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t-class semigroups of integral domains

To Marco Fontana on the occasion of his sixtieth birthday

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Abstract. The t-class semigroup of an integral domain is the semigroup of the isomorphy classes of the t-ideals with the operation induced by ideal t-multiplication. This paper investigates ring-theoretic properties of an integral domain that reflect reciprocally in the Clifford or Boolean property of its t-class semigroup. Contexts (including Lipman and Sally-Vasconcelos stability) that suit best t-multiplication are studied in an attempt to generalize well-known developments on class semigroups. We prove that a Prüfer v-multiplication domain (PVMD) is of Krull type (in the sense of Griffin [24]) if and only if its t-class semigroup is Clifford. This extends Bazzoni and Salce's results on valuation domains [11] and Prüfer domains [7], [8], [9], [10].

1. Introduction

The class semigroup of an integral domain R, denoted $\mathcal{S}(R)$, is the semigroup of nonzero fractional ideals modulo its subsemigroup of nonzero principal ideals [11], [45]. We define the t-class semigroup of R, denoted $\mathcal{S}_t(R)$, to be the semigroup of fractional t-ideals modulo its subsemigroup of nonzero principal ideals, that is, the semigroup of the isomorphy classes of the t-ideals of R with the operation induced by t-multiplication. One may regard $\mathcal{S}_t(R)$ as the t-analogue of $\mathcal{S}(R)$, exactly, as the class group $\mathrm{Cl}(R)$ is the t-analogue of the Picard group $\mathrm{Pic}(R)$. We have $\mathrm{Pic}(R) \subseteqq \mathrm{Cl}(R) \subseteqq \mathcal{S}_t(R) \subseteqq \mathcal{S}(R)$. The first and third containments turn into equality in the class of Prüfer domains as the second does so in the class of Krull domains. More details on the t-operation are provided in the next section.

A commutative semigroup S is said to be Clifford if every element x of S is (von Neumann) regular, i.e., there exists $a \in S$ such that $x^2a = x$. The importance of a Clifford semigroup S resides in its ability to stand as a disjoint union of subgroups G_e, where e ranges over the set of idempotent elements of S, and G_e is the largest subgroup of S with identity equal to e (cf. [30]). The semigroup S is said to be Boolean if for each $x \in S$, $x = x^2$.

Divisibility properties of R are often reflected in group or semigroup-theoretic properties of $\mathrm{Cl}(R)$ or $\mathcal{S}(R)$. If R is a Prüfer domain, $\mathrm{Cl}(R)$ equals its ideal class group, and then R

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is a Bézout domain if and only if $\mathrm{Cl}(R) = 0$. If R is a Krull domain, $\mathrm{Cl}(R)$ equals its usual divisor class group, and then R is a UFD if and only if $\mathrm{Cl}(R) = 0$. So an integral domain R is a UFD if and only if every t-ideal of R is principal. Trivially, Dedekind domains (resp., PIDs) have Clifford (resp., Boolean) class semigroup. In 1994, Zanardo and Zannier proved that all orders in quadratic fields have Clifford class semigroup [45]. They also showed that the ring of all entire functions in the complex plane (which is Bézout) fails to have this property. In 1996, Bazzoni and Salce investigated the structure of $\mathscr{S}(V)$ for any arbitrary valuation domain V, stating that $\mathscr{S}(V)$ is always Clifford [11]. In [7], [8], [9], Bazzoni examined the case of Prüfer domains of finite character, showing that these, too, have Clifford class semigroup. In 2001, she completely resolved the problem for the class of integrally closed domains by proving that "*R is an integrally closed domain with Clifford class semigroup if and only if R is a Prüfer domain of finite character*" [10], Theorem 4.5. It is worth recalling that, in the series of papers [39], [40], [41], Olberding undertook an extensive study of (Lipman and Sally-Vasconcelos) stability conditions which prepared the ground to address the correlation between stability and the theory of class semigroups.

