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DISCONTINUOUS GALERKIN METHOD FOR AN EVOLUTION EQUATION WITH A MEMORY TERM OF POSITIVE TYPE

KASSEM MUSTAPHA AND WILLIAM MCLEAN

ABSTRACT. We consider an initial value problem for a class of evolution equation incorporating a memory term with a weakly singular kernel bounded by $C(t-s)^{\alpha-1}$, where $0 < \alpha < 1$. For the time discretization we apply the discontinuous Galerkin method using piecewise polynomials of degree at most $q-1$, for $q = 1$ or 2. For the space discretization we use continuous piecewise-linear finite elements. The discrete solution satisfies an error bound of order $k^q + h^2\ell(k)$, where k and h are the mesh sizes in time and space, respectively, and $\ell(k) = \max(1, \log k^{-1})$. In the case $q = 2$, we prove a higher convergence rate of order $k^3 + h^2\ell(k)$ at the nodes of the time mesh. Typically, the partial derivatives of the exact solution are singular at $t = 0$, necessitating the use of non-uniform time steps. We compare our theoretical error bounds with the results of numerical computations.

1. Introduction

We study the discretization in time and space of an initial value problem [1, 3, 8, 10, 11, 17, 15]

$$\frac{\partial u}{\partial t} + \mathcal{B}Au = f(t) \quad \text{for } 0 < t < T, \quad \text{with } u(0) = u_0, \tag{1.1}$$

where $\mathcal{B}$ denotes a Volterra integral operator

$$\mathcal{B}v(t) = \int_0^t \beta(t,s)v(s)\,ds$$

and where A is a self-adjoint linear operator with domain $D(A)$ in a real Hilbert space $\mathbb{H}$. We assume that A has a complete eigensystem $\{\lambda_m, \phi_m\}_{m=1}^{\infty}$ such that $0 \le \lambda_0 \le \lambda_1 \le \cdots$ and $\lambda_m \to \infty$ as $m \to \infty$. Thus, A is positive semidefinite. The solution u and source term f take values in $\mathbb{H}$, and the initial data u_0 is an element of $\mathbb{H}$. We let $\langle u, v\rangle$ denote the inner product of u and v in $\mathbb{H}$, and define the bilinear form

$$A(u,v) = \langle Au, v\rangle = \sum_{m=1}^{\infty} \lambda_m \langle u, \phi_m\rangle\langle \phi_m, v\rangle \quad \text{for } u,\, v \in D(A^{1/2}).$$

Concretely, one may take $\mathbb{H} = L_2(\Omega)$ for a bounded, Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ and $A = -\nabla^2$ subject to homogeneous Dirichlet or Neumann boundary conditions.

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In this case, $u = u(x,t)$, $f = f(x,t)$ and $u_0 = u_0(x)$ for $x \in \Omega$ and $t > 0$, with $A(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$.

Throughout the paper, we assume the kernel β to be real-valued and strictly positive definite, that is

$$\int_0^T v(t) \int_0^t \beta(t,s) v(s) \, ds \, dt \geq 0 \quad \text{for all } v \in L_\infty([0,T],\mathbb{R}), \tag{1.2}$$

with equality if and only if v is zero almost everywhere on $[0,T]$. In addition, β may be at worst weakly singular, that is,

$$|\beta(t,s)| \leq C(t-s)^{\alpha-1} \quad \text{for } 0 < s < t < \infty, \text{ with } 0 < \alpha \leq 1,$$

and we assume for simplicity that $\beta(t,s)$ is continuous for $t \neq s$. Of particular interest is the choice

$$\beta(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, \tag{1.3}$$

which makes $\mathcal{B}$ the Riemann–Liouville fractional integration operator of order α and means that the evolution equation in (1.1) is a fractional wave equation [17].

A standard approach [8, 11, 15] to the time discretization of (1.1) uses a combination of finite differences and quadrature. If the kernel $\beta(t,s)$ depends only on the difference $t-s$, then convolution quadrature [3, 9] is a natural choice, allowing the use of fast summation techniques [16]. Another approach [6, 7, 12, 13], again suitable for a convolution kernel, achieves spectral accuracy via numerical inversion of the Laplace transform of the solution. In the present work, we instead apply the discontinuous Galerkin method using piecewise polynomials of degree at most $q-1$, for $q = 1$ or 2.

Since their inception in the early 1970s, discontinuous Galerkin methods have found numerous applications [2], including for the time discretization of parabolic problems [5]. Their advantages include excellent stability properties even for highly non-uniform meshes and suitability for adaptive refinement based on a posteriori error estimates [4] to handle problems with low regularity. Adolfsson, Enelund and Larsson [1] proved a priori and a posteriori error estimates for the piecewise-constant ($q = 1$) discontinuous Galerkin method applied to (1.1), incorporating the use of sparse quadrature to reduce the computational cost of the algorithm. Here, we focus on the piecewise-linear case ($q = 2$) and consider only a priori error bounds.

To set up the time discretization, we begin with a (possibly non-uniform) partition

$$0 = t_0 < t_1 < \cdots < t_N = T \tag{1.4}$$

of the interval $[0,T]$, and denote the nth step-size by $k_n = t_n - t_{n-1}$ and the maximum step-size by $k = \max_{1 \leq n \leq N} k_n$. For $q \geq 1$, we let $\mathbb{P}_q$ denote the space of polynomials of degree strictly less than q with coefficients in $D(A^{1/2})$. For $q = 1$ or 2, our trial space is the set of piecewise-polynomials

$$\mathcal{W}_q = \{\, U : U|_{I_n} \in \mathbb{P}_q \text{ for } 1 \leq n \leq N \,\}, \quad \text{where } I_n = (t_{n-1}, t_n].$$

We follow the usual convention that a function $U \in \mathcal{W}_q$ is left-continuous at each time level t_n, writing

$$U^n = U(t_n) = U(t_n^-), \quad U_+^n = U(t_n^+), \quad [U]^n = U_+^n - U^n. \tag{1.5}$$

For any continuous test function $v : [0,T] \to D(A^{1/2})$, the solution u of (1.1) satisfies

$$\int_{t_{n-1}}^{t_n} \big[\langle u'(t), v(t)\rangle + A\big(\mathcal{B}u(t), v(t)\big)\big]\, dt = \int_{t_{n-1}}^{t_n} \langle f(t), v(t)\rangle\, dt.$$

By comparison, given $U(t)$ for $0 \le t \le t_{n-1}$, the discontinuous Galerkin method determines $U \in \mathcal{W}_q$ on I_n, by requiring that

$$\begin{aligned}\langle U_+^{n-1}, X_+^{n-1}\rangle + \int_{t_{n-1}}^{t_n} \big[\langle U'(t), X(t)\rangle + A\big(\mathcal{B}U(t), X(t)\big)\big]\, dt \\ = \langle U^{n-1}, X_+^{n-1}\rangle + \int_{t_{n-1}}^{t_n} \langle f(t), X(t)\rangle\, dt\end{aligned} \tag{1.6}$$

for all $X \in \mathbb{P}_q(I_n)$. This time-stepping procedure starts from $U^0 \approx u_0$, and after N steps yields the numerical solution $U(t)$ for $0 \le t \le t_N$.

For the piecewise-constant case $q = 1$, since $U'(t) = 0$ and $U(t) = U^n = U_+^{n-1}$ for $t \in I_n$, the discontinuous Galerkin method (1.6) amounts to a generalized backward-Euler scheme

$$\frac{U^n - U^{n-1}}{k_n} + \mathcal{B}^n(AU) = \bar f^n, \tag{1.7}$$

where

$$\mathcal{B}^n(AU) = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_0^t \beta(t,s)AU(s)\, ds\, dt = \sum_{j=1}^{n} \omega_{nj} AU^j k_j,$$

$$\bar f^n = \frac{1}{k_n}\int_{t_{n-1}}^{t_n} f(t)\, dt, \qquad \omega_{nj} = \frac{1}{k_n k_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{\min(t,t_{j-1})} \beta(t,s)\, ds\, dt.$$

Thus, at each time step we must solve an “elliptic” problem

$$U^n + k_n^2 \omega_{nn} AU^n = U^{n-1} + k_n \bar f_n - k_n \sum_{j=1}^{n-1} \omega_{nj} AU^j k_j.$$

For the piecewise-linear case $q = 2$, we define

$$\psi_n^1(t) = \frac{t_n - t}{k_n} \quad \text{and} \quad \psi_n^2(t) = \frac{t - t_{n-1}}{k_n},$$

and use the representation

$$U(t) = U_+^{n-1}\psi_n^1(t) + U^n \psi_n^2(t) \quad \text{for } t \in I_n.$$

By choosing $X(t) = \psi_n^p(t)V$ in (1.6) for $p \in \{1,2\}$ a vector $V \in \mathbb{H}$ (independent of t), we arrive at the 2×2 system

(1.8)

$$(\tfrac12 + \omega_{nn}^{11}A)U_+^{n-1} + (\tfrac12 + \omega_{nn}^{12}A)U^n = U^{n-1} + f^{n1} - \sum_{j=1}^{n-1}\big(\omega_{nj}^{11}AU_+^{j-1} + \omega_{nj}^{12}AU^j\big),$$

$$(-\tfrac12 + \omega_{nn}^{21}A)U_+^{n-1} + (\tfrac12 + \omega_{nn}^{22}A)U^n = f^{n2} - \sum_{j=1}^{n-1}\big(\omega_{nj}^{21}U_+^{j-1} + \omega_{nj}^{22}U^j\big),$$

where

$$\omega_{nj}^{pr} = \int_{t_{n-1}}^{t_n} \psi_n^p(t) \int_{t_{j-1}}^{\min(t,t_j)} \beta(t,s)\psi_j^r(s)\,ds\,dt \quad \text{and} \quad f^{np} = \int_{t_{n-1}}^{t_n} f(t)\psi_n^p(t)\,dt.$$

For a general q, we would have to solve a $q \times q$ system.

