq C\left(C_1(u)[\tau^4+h^{2r+2}]+C_2(u)\tau h^{2r+2}+\tau I_4^n+\tau\|\mathcal{F}(t_n)U^n-\mathcal{F}(t_n)W^n\|_h^2+\tau\sum_{i=n-1}^{n}\|\partial_t\xi^i\|_0^2\right), \tag{4.34}$$

where

$$C_1(u)=\int_{t_{n-1}}^{t_{n+1}}\left(\|u_{tttt}\|_{r+1}^2+\|u_{tt}\|_{r+2}^2\right)ds,\quad C_2(u)=\left\|u_{tt}^n\right\|_{r+1}^2+\|u^n\|_{r+2}^2+\|\mathcal{F}(t^n)u^n\|_{r+1}^2.$$

Using Lemma 4.2 and $\xi^0=0$, we get

$$\tau\sum_{n=1}^{k-1}I_4^n\leq\frac{C(u)}{\epsilon}h^{2r+2}+\left(C\tau\sum_{n=2}^{k-2}\|\xi^n\|_1^2+\epsilon\sum_{i=k-1}^{k}\|\xi^i\|_1^2+C\|\xi^1\|_1^2\right). \tag{4.35}$$

Summing both sides of (4.34) for $n=1,\ldots,k-1$, where $2\leq k\leq N^t$, using (2.5), (2.6), (4.35), $\xi^0=0$, and taking ϵ sufficiently small, we obtain

$$\|\partial_t\xi^{k-1}\|_0^2+\sum_{i=k-1}^{k}\|\xi^i\|_1^2\leq C\tau\sum_{n=1}^{k-1}\|\mathcal{F}(t_n)U^n-\mathcal{F}(t_n)W^n\|_h^2$$
$$+C(u)\left[\tau^4+h^{2r+2}+\tau^{-2}\|\xi^1\|_1^2+\tau\sum_{n=0}^{k-1}\left(\|\partial_t\xi^n\|_0^2+\|\xi^n\|_1^2\right)\right].$$

Hence, for τ sufficiently small, the discrete analogue of Gronwall's inequality and Lemma 4.1 complete the proof. ■

In the next theorem we prove that the quadrature CNFE solution given by (1.12) and (1.13) is uniformly bounded and its rate of convergence to the exact solution is optimal in both time and space.

Theorem 4.4. *If h and τ are sufficiently small and $\tau \le C \min\{h^{\epsilon+d/4}, h\}$ where $\epsilon > 0$ is independent of τ and h, then*

$$\max_{0\le n\le N^t} \|u^n - U^n\|_m \le C(u)(\tau^2 + h^{r+1-m}), \quad m = 0, 1.$$

Proof. It follows from Lemma 3.2 that

$$\|W\|_\infty \le C_1(u), \quad t \in [0, T]. \tag{4.36}$$

Using (4.1), the triangle and inverse inequalities, (4.36), Lemma 4.1, and $\tau \le Ch^{\epsilon+d/4}$, we obtain

$$\|U^1\|_\infty \le \|W^1\|_\infty + \|\xi^1\|_\infty \le \|W^1\|_\infty + Ch^{-d/2}\|\xi^1\|_0 \le C_2(u). \tag{4.37}$$

Next, we use mathematical induction to prove that

$$\|U^n\|_\infty \le C_3(u), \quad n = 1, \ldots, N^t - 1, \tag{4.38}$$

where

$$C_3(u) = \max\{C_1(u), C_2(u)\}. \tag{4.39}$$

It follows from (4.37) and (4.39) that $\|U^1\|_\infty \le C_3(u)$. Assuming that (4.38) is true for $n = 1, \ldots, l$, where l is such that $1 \le l \le N^t - 2$, we will show that (4.38) is also true for $n = l+1$. To this end, the induction hypothesis, (2.1), (2.5), the relation $U^n - W^n = \xi^n - \eta^n + (u^n - W^n)$, (3.2), and (3.4), yield

$$\begin{aligned} \|\mathcal{F}(t_i)U^i - \mathcal{F}(t_i)W^i\|_h &\le C\|U^i - W^i\|_0 \le C(\|\xi^i\|_0 + \|\eta^i\|_0 + \|u^i - W^i\|_0) \\ &\le C\big(\|\xi^i\|_0^2 + h^{2r+2}\|u^i\|_{r+2}^2\big), \quad i = 1, \ldots, l, \end{aligned}$$

and hence, by applying Theorem 4.3 we observe that

$$\|\xi^{k-1}\|_1^2 + \|\xi^k\|_1^2 \le C\tau \sum_{i=1}^{k-1} \|\xi^i\|_1^2 + C(u)(\tau^4 + h^{2r+2}), \quad k = 1, \ldots, l+1. \tag{4.40}$$

Now, an application to the discrete analogue of Gronwall's inequality yields

$$\sum_{i=n}^{n+1} \|\xi^i\|_1^2 \le C(u)(\tau^4 + h^{2r+2}), \quad n = 0, \ldots, l. \tag{4.41}$$

Using the inverse inequality, (4.41), $\tau \le Ch^{\epsilon+d/4}$, and taking h sufficiently small, we obtain

$$\|\xi^{l+1}\|_\infty \le Ch^{-d/2}\|\xi^{l+1}\|_0 \le C(u)(h^{2\epsilon} + h) \le \frac{C_3(u)}{2}. \tag{4.42}$$

TABLE I. Optimal-order L^2 and H^1 convergence rates for $r = 1$ and $h = 1/N = \tau$.