A domain R is called a PVMD (Prüfer v-multiplication domain) if the v-finite v-ideals form a group under the t-multiplication; equivalently, if R_M is a valuation domain for each t-maximal ideal M of R. Ideal t-multiplication converts ring notions such as PID, Dedekind, Bézout (of finite character), Prüfer (of finite character), and integrality to UFD, Krull, GCD (of finite t-character), PVMD (of finite t-character), and pseudo-integrality, respectively. Recall at this point that the PVMDs of finite t-character (i.e., each proper t-ideal is contained in only finitely many t-maximal ideals) are exactly the Krull-type rings introduced and studied by Griffin in 1967–68 [23], [24]. Also pseudo-integrality (which should be termed t-integrality) was introduced and studied in 1991 by D. F. Anderson, Houston and Zafrullah [4]. We'll provide more details about this property which turned to be crucial for our study.

This paper examines ring-theoretic properties of an integral domain which reciprocally reflect in semigroup-theoretic properties of its t-class semigroup. Notions and contexts that suit best t-multiplication are studied in an attempt to parallel analogous developments and generalize well-known results on class semigroups. Recall from [10], [32] that an integral domain R is Clifford regular (resp., Boole regular) if $\mathscr{S}(R)$ is a Clifford (resp., Boolean) semigroup. A first correlation between regularity and stability conditions can be sought through Lipman stability. Indeed, R is called an L-stable domain if $\bigcup_{n \geqq 1} (I^n : I^n) = (I : I)$ for every nonzero ideal I of R [1]. Lipman introduced the notion of stability in the specific setting of one-dimensional commutative semi-local Noetherian rings in order to give a characterization of Arf rings; in this context, L-stability coincides with Boole regularity [37]. By analogy, we call an integral domain R Clifford (resp., Boole) t-regular if $\mathscr{S}_t(R)$ is a Clifford (resp., Boolean) semigroup. Clearly, a Boole t-regular domain is Clifford t-regular.

Section 2 establishes t-analogues of basic results on t-regularity. We notice that a Krull domain (resp., UFD) is Clifford (resp., Boole) t-regular. These two classes of domains serve as a starting ground for t-regularity (as Dedekind domains and PIDs do for regularity). We show that t-regularity stands as a default measure for some classes of Krull-like domains. For instance, it measures how far a t-almost Dedekind domain [33] is from being a Krull domain or a UFD. In particular, we'll see that "UFD = Krull + Boole t-regular". While an integrally closed Clifford regular domain is Prüfer [45], an integrally closed Clifford t-regular domain need not be a PVMD; an example is provided in this regard (cf. Example 2.8). As a prelude to this, our main theorem of this section (Theorem 2.8) investigates the transfer of t-regularity to pseudo-valuation domains; namely, a PVD R is always Clifford t-regular; moreover, R is Boole t-regular if and only if it is issued from a Boole regular valuation ring.

Section 3 seeks a satisfactory t-analogue for Bazzoni's theorem on Prüfer domains of finite character [10], Theorem 4.5 (quoted above). From [4], the pseudo-integral closure of a domain R is defined as $\tilde{R} = \bigcup (I_t : I_t)$, where I ranges over the set of finitely generated ideals of R; and R is said to be pseudo-integrally closed if $R = \tilde{R}$. Clearly $R' \subseteqq \tilde{R} \subseteqq \bar{R}$, where R' and $\bar{R}$ are respectively the integral closure and the complete integral closure of R. In view of Example 2.8 (mentioned above), one has to elevate the "integrally closed" assumption in regularity results to "pseudo-integrally closed". In this vein, we conjecture that "*a pseudo-integrally closed domain is Clifford t-regular if and only if it is a Krull-type domain*". Our main theorem of this section (Theorem 3.2) asserts that "*a* PVMD *is Clifford t-regular if and only if it is a Krull-type domain*". It recovers Bazzoni's theorem and also reveals the fact that in the class of PVMDs, Clifford t-regularity coincides with the finite t-character condition. Moreover, we are able to validate the conjecture in a large class of integral domains (Corollary 3.12).