Regularity results in [10, 11] show, for the specific weakly singular kernel (1.3) and under reasonable assumptions on the data u_0 and $f(t)$, that there exist constants σ and M, with $0 < \sigma \le 1$, such that the exact solution of (1.1) satisfies

$$t\|Au'(t)\| + t^2\|Au''(t)\| \le Mt^{\sigma-1} \quad \text{for } 0 < t \le T, \tag{1.9}$$

and

$$\|u'(t)\| + t\|u''(t)\| \le Mt^{\sigma-1} \quad \text{for } 0 < t \le T. \tag{1.10}$$

This singular behaviour as $t \to 0^+$ may lead to sub-optimal convergence rates if we work with quasi-uniform time meshes. We therefore assume that, for a fixed $\gamma \ge 1$,

$$k_n \le C_\gamma k \min(1, t_n^{1-1/\gamma}) \quad \text{and} \quad t_n \le C_\gamma t_{n-1} \quad \text{for } 2 \le n \le N, \tag{1.11}$$

with

$$c_\gamma k^\gamma \le k_1 \le C_\gamma k^\gamma. \tag{1.12}$$

For instance, we may choose

$$t_n = (n/N)^\gamma T \quad \text{for } 0 \le n \le N. \tag{1.13}$$

We show in Theorem 3.2 that the error $\|U(t) - u(t)\|$ is of order k^q, uniformly for $0 \le t \le T$, provided $\gamma > q/\sigma$ and the initial data satisfy $\|U^0 - u_0\| = O(k^q)$. However, for a quasi-uniform mesh our bound yields a poorer convergence rate of order k^σ. McLean, Thomée and Wahlbin [14] proved this result in the case $q = 1$ using the backward Euler formulation (1.7).

In the piecewise-linear case $q = 2$, faster convergence than $O(k^2)$ is possible at the nodal points t_n. We prove that if β and u are smooth then the nodal error $\|U^n - u(t_n)\|$ is $O(k^3)$. For the weakly-singular kernel (1.3) and for u satisfying (1.9), the nodal error is again $O(k^3)$ provided $\gamma > 3/(\sigma+\alpha)$; see Corollary 4.2. Compare these results with those of Eriksson, Johnson and Thomée [5] for the classical parabolic problem that arises if one takes $\mathcal{B}v = v$ in (1.1): the error is then $O(k^q)$ everywhere on $[0,T]$ and is $O(k^{2q-1})$ at the nodes, for a general $q \ge 1$.

In the concrete setting where $\mathbb{H} = L_2(\Omega)$ and $A = -\nabla^2$, we discretize in space using standard, continuous, piecewise-linear finite elements on a quasi-uniform partition of the domain Ω to obtain a numerical solution U_h. Under the additional regularity assumptions

$$\|u_0\|_2 \le M \quad \text{and} \quad \|u(t)\|_2 + t\|u'(t)\|_2 \le M \quad \text{for } 0 < t \le T, \tag{1.14}$$

where $\|v\|_2 = \|v\|_{H^2(\Omega)}$, we show in Theorem 5.2 that, with

$$\ell(k) = \max(1, \log k^{-1}), \tag{1.15}$$

the error $\|U_h(t) - u(t)\|$ is of order $k^q + h^2\ell(k)$, uniformly for $0 \le t \le T$, provided $\gamma > q/\sigma$. Our final result, Corollary 5.4, establishes an improved error bound of order $k^3 + h^2\ell(k)$ for U_h at the nodes $t = t_n$, when $q = 2$ and β is as in (1.3).

The concluding section of the paper presents the results of some numerical computations that confirm our theoretical error bounds.

2. Stability

An energy argument based on the positive-semidefiniteness of $\mathcal{B}$ and A implies the existence and uniqueness of a mild solution $u \in C([0,T];\mathbb{H})$ to the continuous problem (1.1), and yields a stability estimate [11],

$$\|u(t)\| \le \|u_0\| + 2\int_0^t \|f(s)\|\,ds.$$

To state an analogous result for the discrete problem (1.6), we introduce the notation

$$\|U\|_J = \sup_{t\in J}\|U(t)\|$$

for any subinterval $J \subseteq [0,T]$, and put $J_n = (0,t_n] = \bigcup_{j=1}^n I_j$. Note that the proof makes no assumptions on the mesh (1.4).

Theorem 2.1. *Let $q \in \{1,2\}$. Given $U^0 \in \mathbb{H}$ and $f \in L_1\big((0,T);\mathbb{H}\big)$, there exists a unique $U \in \mathcal{W}_q$ satisfying (1.6) for $n = 1, 2, \ldots, N$. Furthermore, $U(t) \in D(A)$ for $t > 0$ and*

$$\|U\|_{J_n} \le 12\bigg(\|U^0\| + \int_0^{t_n}\|f(t)\|\,dt\bigg) \quad \textit{for } n \ge 1.$$

Proof. Recall the notation (1.5) and assume for the moment that U exists. By choosing $X = U$ in (1.6) and using $\langle U'(t), U(t)\rangle = (d/dt)\|U(t)\|^2/2$, we obtain

$$\tfrac12\big(\|U^j\|^2 + \|U_+^{j-1}\|^2\big) + \int_{t_{j-1}}^{t_j} A\big(\mathcal{B}U(t), U(t)\big)\,dt = \langle U^{j-1}, U_+^{j-1}\rangle + \int_{t_{j-1}}^{t_n}\langle f(t), U(t)\rangle\,dt.$$

Since (1.2) implies

$$\begin{aligned}\sum_{j=1}^n \int_{t_{j-1}}^{t_j} A\big(\mathcal{B}U(t), U(t)\big)\,dt &= \int_0^{t_n}\int_0^t \beta(t,s)A\big(U(s),U(t)\big)\,ds\,dt\\ &= \sum_{m=1}^\infty \lambda_m \int_0^{t_n}\int_0^t \beta(t,s)\langle U(s),\phi_m\rangle\,ds\,\langle\phi_m, U(t)\rangle\,dt \ge 0,\end{aligned}$$

we see that

$$\sum_{j=1}^n\big(\|U^j\|^2 + \|U_+^{j-1}\|^2\big) \le 2\sum_{j=1}^n\langle U^{j-1}, U_+^{j-1}\rangle + 2\int_0^{t_n}\langle f(t), U(t)\rangle\,dt,$$

so

$$\|U^n\|^2 + \|U_+^0\|^2 + \sum_{j=1}^{n-1}\big(\|U^j\|^2 - 2\langle U^j, U_+^j\rangle + \|U_+^j\|^2\big) \le 2\langle U^0, U_+^0\rangle + 2\int_0^{t_n}\langle f(t), U(t)\rangle\,dt$$

and hence

$$\|U^n\|^2 + \|U_+^0\|^2 + \sum_{j=1}^{n-1}\|[U]^j\|^2 \le 2\bigg(\|U^0\|\|U_+^0\| + \int_0^{t_n}\|f(t)\|\|U(t)\|\,dt\bigg). \tag{2.1}$$

Our assumption that $q \in \{1, 2\}$ implies $\|U\|_{I_n} \le \|U^n\| + \|U_+^{j-1}\|$ so

$$\begin{aligned}\|U\|_{I_1}^2 &\le \big(\|U^1\| + \|U_+^0\|\big)^2 \le 2\big(\|U^1\|^2 + \|U_+^0\|^2\big)\\ &\le 4\bigg(\|U^0\|\|U_+^0\| + \int_0^{t_1} \|f(t)\|\|U(t)\|\,dt\bigg),\end{aligned}$$

and, for $n \ge 2$,

$$\begin{aligned}\|U\|_{I_n}^2 &\le \big(\|U^n\| + \|[U]^{n-1}\| + \|U^{n-1}\|\big)^2 \le 4\big(\|U^n\|^2 + \|[U]^{n-1}\|^2\big) + 2\|U^{n-1}\|^2\\ &\le 12\bigg(\|U^0\|\|U_+^0\| + \int_0^{t_n} \|f(t)\|\|U(t)\|\,dt\bigg).\end{aligned}$$

Thus, putting $j_n = \arg\max_{1\le j\le n} \|U\|_{I_j}$, the desired bound follows at once from

$$\|U\|_{J_n}^2 = \|U\|_{I_{j_n}}^2 \le 12\|U\|_{J_n}\bigg(\|U^0\| + \int_0^{t_{j_n}} \|f(t)\|\,dt\bigg),$$

and we see that U is unique.