	$\ell = 1$				$\ell = 2$			
N	$\|e\|_{0,\infty}$	$R(L^2)$	$\|e\|_{1,\infty}$	$R(H^1)$	$\|e\|_{0,\infty}$	$R(L^2)$	$\|e\|_{1,\infty}$	$R(H^1)$
40	8.11 e−04		7.12 e−02		4.28 e−04		7.12 e−02	
80	2.03 e−04	1.99	3.56 e−02	0.99	1.06 e−04	2.00	3.56 e−02	0.99
160	5.07 e−05	1.99	1.78 e−02	0.99	2.66 e−05	2.00	1.78 e−02	1.00
320	1.26 e−05	2.00	8.90 e−03	0.99	6.60 e−06	2.00	8.90 e−03	1.00
640	3.05 e−06	2.05	4.45 e−03	0.99	1.49 e−06	2.14	4.45 e−03	1.00

Using $\xi^{l+1} = U^{+1} - W^{l+1}$, the triangle inequality, (4.36), (4.39), and (4.42) yield

$$\|U^{l+1}\|_\infty \le \|W^{l+1}\|_\infty + \|\xi^{l+1}\|_\infty \le \frac{C_3(u)}{2} + \frac{C_3(u)}{2} = C_3(u),$$

which completes the proof of (4.38) by mathematical induction.

Finally, using (4.38) and following the derivation of (4.40) and (4.41), we obtain

$$\sum_{i=n}^{n+1} \|\xi^i\|_1^2 \le C(u)(\tau^4 + h^{2r+2}), \quad n = 0, \dots, N^t - 1. \tag{4.43}$$

Since $u^n - U^n = \eta^n - \xi^n$, the desired result follows from the triangle inequality, (3.2), (4.43). ■

V. NUMERICAL EXPERIMENTS

In this section, we apply our fully-discrete CNFE scheme (1.12) and (1.13) to a semilinear example problem of the form (1.1)–(1.4) in one dimensional space. We take

$$T = 1, \quad a_{11}(x) = 1, \quad \Omega = (0, 1), \quad u(0, t) = 0 = u(1, t), \quad u(x, 0) = u_t(x, 0) = \sin \pi x,$$

and the inhomogeneous term is

$$f(x, t, u) = t^3 u^2 + [\sin(t) + \cos(t)] \sin(\pi x)(\pi^2 - 1) - t^3([\sin(t) + \cos(t)] \sin(\pi x))^2.$$

In our example, the exact solution is given by $u(x, t) = [\sin(t) + \cos(t)] \sin(\pi x)$.

It is clear that the nonlinearity of the source term f satisfies the Lipschitz condition locally only and this is what was assumed in (2.1).

TABLE II. Optimal-order L^2 and H^1 convergence rates for $r = \ell = 2$ and $h = 1/N = \tau^{3/2}$.

N	$\|e\|_{0,\infty}$	$R(L^2)$	$\|e\|_{1,\infty}$	$R(H^1)$
36	2.1267 e−05		8.9566 e−04	
49	8.4576 e−06	2.9908	4.8150 e−04	2.0131
64	3.8027 e−06	2.9930	2.8338 e−04	1.9848
81	1.8781 e−06	2.9947	1.7621 e−04	2.0169
100	9.9443 e−07	3.0175	1.1558 e−04	2.0011

TABLE III. Optimal-order L^2 and H^1 convergence rates for $r = \ell = 3$ and $h = 1/N = \tau^{1/2}$.

N	$\|e\|_{0,\infty}$	$R(L^2)$	$\|e\|_{1,\infty}$	$R(H^1)$
36	5.8815 e−07		6.8205 e−06	
49	1.7134 e−07	4.0003	2.6542 e−06	3.0612
64	5.8912 e−08	3.9976	1.1806 e−06	3.0333
81	2.2746 e−08	4.0397	5.8057 e−07	3.0130
100	9.8535 e−09	3.9701	3.0787 e−07	3.0102

In our numerical experiments, for several values of N, we used uniform spatial and time partition with the step size $h = 1/N$ and $\tau = 1/\sqrt{N^{r+1}}$ to check the rate of convergence ($R(H^k)$) in the H^k norm for $k = 0, 1$. We chose the approximation space $S_h \subset H_0^1(0, 1)$ to be the space of all continuous piecewise polynomials of degree at most r.

Tables I–III show the errors $\|e\|_{k,\infty} = \max_{0\le n\le N} \|u^n - U^n\|_k$ in the H^k norm for $k = 0, 1$. We calculated the errors $\|e\|_{k,\infty}$ using $r + 1$ Gauss quadrature points on each interval of the finest spatial mesh.

For a fixed h, we computed U^0 and U^1 by solving (1.13). Then, we computed the sparse LU-factorization of the time-independent symmetric matrix resulting from (1.12). We used this factorization in (1.12) to compute U^{n+1}.

In our calculation, we choose the nodes $\{\sigma_{j,K}\}_{j=1}^{\ell}$ in (1.11) to be the Gauss quadrature points on each cell and $\{w_{j,K}\}_{j=1}^{\ell}$ are the associated weights. In the current case, our analysis requires ℓ to be greater than r in order to obtain the quadrature error formula (2.2) which is needed to prove the optimal-order convergence rates of the quadrature CNFE scheme in the H^k norm for $k = 0, 1$, see Theorem 4.4. However, for $\ell = r$ results in Tables I–III confirm the optimal convergence results given in Theorem 4.4. Theoretically it is difficult to prove such practically expected results for semilinear hyperbolic problems. For $\ell = r + 1$, all the assumptions which are given in Section II on the quadrature rule are satisfied, and hence we expect to obtain optimal order convergence rates in L_2 and H^1 norms. Such results are demonstrated numerically below. It is clear from Table I that for $\ell = r + 1$, we obtained better convergence rates and error bounds.

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