Section ~~5~~ is devoted to generating examples. We treat the possible transfer of the PVMD notion endowed with the finite t-character condition to pullbacks and polynomial rings. Original families of integral domains with Clifford t-class semigroup stem from our results.

All rings considered in this paper are integral domains. For the convenience of the reader, Figure 1 displays a diagram of implications summarizing the relations between the main classes of integrally closed domains that provide a suitable environment for our study. It also places (t-)regularity in a ring-theoretic perspective.

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2. Basic results on *t*-regularity

Let R be a domain with quotient field K. We first review some terminology related to the v- and t-operations. For a nonzero fractional ideal I of R, let I^{-1} denote $(R : I) = \{x \in K \mid xI \subseteqq R\}$. The v- and t-closures of I are defined, respectively, by $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup J_v$ where J ranges over the set of finitely generated subideals of I. The (nonzero) ideal I is said to be divisorial or a v-ideal if $I_v = I$, and a t-ideal if $I_t = I$. Under the ideal t-multiplication $(I, J) \mapsto (IJ)_t$, the set $F_t(R)$ of fractional t-ideals of R is a semigroup with unit R. An invertible element for this operation is called a t-invertible t-ideal of R. So that the set $\mathrm{Inv}_t(R)$ of t-invertible fractional t-ideals of R is a group with unit R. For more basic details about star operations, we refer the reader to [22], Sections 32 and 34. Let $F(R)$, $\mathrm{Inv}(R)$, and $P(R)$ denote the sets of nonzero, invertible, and nonzero principal fractional ideals of R, respectively. Under this notation, the (t-)class groups and semigroups are defined as follows: $\mathrm{Pic}(R) = \mathrm{Inv}(R)/P(R)$, $\mathrm{Cl}(R) = \mathrm{Inv}_t(R)/P(R)$, $\mathscr{S}(R) = F(R)/P(R)$, and $\mathscr{S}_t(R) = F_t(R)/P(R)$.