If $\mathbb{H}$ is finite dimensional, then the existence of U follows from uniqueness because the square linear system (1.8) must be uniquely solvable. If $\mathbb{H}$ is infinite dimensional, then we can construct U by expanding in the eigenfunctions of A, because the 2×2 matrix

$$\begin{bmatrix} (\frac12 + \omega_{nn}^{11}\lambda) & (\frac12 + \omega_{nn}^{12}\lambda)\\ (-\frac12 + \omega_{nn}^{21}\lambda) & (\frac12 + \omega_{nn}^{22}\lambda)\end{bmatrix} \tag{2.2}$$

is non-singular for all $\lambda \ge 0$. Moreover, defining $V(t) = 0$ for $t \notin I_n$, we see that the quadratic form

$$\begin{bmatrix} V_+^{n-1} & V^n\end{bmatrix}\begin{bmatrix}\omega_{nn}^{11} & \omega_{nn}^{12}\\ \omega_{nn}^{21} & \omega_{nn}^{22}\end{bmatrix}\begin{bmatrix} V_+^{n-1}\\ V^n\end{bmatrix} = \int_0^T V(t)\mathcal{B}V(t)\,dt$$

is strictly positive-definite and so the determinant of (2.2) is bounded below by $c\lambda^2$ for λ sufficiently large. Thus, Cramer's rule shows that the norm of the inverse matrix is $O(\lambda^{-1})$ as $\lambda \to \infty$, and a simple inductive argument gives U_+^{n-1}, $U^n \in D(A)$ for $1 \le n \le N$, implying that $U(t) \in D(A)$ for $0 < t \le T$. □

The proof above also yields a bound for the jumps in the numerical solution.

Corollary 2.2.

$$\sum_{j=1}^{n-1} \|[U]^j\|^2 \le 24\bigg(\|U^0\| + \int_0^{t_n} \|f(t)\|\,dt\bigg)^2.$$

Proof. Apply (2.1). □

3. Error from the time discretization

For our error analysis, we reformulate the discontinuous Galerkin method in terms of a global bilinear form

$$G_N(U,X) = \langle U^0_+, X^0_+\rangle + \sum_{n=1}^{N-1} \langle [U]^n, X^n_+\rangle + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \bigl[\langle U'(t), X(t)\rangle + A\bigl(\mathcal{B}U(t), X(t)\bigr)\bigr]\, dt. \tag{3.1}$$

Summing the equations (1.6) gives

$$\begin{aligned} G_N(U,X) &= \langle U^0, X^0_+\rangle + \int_0^{t_N} \langle f(t), X(t)\rangle\, dt \quad \text{for all } X \in \mathcal{W}_q, \\ U(0) &= u_0, \end{aligned} \tag{3.2}$$

and conversely, by choosing X to be identically zero outside I_n, we see that if $U \in \mathcal{W}_q$ satisfies (3.2) then (1.6) holds for each n. Since the exact solution u has no jumps,

$$G_N(u,X) = \langle u_0, X^0_+\rangle + \int_0^{t_N} \langle f(t), X(t)\rangle\, dt$$

and thus

$$G_N(U-u,X) = \langle U^0 - u_0, X^0_+\rangle \quad \text{for all } X \in \mathcal{W}_q. \tag{3.3}$$

Integration by parts yields an alternative expression for the bilinear form (3.1),

$$G_N(U,X) = \langle U^N, X^N\rangle - \sum_{n=1}^{N-1} \langle U^n, [X]^n\rangle + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \bigl[-\langle U(t), X'(t)\rangle + A\bigl(\mathcal{B}U(t), X(t)\bigr)\bigr]\, dt. \tag{3.4}$$

For any continuous function $u : I_n \to \mathbb{H}$ we define an interpolant $\Pi u : I_n \to \mathbb{P}_q$ by requiring

$$\Pi u(t_n) = u(t_n) \quad \text{and} \quad \int_{t_{n-1}}^{t_n} [u(t) - \Pi u(t)] t^p\, dt = 0 \quad \text{for } p = 0,\, 1,\, \ldots,\, q-2. \tag{3.5}$$

At $t = 0$, we define $\Pi u(0) = u(0)$.

For the piecewise-constant case $q = 1$, the second condition of (3.5) is vacuous and we have simply

$$\Pi u(t) = u(t_n) \quad \text{for } t \in I_n,$$

so if u is differentiable then we may represent the interpolation error as

$$\Pi u(t) - u(t) = \int_t^{t_n} u'(s)\, ds \quad \text{for } t \in I_n. \tag{3.6}$$

In the piecewise-linear case $q = 2$, the interpolation error $u - \Pi u$ must be orthogonal to constants on the interval I_n, and we find that

$$\Pi u(t) = u(t_n) + \frac{u(t_n) - \bar{u}^n}{k_n/2}(t - t_n) \quad \text{for } t \in I_n, \tag{3.7}$$

where $\bar{u}^n = k_n^{-1}\int_{t_{n-1}}^{t_n} u(t)\,dt$. Elementary calculations then show that for $t \in I_n$,

$$\begin{aligned}\Pi u(t) - u(t) &= \int_t^{t_n} u'(s)\,ds - 2\frac{t_n - t}{k_n^2}\int_{t_{n-1}}^{t_n}(s - t_{n-1})u'(s)\,ds \\ &= \int_t^{t_n}(t_n - s)u''(s)\,ds + \frac{t_n - t}{k_n^2}\int_{t_{n-1}}^{t_n}(s - t_{n-1})^2 u''(s)\,ds.\end{aligned} \tag{3.8}$$

Using Π, we decompose the error into two terms,

$$U - u = (U - \Pi u) + (\Pi u - u) \tag{3.9}$$

and estimate each term separately. The representations (3.6) and (3.8) immediately yield bounds for the second term

$$\|\Pi u - u\|_{I_n} \le C k_n^{r-1}\int_{t_{n-1}}^{t_n}\|u^{(r)}(t)\|\,dt \quad \text{for } 1 \le r \le q \le 2, \tag{3.10}$$

and we handle the first term as follows.

Theorem 3.1. *Let $q \in \{1, 2\}$. If u is the solution of the initial value problem* (1.1)*, and if U is the approximate solution obtained by the discontinuous Galerkin method* (1.6)*, then*

$$\|U - \Pi u\|_{J_n} \le C\|U^0 - u_0\| + C t_n^\alpha \int_0^{t_1} t\|Au'(t)\|\,dt + C t_n^\alpha \sum_{j=2}^n k_j^q \int_{t_{n-1}}^{t_n}\|Au^{(q)}(t)\|\,dt.$$

Proof. For brevity, we put

$$\theta = U - \Pi u \quad \text{and} \quad \eta = \Pi u - u.$$

The Galerkin orthogonality relation (3.3) implies that

$$G_N(\theta, X) = \langle U^0 - u(0), X_+^0\rangle - G_N(\eta, X) \quad \text{for all } X \in \mathcal{W}_q,$$

and by the construction of the interpolant we have $\eta^n = 0$ for all $n \ge 1$. Hence, using the alternative expression (3.4) for G_N,

$$G_N(\eta, X) = \sum_{n=1}^N \int_{t_{n-1}}^{t_n}\big[-\langle \eta(t), X'(t)\rangle + A\big(\mathcal{B}\eta(t), X(t)\big)\big]\,dt.$$

Moreover, $\int_{t_{n-1}}^{t_n}\langle \eta(t), X'(t)\rangle\,dt = 0$ if $q = 1$ because $X'(t)$ is identically zero on I_n. The same conclusion holds if $q = 2$ because $X'(t)$ is constant on I_n and hence is orthogonal to the interpolation error. Therefore, $\theta \in \mathcal{W}_q$ satisfies

$$G_N(\theta, X) = \langle U^0 - u(0), X_+^0\rangle - \int_0^T \langle \mathcal{B}A\eta(t), X(t)\rangle\,dt \quad \text{for all } X \in \mathcal{W}_q, \tag{3.11}$$

which has the same form as the equation (3.2) satisfied by U, so we may apply the stability result of Theorem 2.1 and conclude that

$$\|\theta\|_{J_n} \le 12\bigg(\|U^0 - u(0)\| + \int_0^{t_n}\|\mathcal{B}A\eta(t)\|\,dt\bigg) \quad \text{for } 1 \le n \le N.$$

Reversing the order of integration, we find that

$$\begin{aligned}\int_0^{t_n} \|\mathcal{B}A\eta(t)\|\,dt &\le \int_0^{t_n}\int_0^t |\beta(t,s)|\|A\eta(s)\|\,ds\,dt\\ &\le C\int_0^{t_n}\int_s^{t_n}(t-s)^{\alpha-1}\,dt\,\|A\eta(s)\|\,ds\\ &= C_\alpha\int_0^{t_n}(t_n-s)^\alpha\|A\eta(s)\|\,ds \le Ct_n^\alpha\int_0^{t_n}\|A\eta(s)\|\,ds,\end{aligned}$$

so

$$\|U-\Pi u\|_{J_n} \le C_\alpha\biggl(\|U^0-u_0\| + t_n^\alpha\int_0^{t_n}\|A\eta(t)\|\,dt\biggr) \quad \text{for } 1\le n\le N.$$

When $q=2$, the desired bound follows using the formula (3.8):

$$\begin{aligned}\int_0^{t_1}\|A\eta(t)\|\,dt &\le \int_0^{t_1}\biggl(\int_t^{t_1}\|Au'(s)\|\,ds + 2\frac{t_1-t}{k_1^2}\int_0^{t_1}s\|Au'(s)\|\,ds\biggr)\,dt\\ &= 2\int_0^{t_1}s\|Au'(s)\|\,ds\end{aligned}$$

and

$$\int_{t_1}^{t_n}\|A\eta(t)\|\,dt \le \sum_{j=2}^n k_j\|Au-\Pi Au\|_{I_j} \le C\sum_{j=2}^n k_j^2\int_{t_{j-1}}^{t_j}\|Au''(t)\|\,dt.$$

Similar, but simpler, estimates lead to the result for $q=1$. □

The next theorem shows that we can obtain $O(k^q)$ accuracy for all $t\in[0,T]$ provided the mesh grading, as determined by the parameter $\gamma\ge 1$, is sufficiently strong.

Theorem 3.2. *Let $q\in\{1,2\}$ and assume that the step sizes are such that* (1.11) *and* (1.12) *hold. If the exact solution u satisfies the regularity estimates* (1.9) *and* (1.10)*, then*

$$\|U-u\|_{J_n} \le C\|U^0-u_0\| + CM\times\begin{cases} k^{\gamma\sigma}, & 1\le\gamma<q/\sigma,\\ k^q\log(t_n/t_1), & \gamma=q/\sigma,\\ t_n^{\sigma-q/\gamma}k^q, & \gamma>q/\sigma.\end{cases}$$

Proof. From (3.10), the assumptions (1.10) and (1.12) give

$$\|\Pi u-u\|_{I_1} \le C\int_0^{t_1}\|u'(t)\|\,dt \le CM\int_0^{t_1}t^{\sigma-1}\,dt \le CMt_1^\sigma \le CMk^{\gamma\sigma} \tag{3.12}$$

and, using (1.11) for $n\ge 2$,

$$\begin{aligned}\|\Pi u-u\|_{I_n} &\le Ck_n^{q-1}\int_{t_{n-1}}^{t_n}\|u^{(q)}(t)\|\,dt \le CMk_n^{q-1}\int_{t_{n-1}}^{t_n}t^{\sigma-q}\,dt\\ &\le CMk_n^q t_n^{\sigma-q} \le CMk^q t_n^{\sigma-q/\gamma},\end{aligned} \tag{3.13}$$

so we may bound the interpolation error as follows,

$$\|\Pi u-u\|_{J_n} = \max_{1\le j\le n}\|\Pi u-u\|_{I_n} \le CM\times\begin{cases} k^{\gamma\sigma}, & 1\le\gamma\le q/\sigma,\\ t_n^{\sigma-q/\gamma}k^q, & \gamma\ge q/\sigma.\end{cases}$$

Next, by (1.9) and (1.12),

$$\int_0^{t_1} t\|Au'(t)\|\,dt \le CM\int_0^{t_1} t^{\sigma-1}\,dt \le CMk^{\gamma\sigma},$$

and, using (1.11),

$$\begin{aligned}\sum_{j=2}^n k_j^q \int_{t_{j-1}}^{t_j} \|Au^{(q)}(t)\|\,dt &\le CM\sum_{j=2}^n k_j^q \int_{t_{j-1}}^{t_j} t^{\sigma-1-q}\,dt\\ &\le CMk^q \sum_{j=2}^n t_j^{(1-1/\gamma)q}\int_{t_{j-1}}^{t_j} t^{\sigma-1-q}\,dt \le CMk^q\int_{t_1}^{t_n} t^{\sigma-q/\gamma-1}\,dt\end{aligned}$$

and the result follows from (3.9) and Theorem 3.1, after noting that

$$\int_{t_1}^{t_n} t^{\sigma-q/\gamma-1}\,dt \le \begin{cases} Ct_1^{-(q/\gamma-\sigma)} \le Ck^{-(q-\gamma\sigma)}, & 1\le\gamma<q/\sigma,\\ C\log(t_n/t_1), & \gamma=q/\sigma,\\ Ct_n^{\sigma-q/\gamma}, & \gamma>q/\sigma.\end{cases}$$

□

4. Superconvergence at the nodes

We now show that for $q=2$ the numerical solution achieves a faster convergence rate at $t=t_n$, depending on the quantities

$$\epsilon_{nj} = \max_{t\in I_j}\biggl(\int_t^{t_j}|\beta(s,t)|\,ds + \int_{t_j}^{t_n}\bigl|\beta(s,t)-\beta(s,t_j)\bigr|\,ds\biggr) \quad \text{for } 1\le j\le n\le N. \tag{4.1}$$

Theorem 4.1. *If u is the solution of the initial value problem* (1.1)*, and U is the approximate solution obtained by the piecewise-linear ($q=2$) discontinuous Galerkin method* (1.6)*, then*

$$\|U^n-u(t_n)\| \le C\biggl(\|U^0-u_0\| + \epsilon_{n1}\int_0^{t_1} t\|Au'(t)\|\,dt + \sum_{j=2}^n \epsilon_{nj}k_j^2\int_{t_{j-1}}^{t_j}\|Au''(t)\|\,dt\biggr).$$

Proof. Let z be the solution of the dual problem

$$-z' + \mathcal{B}^*Az = 0 \quad \text{for } 0\le t\le T, \quad \text{with } z(T)=z_T, \tag{4.2}$$

where $\mathcal{B}^*v(t) = \int_t^T \beta(s,t)v(s)\,ds$. Since z has no jumps and since

$$\int_0^T \bigl[\langle -v(t), z'(t)\rangle + A\bigl(\mathcal{B}v(t), z(t)\bigr)\bigr]\,dt = \int_0^T \langle v(t), -z'(t)+\mathcal{B}^*Az(t)\rangle\,dt = 0,$$

the formula (3.4) yields the identity

$$G_N(v,z) = \langle v(T), z_T\rangle$$

for all piecewise-continuous $v(t)$. Let $Z\in\mathcal{W}_2$ denote the approximate solution of (4.2) given by the discontinuous Galerkin method

$$G_N(V,Z) = \langle V^N, z_T\rangle \quad \text{for all } V\in\mathcal{W}_2,$$

and let $\theta = U-\Pi u$ and $\eta = \Pi u - u$, as before, so that $U-u=\theta+\eta$. Since $u(t_N)=\Pi u(t_N)$, by taking $V=\theta$ we see from (3.3) that

$$\langle U^N-u(t_N), z_T\rangle = \langle\theta^N, z_T\rangle = G_N(\theta,Z) = \langle U^0-u_0, Z_+^0\rangle - G_N(\eta,Z). \tag{4.3}$$

Moreover, $\eta^n = 0$ and $\int_{t_{n-1}}^{t_n} \langle \eta(t), Z'(t) \rangle \, dt = 0$ for all n, so the formula (3.4) shows that

$$G_N(\eta, Z) = \sum_{n=1}^{N} \delta_n, \quad \text{where} \quad \delta_n = \int_{t_{n-1}}^{t_n} \langle A\eta(t), \mathcal{B}^* Z(t) \rangle \, dt.$$

The orthogonality property of Π implies that

$$\delta_n = \int_{t_{n-1}}^{t_n} \langle A\eta(t), \mathcal{B}^* Z(t) - \mathcal{B}^* Z(t_n) \rangle \, dt,$$

and for $t \in I_n$,

$$\begin{aligned} \|\mathcal{B}^* Z(t) - \mathcal{B}^* Z(t_n)\| &= \left\| \int_t^T \beta(s,t) Z(s) \, ds - \int_{t_n}^T \beta(s,t_n) Z(s) \, ds \right\| \\ &= \left\| \int_t^{t_n} \beta(s,t) Z(s) \, ds + \int_{t_n}^T \big[\beta(s,t) - \beta(s,t_n)\big] Z(s) \, ds \right\| \le \epsilon_{Nn} \|Z\|_{J_N}, \end{aligned}$$

so

$$\|\delta_n\| \le \epsilon_{Nn} \|Z\|_{J_N} \int_{t_{n-1}}^{t_n} \|A\eta(t)\| \, dt.$$

Using (3.10) we see that

$$\int_{t_{n-1}}^{t_n} \|A\eta(t)\| \, dt \le C k_n^2 \int_{t_{n-1}}^{t_n} \|Au''(t)\| \, dt \quad \text{for } n \ge 2,$$

and using (3.8) we find

$$\int_0^{t_1} \|A\eta(t)\| \, dt \le C \int_0^{t_1} t \|Au'(t)\| \, dt.$$

Stability of the discontinuous Galerkin method for the dual problem means that $\|Z_+^0\| + \|Z\|_{J_N} \le C\|z_T\|$. Thus,

$$\begin{aligned} |\langle U^N - u(t_N), z_T \rangle| \le C\bigg(&\|U^0 - u_0\| + \epsilon_{N1} \int_0^{t_1} t\|Au'(t)\| \, dt \\ &+ \sum_{n=2}^{N} \epsilon_{Nn} k_n^2 \int_{t_{n-1}}^{t_n} \|Au''(t)\| \, dt \bigg) \|z_T\|, \end{aligned}$$

and since $z_T \in \mathbb{H}$ is arbitrary we obtain the desired bound for $\|U^N - u(t_N)\|$. □

If $\beta(t,s)$ and $u(t)$ are smooth, then $\epsilon_{nj} = O(k)$ and so $\|U^n - u(t_n)\| = O(k^3)$. For the specific non-smooth kernel (1.3), we have the same convergence rate provided the mesh grading is sufficiently strong.