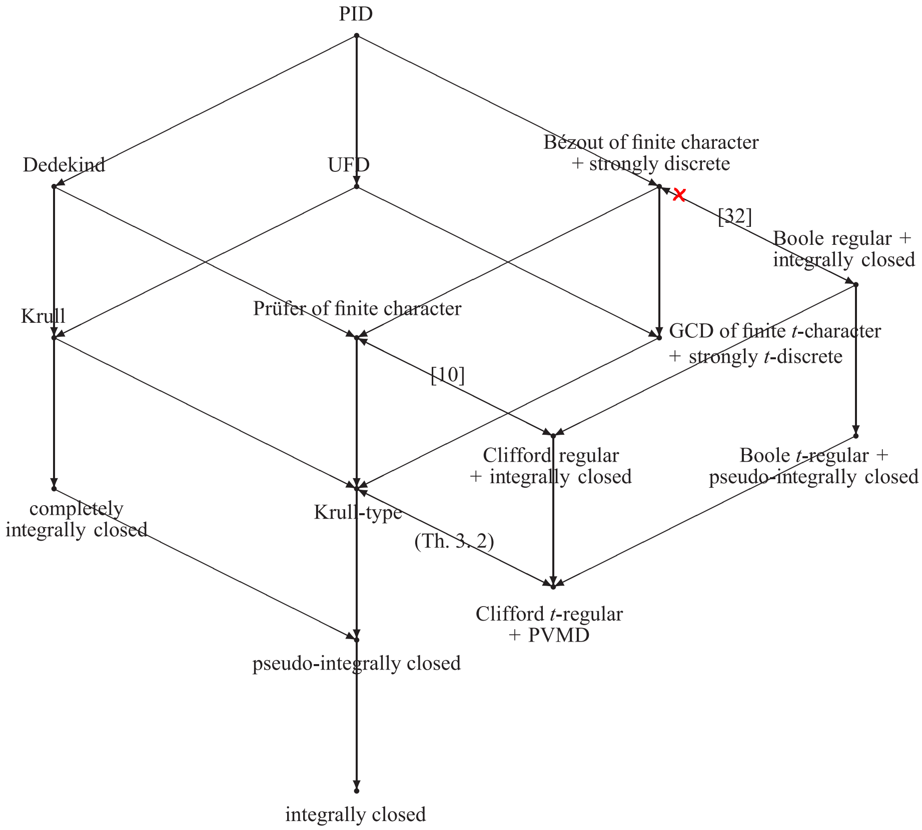

Figure 1. A ring-theoretic perspective for (t-)regularity

Recall two basic properties of the t-operation which will be used (in different forms) throughout the paper. For any two nonzero ideals I and J of a domain R, we have $(IJ)_t = (I_tJ)_t = (IJ_t)_t = (I_tJ_t)_t$. Also one can easily check that $(I_t : J) = (I_t : J_t)$. In particular, we have $I^{-1} = (R : I) = (R : I_t)$ and, if I is a t-ideal, $(I : I^2) = \big(I : (I^2)_t\big)$. Actually, these properties hold for any star operation.

Throughout, we shall use $\mathrm{qf}(R)$ to denote the quotient field of a domain R and $\bar{I}$ to denote the isomorphy class of an ideal I of R in $\mathscr{S}_t(R)$.

Our first result displays necessary and/or sufficient ideal-theoretic conditions for the isomorphy class of an ideal to be regular in the t-class semigroup.

Lemma 2.1. *Let I be a t-ideal of a domain R. Then*:

(1) *$\bar{I}$ is regular in $\mathscr{S}_t(R)$ if and only if $I = \big(I^2(I : I^2)\big)_t$.*

(2) *If I is t-invertible, then $\bar{I}$ is regular in $\mathscr{S}_t(R)$.*

Proof. (1) Assume $\bar{I}$ is regular in $\mathscr{S}_t(R)$. Then there exist a fractional t-ideal J of R and $0 \neq c \in \mathrm{qf}(R)$ such that $I = c(JI^2)_t = (cJI^2)_t$. We may denote cJ by J, that is,

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$I = (JI^2)_t$. Since $JI^2 \subseteq (JI^2)_t = I$, $J \subseteq (I : I^2)$. So $I = (JI^2)_t \subseteq \big(I^2(I : I^2)\big)_t \subseteq I$ and hence $I = \big(I^2(I : I^2)\big)_t$. The converse is trivial.

(2) Assume $(II^{-1})_t = R$. Then $R \subseteq (I : I) \subseteq (II^{-1} : II^{-1}) \subseteq \big((II^{-1})_t : (II^{-1})_t\big) = R$. So $(I : I^2) = \big((I : I) : I\big) = I^{-1}$. Hence

$$\big(I^2(I : I^2)\big)_t = (I^2I^{-1})_t = \big(I(II^{-1})\big)_t = \big(I(II^{-1})_t\big)_t = I.$$

By (1), $\bar{I}$ is regular in $\mathscr{S}_t(R)$. □

Next, we show that Krull domains (resp., UFDs) are Clifford (resp., Boole) *t*-regular. Further, we identify *t*-regularity as a default condition for some classes of Krull-like domains towards the Krull (or UFD) property. Recall at this point that a domain R is Krull if and only if every *t*-ideal of R is *t*-invertible.

Proposition 2.2. (1) *Any Krull domain is Clifford t-regular.*

(2) *A domain R is a* UFD *if and only if R is Krull and Boole t-regular.*

Proof. (1) Follows from Lemma 2.1(2).

(2) Clearly, a UFD is Boole *t*-regular. We only need to prove the "if" assertion. Assume R is Krull and Boole *t*-regular and let I be a *t*-ideal of R. There exists $0 \neq c \in \mathrm{qf}(R)$ such that $(I^2)_t = cI$. Then $(I : I^2) = \big(I : (I^2)_t\big) = (I : cI) = c^{-1}(I : I)$. R is completely integrally closed, then $(I : I) = R$, so that $(I : I^2) = (R : I) = I^{-1}$. Therefore $I^{-1} = c^{-1}R$, and hence $II^{-1} = c^{-1}I$. Since I is *t*-invertible, $R = (II^{-1})_t = (c^{-1}I)_t = c^{-1}I$, hence $I = cR$. It follows that $\mathrm{Cl}(R) = \mathscr{S}_t(R) = 0$, i.e., R is a UFD. □