Corollary 4.2. *Let* $q = 2$ *and* $\beta(t,s) = (t-s)^{\alpha-1}/\Gamma(\alpha)$, *with* $0 < \alpha < 1$. *If* u *satisfies the regularity assumption* (1.9), *and if the time mesh satisfies the conditions* (1.11) *and* (1.12), *then, with* $\gamma^* = 3/(\sigma + \alpha)$,

$$\|U^n - u(t_n)\| \le C\|U^0 - u_0\| + CM \times \begin{cases} k^{\gamma(\sigma+\alpha)}, & 1 \le \gamma < \gamma^*, \\ k^3, & \gamma \ge \gamma^*. \end{cases}$$

Proof. Noting that $\beta(s,t) \ge \beta(s,t_j) > 0$ for $t \in I_j$, we have

$$\begin{aligned}\int_t^{t_j} |\beta(s,t)|\,ds + \int_{t_j}^{t_n} \bigl|\beta(s,t)-\beta(s,t_j)\bigr|\,ds \\ &= \int_t^{t_j} \beta(s,t)\,ds + \int_{t_j}^{t_n} \bigl(\beta(s,t)-\beta(s,t_j)\bigr)\,ds \\ &= \int_t^{t_n} \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}\,ds - \int_{t_j}^{t_n} \frac{(s-t_j)^{\alpha-1}}{\Gamma(\alpha)}\,ds = \frac{(t_n-t)^\alpha-(t_n-t_j)^\alpha}{\Gamma(\alpha+1)},\end{aligned}$$

and so $\epsilon_{nj} = \bigl[(t_n-t_{j-1})^\alpha - (t_n-t_j)^\alpha\bigr]/\Gamma(\alpha+1)$. Since $X^\alpha - Y^\alpha \le (X-Y)^\alpha$ for any $X \ge Y \ge 0$, we see that $\epsilon_{nj} \le k_j^\alpha/\Gamma(\alpha+1)$, with equality when $j=n$. However, for $j<n$ we obtain a sharper bound by applying the mean value theorem: $\epsilon_{nj} \le (t_n-t_j)^{\alpha-1}k_j/\Gamma(\alpha)$. Thus, using (1.9), (1.11) and (1.12), we have

$$\epsilon_{n1}\int_0^{t_1} t\|Au'(t)\|\,dt \le CMk_1^\alpha \int_0^{t_1} t^{\sigma-1}\,dt \le CMk_1^{\alpha+\sigma} \le CMk^{\gamma(\alpha+\sigma)}$$

and, for $2 \le j \le n-1$,

$$\begin{aligned}&\epsilon_{nj}k_j^2\int_{t_{j-1}}^{t_j} \|Au''(t)\|\,dt \le CMk_j^3(t_n-t_j)^{\alpha-1}\int_{t_{j-1}}^{t_j} t^{\sigma-3}\,dt\\ &\le CMk^3(t_j)^{3(1-1/\gamma)}(t_n-t_j)^{\alpha-1}\int_{t_{j-1}}^{t_j} t^{\sigma-3}\,dt \le CMk^3\int_{t_{j-1}}^{t_j}(t_n-t)^{\alpha-1}t^{\sigma-3/\gamma}\,dt,\end{aligned}$$

with

$$\begin{aligned}\epsilon_{nn}k_n^2\int_{t_{n-1}}^{t_n} \|Au''(t)\|\,dt &\le CMk_n^{2+\alpha}\int_{t_{n-1}}^{t_n} t^{\sigma-3}\,dt \le CMk_n^{3+\alpha}t_n^{\sigma-3}\\ &\le CMk^3\bigl(k_n/t_n\bigr)^\alpha t_n^{\alpha+\sigma-3/\gamma} \le CMt_n^{\alpha+\sigma-3/\gamma}k^3.\end{aligned}$$

Using the substitution $t = t_nz$, we find that

$$\begin{aligned}\sum_{j=2}^{n-1}\epsilon_{nj}k_j^2\int_{t_{j-1}}^{t_j}\|Au''(t)\|\,dt &\le CMk^3\int_{t_1}^{t_{n-1}}(t_n-t)^{\alpha-1}t^{\sigma-3/\gamma}\,dt\\ &= CMk^3t_n^{\alpha+\sigma-3/\gamma}\int_{t_1/t_n}^{t_{n-1}/t_n}(1-z)^{\alpha-1}z^{\sigma-3/\gamma}\,dz,\end{aligned}$$

and elementary calculation yields

$$\int_{t_1/t_n}^{t_{n-1}/t_n}(1-z)^{\alpha-1}z^{\sigma-3/\gamma}\,dz \le C\times\begin{cases}(t_1/t_n)^{1+\sigma-3/\gamma} & \sigma-3/\gamma<-1,\\ \log(t_n/t_1), & \sigma-3/\gamma=-1,\\ 1, & \sigma-3/\gamma>-1.\end{cases}$$

Theorem 4.1 now shows that the nodal error $\|U^n-u(t_n)\|$ is bounded by $C\|U^0-u_0\|$ plus

$$CMk^{\gamma(\alpha+\sigma)} + CM\times\begin{cases}t_n^{\alpha-1}k^{\gamma(1+\sigma)}, & 1\le\gamma<3/(1+\sigma),\\ t_n^{\alpha-1}k^3\log(t_n/t_1), & \gamma=3/(1+\sigma),\\ t_n^{\alpha+\sigma-3/\gamma}k^3, & \gamma>3/(1+\sigma),\end{cases}$$

which is $O(k^3)$ for $\gamma \ge \gamma^* = 3/(\alpha+\sigma) > 3/(1+\sigma)$. If $1 \le \gamma < 3/(1+\sigma)$ then

$$t_n^{\alpha-1}k^{\gamma(1+\sigma)} = k^{\gamma(\alpha+\sigma)}(k^\gamma/t_n)^{1-\alpha} \le Ck^{\gamma(\alpha+\sigma)}(t_1/t_n)^{1-\alpha} \le Ck^{\gamma(\alpha+\sigma)}.$$

Likewise, if $\gamma = 3/(1+\sigma)$ then

$$t_n^{\alpha-1}k^3\log(t_n/t_1) = k^{\gamma(\alpha+\sigma)}(k^\gamma/t_n)^{1-\alpha}\log(t_n/t_1) \le Ck^{\gamma(\alpha+\sigma)}(t_1/t_n)^{1-\alpha}\log(t_n/t_1) \le Ck^{\gamma(\alpha+\sigma)},$$

and in the remaining case $3/(1+\sigma) < \gamma < \gamma^* = 3/(\alpha+\sigma)$ we have

$$t_n^{\alpha+\sigma-3/\gamma}k^3 = k^{\gamma(\alpha+\sigma)}(k^\gamma/t_n)^{3/\gamma-(\alpha+\sigma)} \le Ck^{\gamma(\alpha+\sigma)}(t_1/t_n)^{3/\gamma-(\alpha+\sigma)} \le Ck^{\gamma(\alpha+\sigma)}.$$

□

5. Space discretization

We assume now that $\mathbb{H} = L_2(\Omega)$ for a bounded, convex domain Ω, and that A is a strongly-elliptic, second-order, self-adjoint partial differential operator. Let the boundary $\partial\Omega$ consist of two pieces Γ_{D} and Γ_{N}, on which we impose homogeneous Dirichlet and Neumann boundary conditions, respectively, and define

$$H^1_{\mathrm{D}}(\Omega) = D(A^{1/2}) = \{\, v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{D}} \,\}.$$

Let $S_h \subseteq H^1_{\mathrm{D}}(\Omega)$ be a continuous piecewise-linear finite element space based on a quasi-uniform partition of the domain Ω, with h denoting the maximum diameter of the elements, and assume that the partition is aligned with Γ_{D} and Γ_{N}. We then have the approximation property

$$\min_{\chi\in S_h}\bigl(\|v-\chi\| + h\|\nabla(v-\chi)\|\bigr) \le Ch^2\|v\|_2 \quad \text{for } v \in H^1_{\mathrm{D}}(\Omega)\cap H^2(\Omega),$$

where we use the abbreviation $\|v\|_r = \|v\|_{H^r(\Omega)}$.

Based on the weak formulation of the initial value problem (1.1), we define a spatially-discrete, approximate solution $u_h : [0,T] \to S_h$ by requiring

$$\langle u_h'(t), \chi\rangle + \int_0^t \beta(t,s)A\bigl(u_h(s),\chi\bigr)\,ds = \langle f(t),\chi\rangle \quad \text{for } 0 \le t \le T \text{ and all } \chi \in S_h,$$

with $u_h(0) = u_{0h} \approx u_0$ for a suitable $u_{0h} \in S_h$. This semi-discrete solution satisfies the error bound [11, Theorem 2.1]

$$\|u_h(t) - u(t)\| \le \|u_{0h} - u_0\| + Ch^2\int_0^t \|u'(s)\|_2\,ds \quad \text{for } 0 \le t \le T.$$

Let $\mathbb{P}_q(S_h)$ denote the space of polynomials of degree strictly less than q with coefficients in S_h, and define the corresponding trial space of piecewise-polynomials $\mathcal{W}_q(S_h)$. Thus, a function $X(x,t)$ in $\mathcal{W}_q(S_h)$ is continuous in x but may be discontinuous at $t = t_n$.

Applying the discontinuous Galerkin method in time, we arrive at a fully-discrete numerical solution $U_h : [0,T] \to \mathcal{W}_q(S_h)$ defined by

$$\begin{aligned} G_N(U_h, X) &= \langle U_h^0, X_+^0\rangle + \int_0^{t_N} \langle f(t), X(t)\rangle\,dt \quad \text{for all } X \in \mathcal{W}_q(S_h), \\ U_h(0) &= U_h^0, \end{aligned} \tag{5.1}$$

for a suitable $U_h^0 \in S_h$ with $U_h^0 \approx u_0$; cf. (3.2). In place of (3.9), we now decompose the error as

$$U_h - u = (U_h - \Pi R_h u) + (\Pi R_h u - u), \tag{5.2}$$

where $R_h : H^1_{\mathrm{D}}(\Omega) \to S_h$ is the Ritz projector for the (strictly) positive-definite bilinear form $A(u,v) + \langle u, v\rangle$; thus,

$$A(R_hv,\chi) + \langle R_hv,\chi\rangle = A(v,\chi) + \langle v,\chi\rangle \quad \text{for all } \chi \in S_h. \tag{5.3}$$

(The term $\langle u, v\rangle$ in the bilinear form is needed only if A has a zero eigenvalue.)

Theorem 5.1. *Let $q \in \{1, 2\}$. If u is the solution of the initial value problem* (1.1), *and if $U_h \in \mathcal{W}_q(S_h)$ is the approximate solution defined by* (5.1), *then*

$$\|U_h - \Pi R_hu\|_{J_n} \le C_T\biggl(\|U^0_h - R_hu_0\| + \|u_0 - R_hu_0\| + \int_0^{t_n} \|u'(t) - R_hu'(t)\|\,dt + \int_0^{t_1} t\|Au'(t)\|\,dt + \sum_{j=2}^{n} k_j^q \int_{t_{j-1}}^{t_j} \|Au^{(q)}(t)\|\,dt\biggr) \quad \textit{for } 1 \le n \le N.$$

Proof. The Galerkin orthogonality property (3.3) now takes the form

$$G_N(U_h - u, X) = \langle U^0_h - u_0, X^0_+\rangle \quad \text{for all } X \in \mathcal{W}_q(S_h), \tag{5.4}$$

and for brevity we let

$$W = \Pi R_hu \quad \text{and} \quad \xi = R_hu - u.$$

Adapting the proof of Theorem 3.1, we see from (5.4) that

$$G_N(U_h - W, X) = \langle U^0_h - u_0, X^0_+\rangle - G_N(W - u, X) \quad \text{for all } X \in \mathcal{W}_q(S_h), \tag{5.5}$$

and, because $W^n = R_hu(t_n)$, the formula (3.4) gives

$$G_N(W - u, X) = \langle \xi^N, X^N\rangle - \sum_{n=1}^{N-1} \langle \xi^n, [X]^n\rangle + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \bigl[-\langle W - u, X'\rangle + A\bigl(\mathcal{B}(W - u), X\bigr)\bigr]\,dt.$$

Since $\int_{t_{n-1}}^{t_n} \langle W - R_hu, X'\rangle\,dt = \int_{t_{n-1}}^{t_n} \langle \Pi(R_hu) - (R_hu), X'\rangle\,dt = 0$, an integration by parts shows that

$$\begin{aligned}\int_{t_{n-1}}^{t_n} -\langle W - u, X'\rangle\,dt &= \int_{t_{n-1}}^{t_n} -\langle R_hu - u, X'\rangle\,dt = \int_{t_{n-1}}^{t_n} -\langle \xi, X'\rangle\,dt \\ &= -\langle \xi^n, X^n\rangle + \langle \xi^{n-1}, X^{n-1}_+\rangle + \int_{t_{n-1}}^{t_n} \langle \xi', X\rangle\,dt,\end{aligned}$$

and, with $\eta = \Pi u - u$, the definition of the Ritz projector gives

$$\begin{aligned}A\bigl((W - u)(s), X(t)\bigr) &= A\bigl(R_h\Pi u(s) - u(s), X(t)\bigr) \\ &= \bigl[A\bigl((\Pi u - u)(s), X(t)\bigr) + \langle \Pi(u - R_hu)(s), X(t)\rangle\bigr] \\ &= \langle A\eta(s) - \Pi\xi(s), X(t)\rangle,\end{aligned}$$

so

$$G_N(W - u, X) = \langle \xi^0, X^0_+\rangle + \int_0^{t_N} \langle \xi' + \mathcal{B}A\eta - \mathcal{B}\Pi\xi, X\rangle\,dt.$$

Thus, from (5.5),

$$G_N(U_h - W, X) = \langle U^0_h - R_hu_0, X^0_+\rangle - \int_0^{t_N} \langle \xi' + \mathcal{B}A\eta - \mathcal{B}\Pi\xi, X\rangle\,dt$$

for all $X \in \mathcal{W}_q(S_h)$, cf. (3.11). Stability of the discontinuous Galerkin method (Theorem 2.1 with $\mathbb{H} = S_h$) now yields the estimate

$$\|U_h - W\|_{J_n} \le 12\biggl(\|U_h^0 - R_h u_0\| + \int_0^{t_n} \|\xi' + \mathcal{B}A\eta - \mathcal{B}\Pi\xi\|\,dt\biggr).$$

We already estimated the term $\int_0^{t_n} \|\mathcal{B}A\eta\|\,dt$ in Theorem 3.1, and for the remaining terms in the integral we apply the bound $\|\Pi u\|_{I_n} \le C\|u\|_{I_n}$ and arrive at

$$\int_0^{t_n} \|\xi'(t) - \mathcal{B}\Pi\xi\|\,dt \le \int_0^{t_n} \|\xi'\|\,dt + Ct_n^{\alpha+1}\|\xi\|_{J_n} \le C_T\biggl(\|\xi(0)\| + \int_0^{t_n} \|\xi'\|\,dt\biggr).$$

□

We can now show that the space discretisation leads to an additional error of order $h^2\ell(k)$ compared with the error bound of Theorem 3.2; recall from (1.15) that $\ell(k) = \max(1, \log k^{-1})$.

Theorem 5.2. *Let $q \in \{1, 2\}$ and assume that the time mesh is such that* (1.11) *and* (1.12) *hold. If the exact solution u satisfies the regularity estimates* (1.9), (1.10) *and* (1.14)*, then, for $1 \le n \le N$ and with $C = C_T$,*

$$\|U_h - u\|_{J_n} \le C\|U_h^0 - u_0\| + CMh^2\ell(t_n/t_1) + CM \times \begin{cases} k^{\gamma\sigma}, & 1 \le \gamma < q/\sigma,\\ k^q\ell(t_n/t_1), & \gamma = q/\sigma,\\ k^q, & \gamma > q/\sigma. \end{cases}$$

Proof. Recall that

$$\|v - R_h v\| \le Ch^r\|v\|_r \quad \text{for } 0 \le r \le 2,$$

and, in particular, that $\|R_h v\| \le C\|v\|$. To estimate the second term $\Pi R_h u - u$ in the decomposition (5.2), we again write $\xi = R_h u - u$ and observe that

$$\|\Pi R_h u - u\|_{J_n} = \|R_h(\Pi u - u) + \xi\|_{J_n} \le C\|\Pi u - u\|_{J_n} + \|\xi(0)\| + \int_0^{t_n} \|\xi'\|\,dt.$$

In view of Theorems 3.2 and 5.1, and the fact that $\|U_h^0 - R_h u_0\| \le \|U_h^0 - u_0\| + \|u_0 - R_h u_0\|$, it suffices to note that $\|\xi(0)\| \le Ch^2\|u_0\|_2$ and, using (1.10), (1.11), (1.12) and (1.14),

$$\begin{aligned}\int_0^{t_n} \|\xi'(t)\|\,dt &\le C\int_0^{t_1} \|u'(t)\|\,dt + \int_{t_1}^{t_n} Ch^2\|u'(t)\|_2\,dt\\ &\le CM\int_0^{t_1} t^{\sigma-1}\,dt + CMh^2\int_{t_1}^{t_n} t^{-1}\,dt \le CMk^{\gamma\sigma} + CMh^2\ell(t_n/t_1).\end{aligned}$$

□

Next, we prove a spatially-discrete version of Theorem 4.1, showing superconvergence at $t = t_n$.