Recall from [33] that a domain R is said to be *t*-almost Dedekind if R_M is a rank-one DVR for each *t*-maximal ideal M of R. This notion falls strictly between the classes of Krull domains and PVMDs. Our next result shows that *t*-regularity measures how far a *t*-almost Dedekind domain or completely integrally closed domain is from being Krull or a UFD. A domain R is said to be strongly *t*-discrete if it has no *t*-idempotent *t*-prime ideals, i.e., for every *t*-prime ideal P of R, $(P^2)_t \subsetneq P$ (cf. [15]).

Proposition 2.3. *Let R be a domain. The following statements are equivalent*:

(i) *R is Krull* (*resp.*, *a* UFD).

(ii) *R is t-almost Dedekind and Clifford* (*resp.*, *Boole*) *t-regular.*

(iii) *R is strongly t-discrete*, *completely integrally closed, and Clifford* (*resp.*, *Boole*) *t-regular.*

Proof. (i) ⇒ (ii) Straightforward.

(ii) ⇒ (i) Suppose there exists a *t*-ideal I of R which is not *t*-invertible. Then $J = (II^{-1})_t$ is a proper trace *t*-ideal of R with $J^{-1} = (J : J)$. Further, R is completely integrally closed since $R = \bigcap R_M$, where M ranges over the *t*-maximal ideals of R [33], Proposition 2.9. Therefore $J^{-1} = (J : J) = R$, so that

$$J^2(J:J^2) = J^2\big((J:J):J\big) = J^2J^{-1} = J^2.$$

Now, $\bar{J}$ is regular in $\mathscr{S}_t(R)$, then $J = \big(J^2(J:J^2)\big)_t = (J^2)_t$. By induction, we get $J = (J^n)_t$, for each $n \geqq 1$. By [33], Proposition 2.54, $J = \bigcap_{n\geqq 1}(J^n)_t = (0)$, the desired contradiction.

(i) $\Rightarrow$ (iii) Let P be a t-prime ideal of R. Since R is Krull, $(PP^{-1})_t = R$. Suppose P is t-idempotent, i.e., $(P^2)_t = P$. Then $\big((P^2)_tP^{-1}\big)_t = (PP^{-1})_t = R$. Hence $P = \big((PP^{-1})_tP\big)_t = (P^2P^{-1})_t = \big((P^2)_tP^{-1}\big)_t = R$, absurd.

(iii) $\Rightarrow$ (i) Suppose there is a t-ideal I of R such that $J = (II^{-1})_t \subsetneqq R$. Here too we have $J^{-1} = (J:J) = R$. Let M be a t-maximal ideal of R containing J. Necessarily, $(M:M) = M^{-1} = R$. Therefore $(M:M^2) = \big((M:M):M\big) = (R:M) = M^{-1} = R$. So $M^2(M:M^2) = M^2$. Since R is Clifford t-regular, then $M = \big(M^2(M:M^2)\big)_t = (M^2)_t$ and hence M is t-idempotent, absurd.

The Boolean statements follow readily from the Clifford statements combined with Proposition 2.2, completing the proof. □

Notice that the ring of all entire functions in the complex plane is (Bézout) strongly (t-)discrete [18], Corollary 8.1.6, and completely integrally closed, but it is not (t-)almost Dedekind (since it has an infinite Krull dimension). Also the "strongly t-discrete" assumption in (iii) is not superfluous, since a non-discrete rank-one valuation domain is completely integrally closed and Clifford (t-)regular [11], but it is not Krull.

The next result establishes the transfer of t-regularity to polynomial rings. Recall at this point that Clifford or Boole regularity of a polynomial ring $R[X]$ forces R to be a field [32], Corollary 2.5.