Theorem 5.3. *Let $q = 2$ and define ϵ_{nj} by* (4.1)*. If u is the solution of* (1.1) *and if $U_h \in \mathcal{W}_2(S_h)$ is the approximate solution given by* (5.1)*, then*

$$\|U_h^n - u(t_n)\| \le C_T\biggl(\|U_h^0 - u_0\| + \|u_0 - R_h u_0\| + \int_0^{t_n} \|u'(t) - R_h u'(t)\|\,dt + \epsilon_{n1}\int_0^{t_1} t\|Au'(t)\|\,dt + \sum_{j=2}^{n} \epsilon_{nj} k_j^2 \int_{t_{n-1}}^{t_n} \|Au''(t)\|\,dt\biggr) \quad \text{for } 1 \le n \le N.$$

Proof. We adapt the proof of Theorem 4.1, letting $Z \in \mathcal{W}_2(S_h)$ denote the solution of

$$G_N(V, Z) = \langle V^N, z_T\rangle \quad \text{for all } V \in \mathcal{W}_2(S_h),$$

and writing

$$W = \Pi R_h u, \quad \eta = \Pi u - u, \quad \xi = R_h u - u.$$

The Galerkin orthogonality (5.4) implies, cf. (4.3),

$$\begin{aligned}\langle U_h^N - R_h u(t_N), z_T\rangle &= \langle U_h^N - W^N, z_T\rangle = G_N(U_h - W, Z)\\ &= G_N(U_h - u, Z) + G_N(u - W, Z)\\ &= \langle U_h^0 - u_0, Z_+^0\rangle - G_N(\eta, Z) - G_N(\Pi\xi, Z),\end{aligned}$$

and by the triangle inequality,

$$\|U_h^N - u(t_N)\| \le \|U_h^N - R_h u(t_N)\| + \|\xi(0)\| + \int_0^{t_N} \|\xi'(t)\|\,dt,$$

so it suffices to prove that

$$|G_N(\Pi\xi, Z)| \le C_T\|z_T\|\biggl(\|\xi(0)\| + \int_0^{t_N} \|\xi'(t)\|\,dt\biggr). \tag{5.6}$$

From the definition (3.1) of G_N,

$$G_N(\Pi\xi, Z) = \langle \Pi\xi_+^0, Z_+^0\rangle + \sum_{n=1}^{N-1} \langle [\Pi\xi]^n, Z_+^n\rangle + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \bigl[\langle (\Pi\xi)'(t), Z(t)\rangle + A\bigl(\mathcal{B}\Pi\xi(t), Z(t)\bigr)\bigr]\,dt,$$

and from the definition (5.3) of R_h,

$$A\bigl(\mathcal{B}\Pi\xi(t), Z(t)\bigr) = -\langle \mathcal{B}\Pi\xi(t), Z(t)\rangle.$$

Integrating by parts, applying the orthogonality and interpolation propertes of Π and noting that $\xi_+^{n-1} = \xi(t_{n-1}) = (\Pi\xi)^{n-1}$, we have

$$\begin{aligned}\int_{t_{n-1}}^{t_n} \langle (\Pi\xi)'(t), Z(t)\rangle\,dt &= \langle (\Pi\xi)^n, Z^n\rangle - \langle (\Pi\xi)_+^{n-1}, Z_+^{n-1}\rangle - \int_{t_{n-1}}^{t_n} \langle \Pi\xi(t), Z'(t)\rangle\,dt\\ &= \langle \xi^n, Z^n\rangle - \langle (\Pi\xi)_+^{n-1}, Z_+^{n-1}\rangle - \int_{t_{n-1}}^{t_n} \langle \xi(t), Z'(t)\rangle\,dt\\ &= \langle \xi_+^{n-1} - (\Pi\xi)_+^{n-1}, Z_+^{n-1}\rangle + \int_{t_{n-1}}^{t_n} \langle \xi'(t), Z(t)\rangle\,dt\\ &= -\langle [\Pi\xi]^{n-1}, Z_+^{n-1}\rangle + \int_{t_{n-1}}^{t_n} \langle \xi'(t), Z(t)\rangle\,dt.\end{aligned}$$

Thus,

$$G_N(\Pi\xi, Z) = \langle \xi(0), Z_+^0\rangle + \int_{t_{n-1}}^{t_n} \langle \xi'(t) - \mathcal{B}\Pi\xi(t), Z(t)\rangle\,dt,$$

and hence, noting that $\|Z\|_{J_N} \le C\|z_T\|$ by stability of the dual problem,

$$|G_N(\Pi\xi, Z)| \le C\|z_T\|\biggl(\|\xi(0)\| + \int_0^{t_N} \bigl[\|\xi'(t)\| + \|\mathcal{B}\Pi\xi(t)\|\bigr]\,dt\biggr).$$

The representation (3.7) implies that $\|\Pi u\|_{I_n} \le 3\|u\|_{I_n}$. Thus,

$$\int_0^{t_N} \|\mathcal{B}\Pi\xi(t)\|\,dt \le \int_0^{t_N}\int_0^t |\beta(t,s)|\|\Pi\xi(s)\|\,ds\,dt \le C_T\|\xi\|_{J_N},$$

so (5.6) follows using the bound $\|\xi\|_{J_N} \le \|\xi(0)\| + \int_0^{t_N}\|\xi'(t)\|\,dt$. □

Corollary 5.4. *Let $q = 2$, assume β is the weakly singular kernel* (1.3) *and suppose that that the time mesh satisfies* (1.11) *and* (1.12). *If the regularity estimates* (1.9), (1.10) *and* (1.14) *hold, then, with $\gamma^* = 3/(\sigma+\alpha)$ and $C = C_T$,*

$$\|U_h^n - u(t_n)\| \le C\|U_h^0 - u_0\| + CMh^2\ell(k) + CM \times \begin{cases} k^{\gamma(\sigma+\alpha)}, & 1 \le \gamma < \gamma^*,\\ k^3, & \gamma \ge \gamma^*,\end{cases}$$

for $1 \le n \le N$.

Proof. Apply the estimates from the proofs of Theorem 5.2 and Corollary 4.2. □

6. Numerical Experiments

We now apply the discontinuous Galerkin method (1.6) and its spatially-discrete version (5.1) to some problems of the form (1.1). In each case the time interval is $[0,T] = [0,1]$ and we employ a time mesh of the form (1.13) for various choices of the mesh grading parameter $\gamma \ge 1$. We consider only the piecewise-linear case $q = 2$.

6.1. **Scalar examples.** To demonstrate the effect of the time discretization by itself, with no additional errors arising from a spatial discretization, we first consider a purely time-dependent problem

$$\frac{du}{dt} + \int_0^t \beta(t-s)u(s)\,ds = f(t) \quad \text{for } 0 < t < T \text{ with } u(0) = u_0,$$

and with a weakly singular kernel $\beta(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $0 < \alpha < 1$. Using the Mittag–Leffler function $E_\mu(x) = \sum_{p=0}^\infty x^p/\Gamma(1+p\mu)$, we may write the exact solution as

$$u(t) = E_{\alpha+1}(-t^{\alpha+1})u_0 + \int_0^t E_{\alpha+1}(-s^{\alpha+1})f(t-s)\,ds;$$

see [10]. Choosing initial data $u_0 = 0$ and a source term $f = (\alpha+1)t^\alpha$, we find that

$$u(t) = -\Gamma(\alpha+2)\sum_{p=1}^\infty \frac{(-t)^{(\alpha+1)p}}{\Gamma(1+(\alpha+1)p)} = \Gamma(\alpha+2)\bigl(1 - E_{\alpha+1}(-t^{\alpha+1})\bigr). \tag{6.1}$$

To tabulate our numerical results, we introduce a finer grid

$$\mathcal{G}^{N,m} = \{\, t_{j-1} + \ell k_j/m : j = 1, 2, \ldots, N \text{ and } \ell = 0, 1, \ldots, m \,\} \tag{6.2}$$

and an associated norm $\|v\|_\infty^{N,m} = \max_{t\in\mathcal{G}^{N,m}} |v(t)|$. Thus, $\|U-u\|^{N,1}$ is the maximum error at the nodes whereas, for larger values of m, the norm $\|U-u\|^{N,m}$ approximates the uniform error $\|U-u\|_{L_\infty(0,T)}$.

N	$\gamma = 1$		$\gamma = 1.25$		$\gamma = 1.45$		$\gamma = 2$	
40	2.26e-04		6.25e-05		3.38e-05		4.27e-05	
80	8.61e-05	1.39	1.86e-05	1.74	8.59e-06	1.97	1.09e-05	1.96
160	3.26e-05	1.39	5.53e-06	1.74	2.16e-06	1.98	2.77e-06	1.98
320	1.23e-05	1.39	1.64e-06	1.74	5.55e-07	1.99	6.96e-07	1.99
640	4.69e-06	1.39	4.89e-07	1.74	1.36e-07	1.99	1.74e-07	1.99

TABLE 1. The error $\|U-u\|_\infty^{N,5}$ with different mesh gradings, when $\alpha = 0.4$. We observe $O(k^{(\alpha+1)\gamma})$ convergence if $1 \le \gamma < 2/(\alpha+1) \approx 1.4286$, and $O(k^2)$ convergence if $\gamma > 2/(\alpha+1)$.

N	$\gamma = 1$		$\gamma = 1.25$		$\gamma = 1.5$	
40	2.73e-07		6.34e-08		9.51e-08	
80	5.37e-08	1.34	9.06e-09	2.80	1.38e-08	2.78
160	1.03e-08	1.37	1.25e-09	2.85	1.94e-09	2.84
320	1.97e-09	2.39	1.69e-10	2.88	2.65e-10	2.87
640	3.74e-10	2.39	2.26e-11	2.90	3.57e-11	2.90
1280	7.11e-11	2.39	2.98e-12	2.93	4.73e-12	2.92

TABLE 2. The nodal error $\|U-u\|_\infty^{N,1}$ with different mesh gradings, when $\alpha = 0.2$. We observe $O(k^{\gamma(2\alpha+2)})$ convergence for $1 \le \gamma < \gamma^* = 3/(2\alpha+2) = 1.25$, and $O(k^3)$ convergence for $\gamma \ge \gamma^*$.

Since the exact solution (6.1) behaves like $t^{\alpha+1}$ as $t \to 0^+$, we see that the first regularity condition (1.9) holds for any $\sigma \le \alpha + 2$ and the second condition (1.10) holds for any $\sigma \le \alpha+1$. Thus, from Theorem 3.2 we expect $\|U-u\|_{J_n}$ to be $O(k^{\gamma\sigma})$ for $1 \le \gamma < 2/(\alpha+1)$, and $O(k^2)$ for $\gamma > 2/(\alpha+1)$. Results for $\alpha = 0.4$, shown in Table 1, are consistent with these error bounds.

In Corollary 4.2 we may take $\sigma = \alpha + 2$, leading to $\gamma^* = 3/(2\alpha+2)$ and an expected nodal error of order k^3 for any $\gamma \ge \gamma^*$. This predicted behaviour is consistent with the numerical results in Table 2, where $\alpha = 0.2$.

We also consider an example with the smooth kernel $\beta(t) = e^{-2t}$. The exact solution has the form

$$u(t) = W(t)u_0 + \int_0^t W(t-s)f(s)\,ds, \quad \text{where} \quad W(t) = (1+t)e^{-t},$$

see [11, Section 6], so for the particular choices $u_0 = 1$ and $f(t) = te^t$ we have

$$u(t) = (1+t)e^{-t} + \int_0^t (1+s)e^{-s}(t-s)e^{t-s}\,ds = \frac{3t}{2}\cosh t - \sinh t + (1+t/2)e^{-t}. \tag{6.3}$$

Table 3 shows that, for a uniform mesh, we obtain $O(k^2)$ convergence globally and $O(k^3)$ convergence at the nodes, as expected from the error bounds in Theorems 3.1 and 4.1.

N	$\|U-u\|_\infty^{N,5}$		$\|U-u\|_\infty^{N,1}$	
40	1.56e-04		2.55e-07	
80	3.92e-05	1.991	3.20e-08	2.998
160	9.83e-06	1.995	4.00e-09	2.999
320	2.46e-06	1.997	5.00e-10	2.999
640	6.16e-07	1.998	6.25e-11	3.000

Table 3. Global and nodal errors for a uniform mesh ($\gamma = 1$) when $\beta(t) = e^{-2t}$. We observe $O(k^2)$ and $O(k^3)$ convergence, respectively.

6.2. **A problem in one space dimension.** Let

$$\beta(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \qquad \Omega = (0,1), \qquad Au = -u_{xx},$$

and assume that $u = u(x,t)$ satisfies homogeneous Dirichlet boundary conditions $u(0,t) = 0 = u(1,t)$ for all $t \in [0,T] = [0,1]$. The solution operator for the homogeneous problem ($f \equiv 0$) is given in terms of the Mittag–Leffler function and the eigensystem of A by

$$\mathcal{E}(t)v = \sum_{m=1}^{\infty} \langle v, \phi_m\rangle E_{\alpha+1}(-\lambda_m t^{\alpha+1})\phi_m, \quad \lambda_m = (m\pi)^2, \quad \phi_m(x) = \sqrt{2}\sin(m\pi x),$$

and for the inhomogeneous problem a Duhamel principle yields an integral representation

$$u(t) = \mathcal{E}(t)u_0 + \int_0^t \mathcal{E}(t-s)f(s)\,ds;$$

see [10] or [11]. We choose $u_0(x) = \phi_1(x)/\sqrt{2} = \sin(\pi x)$ for the initial data and $f(t,x) = (\alpha+1)\,t^\alpha \sin(\pi x)$ for the inhomogeneous term, and find that

$$u(t) = \left\{E_{\alpha+1}(-\pi^2 t^{\alpha+1})\left(1 - \frac{\Gamma(\alpha+2)}{\pi^2}\right) + \frac{\Gamma(\alpha+2)}{\pi^2}\right\}\sin(\pi x). \tag{6.4}$$

Thus, the first regularity condition (1.9) holds for $\sigma \le \alpha + 2$, the second condition (1.10) holds for $\sigma \le \alpha + 1$, and the additional assumption (1.14) is also satisfied.

We apply our fully discrete scheme (5.1) with a time mesh of the form (1.13) and a uniform spatial mesh with N_x subintervals, each of length $h = 1/N_x$. We choose U_h^0 to be the L_2 projection of the initial data u_0 onto the space of continuous, piecewise-linear functions S_h. Taking $\sigma = \alpha + 1$ in Theorem 5.2 we see that the global error $\|U_h - u\|_{L_\infty(L_2)}$ is of order $h^2\ell(k) + k^{\gamma(\alpha+1)}$ for $1 \le \gamma < 2/(\alpha+1)$, and of order $h^2\ell(k) + k^2$ for $\gamma > 2/(\alpha+1)$. With $\alpha = 0.4$ and defining the norm $\|v\|_\infty^{N,m} = \max_{t\in\mathcal{G}^{N,m}} \|v\|_{L_2(\Omega)}$, we obtain the results shown in Table 4, which are consistent with our theoretical error bounds. Putting $\sigma = \alpha + 2$ in Corollary 5.4 gives $\gamma^* = 3/(2\alpha+2)$ so we expect the nodal error $\|U_h(t_n) - u(t_n)\|$ to be of order $h^2\ell(k) + k^{\gamma(2\alpha+2)\gamma}$ for $1 \le \gamma < \gamma^*$ and of order $h^2\ell(k) + k^3$ for $\gamma \ge \gamma^*$. We observe this behaviour in Table 5 for the case $\alpha = 0.3$.

$N = N^x$	$\gamma = 1$		$\gamma = 1.45$		$\gamma = 2$	
20	2.69e-03		1.26e-03		1.38e-03	
40	1.07e-03	1.31	3.23e-04	1.96	3.54e-04	1.96
80	4.17e-04	1.36	8.15e-05	1.98	8.97e-05	1.98
160	1.59e-04	1.38	2.04e-05	1.99	2.25e-05	1.99
320	6.07e-05	1.39	5.12e-06	1.99	5.65e-06	1.99

TABLE 4. The error $\|U_h - u\|_\infty^{N,5}$ with different mesh gradings, when $\alpha = 0.4$. Taking $N_x = N$, we observe convergence of order $h^2\ell(k)+k^{(\alpha+1)\gamma}$ for $1 \le \gamma \le 2/(\alpha+1) \approx 1.4286$, and order $h^2\ell(k)+k^2$ for $\gamma > 2/(\alpha+1)$.

	$N^x = N$		$N^x = N^{3/2}$					
N	$\gamma = 1$		$\gamma = 1$		$\gamma = 1.2$		$\gamma = 1.4$	
36	2.25e-04		1.67e-05		6.67e-06		6.14e-06	
49	1.22e-04	1.98	8.03e-06	2.37	2.71e-06	2.91	2.40e-06	3.03
64	7.23e-05	1.98	4.18e-06	2.44	1.25e-06	2.90	1.07e-06	3.03
81	4.53e-05	1.96	2.33e-06	2.48	6.30e-07	2.91	5.25e-07	3.02
100	2.98e-05	2.00	1.37e-06	2.51	3.40e-07	2.91	2.77e-07	3.02
121	2.04e-05	1.99	8.34e-07	2.61	1.95e-07	2.92	1.56e-07	3.02

TABLE 5. The nodal error $\|U_h - u\|_\infty^{N,1}$ for various mesh gradings, when $\alpha = 0.3$. We observe convergence of order $h^2\ell(k) + k^{\gamma(2\alpha+1)}$ for $1 \le \gamma < \gamma^* = 3/(2\alpha+2) \approx 1.1538$, and of order $h^2\ell(k) + k^3$ for $\gamma \ge \gamma^*$.

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Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia.
E-mail address: kassem@kfupm.edu.sa

School of Mathematics and Statistics, The University of New South Wales, Sydney 2052, Australia.
E-mail address: w.mclean@unsw.edu.au