Proposition 2.4. *Let R be an integrally closed domain and X an indeterminate over R. Then R is Clifford* (*resp.*, *Boole*) *t-regular if and only if so is $R[X]$.*

Proof. Assume that R is Clifford t-regular and let J be a t-ideal of $R[X]$ with $I = J \cap R$. If $I \neq 0$, then I is a t-ideal of R and hence $J = I[X]$. If $I = (0)$, then $J = fA[X]$ for some $f \in R[X]$ and A a fractional t-ideal of R [42]. So that $J^2(J:J^2)$ equals $\big(I^2(I:I^2)\big)[X]$ or $f\big(A^2(A:A^2)\big)[X]$. In both cases, $\big(J^2(J:J^2)\big)_t = J$ by [33], Proposition 2.3(1) (which ensures that the t-operation is stable under ideal extension). Therefore $\bar{J}$ is regular in $\mathscr{S}_t(R[X])$. Conversely, if I is a t-ideal of R, consider the t-ideal $I[X]$ of $R[X]$ and apply the same techniques backward. Similar arguments as above lead to the conclusion for the Boolean statement. □

The next result establishes the transfer of t-regularity to two types of overrings.

Proposition 2.5. *Let R be a Clifford* (*resp.*, *Boole*) *t-regular domain. Then*:

(1) *R_S is Clifford* (*resp.*, *Boole*) *t-regular, for any multiplicative subset S of R.*

(2) *$(I_v : I_v)$ is Clifford* (*resp.*, *Boole*) *t-regular, for any nonzero ideal I of R.*

For the proof, we need the following lemma.

Lemma 2.6. *Let R be a domain, I a fractional ideal of R, and S a multiplicative subset of R. Then $I_t \subseteqq (IR_S)_{t_1}$, where t_1 denotes the t-operation with respect to R_S.*

Proof. Let $x \in I_t$. Then there exists a finitely generated ideal A of R such that $A \subseteqq I$ and $x(R : A) \subseteqq R$. Hence $x(R_S : AR_S) = x(R : A)R_S \subseteqq R_S$. Therefore $x \in (AR_S)_{t_1} \subseteqq (IR_S)_{t_1}$. $\square$

Proof of Proposition 2.5. (1) If J is a t-ideal of R_S, then $I = J \cap R$ is a t-ideal of R by Lemma 2.6. Since R is Clifford (resp., Boole) t-regular, then $I = \big(I^2(I : I^2)\big)_t$ (resp., $(I^2)_t = cI$ for some nonzero $c \in \mathrm{qf}(R)$). Hence

$$J = IR_S = \big(I^2(I : I^2)\big)_t R_S \subseteqq \big(\big(I^2(I : I^2)\big)R_S\big)_{t_1} \subseteqq \big(J^2(J : J^2)\big)_{t_1} \subseteqq J$$

(resp., $cJ = cIR_S = (I^2)_t R_S \subseteqq (I^2R_S)_{t_1} = (J^2)_{t_1} \subseteqq cJ$, since $I^2 \subseteqq (I^2)_t = cI$ and then $J^2 \subseteqq cJ$). Therefore $J = \big(J^2(J : J^2)\big)_{t_1}$ (resp., $(J^2)_{t_1} = cJ$). It follows that R_S is Clifford (resp., Boole) t-regular.

(2) Let I be a nonzero ideal of R and set $T = (I_v : I_v)$. Since

$$T = (II^{-1})^{-1} = (II^{-1} : II^{-1}) = \big((II^{-1})_v : (II^{-1})_v\big),$$

without loss of generality, we may assume that I is a trace v-ideal of R, that is $T = I^{-1} = (I : I)$. Also denote by v_1 and t_1 the v- and t-operations with respect to T. Let J be a nonzero ideal of T. Then J is a fractional ideal of R and we claim that $J_t \subseteqq J_{t_1}$. Indeed, let $x \in J_t$. Then there exists a finitely generated (fractional) ideal A of R such that $A \subseteqq J$ and $x(R : A) \subseteqq R$. Let $z \in (T : AT)$. Then $zAI \subseteqq I \subseteqq R$, hence $zI \subseteqq (R : A)$, whence $xzI \subseteqq x(R : A) \subseteqq R$ and $xz \in I^{-1} = T$. Therefore $x(T : AT) \subseteqq T$, and hence $x \in (AT)_{v_1} = (AT)_{t_1} \subseteqq J_{t_1}$. Consequently, if J is a t-ideal of T, then it's a t-ideal of R. Since R is Clifford (resp., Boole) t-regular, then $J = \big(J^2(J : J^2)\big)_t \subseteqq \big(J^2(J : J^2)\big)_{t_1} \subseteqq J$ (resp., $cJ = (J^2)_t \subseteqq (J^2)_{t_1} \subseteqq cJ$, since $J^2 \subseteqq (J^2)_t = cJ$, for some nonzero $c \in \mathrm{qf}(R) = \mathrm{qf}(T)$). Hence $J = \big(J^2(J : J^2)\big)_{t_1}$ (resp., $(J^2)_{t_1} = cJ$) and therefore T is Clifford (resp., Boole) t-regular. $\square$

We close this section with an investigation of the integrally closed setting. In this vein, recall Zanardo-Zannier's crucial result that an integrally closed Clifford regular domain is necessarily Prüfer [45]. In [32], we stated an analogue for Boole regularity, that is, an integrally closed Boole regular domain is Bézout. Next, we show that an integrally closed Clifford (or Boole) t-regular domain need not be a PVMD, the natural context for t-regularity. Our family of such examples stems from the following theorem on the inheritance of t-regularity by PVDs (i.e., pseudo-valuation domains). We refer the reader to [25] for the definition and the main properties of PVDs.

Theorem 2.7. *Let R be a* PVD. *Then*:

(1) *R is Clifford t-regular.*

(2) *R is Boole t-regular if and only if its associated valuation overring is Boole regular.*

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Proof. (1) We may assume that R is not a valuation domain. [3], Proposition 2.6 characterizes PVDs in terms of pullbacks. The aforementioned proposition states that R is a PVD if and only if $R = \phi^{-1}(k)$ for some subfield k of $K = V/M$, where V is the associated valuation overring of R, M its maximal ideal and ϕ the canonical homomorphism from V onto K. Now, let I be a t-ideal of R. If I is an ideal of V, we are done (since V is Clifford regular). If I is not an ideal of V, then $I = c\phi^{-1}(W)$, where $0 \neq c \in M$ and W is a k-vector space such that $k \subseteq W \subset K$ (cf. [6], Theorem 2.1(n)). Assume $k \subsetneqq W$. Then $(k : W) = (0)$. Hence $I^{-1} = (R : I) = (\phi^{-1}(k) : c\phi^{-1}(W)) = c^{-1}\phi^{-1}(k : W) = c^{-1}M$ by [27], Proposition 6. Since R is a PVD which is not a valuation domain, by [29], Proposition 4.3, R is a TV-domain (i.e. the t- and v-operations coincide in R). Hence $I = I_t = I_v = (R : c^{-1}M) = cM^{-1} = cV$ is an ideal of V, a contradiction. Therefore $k = W$ and then $I = cR$ is a principal ideal of R. So $\bar{I}$ is regular in $\mathcal{S}_t(R)$, as desired.

(2) Assume that R is Boole t-regular. By Proposition 2.5, $V = (M : M) = (M_v : M_v)$ is Boole regular (the t-operation on V is trivial). Conversely, assume that V is Boole regular. Similar arguments as above lead to the conclusion. □

Contrast this result with [32], Theorem 5.1, which asserts that a PVD R associated to a valuation (resp., strongly discrete valuation) domain (V, M) is Clifford (resp., Boole) regular if and only if $[V/M : R/M] = 2$.

Example 2.8. There exists an integrally closed Boole (hence Clifford) t-regular domain which is not a PVMD. Indeed, let k be a field and let X and Y be two indeterminates over k. Let $R = k + M$ be the PVD associated to the rank-one DVR $V = k(X)[[Y]] = k(X) + M$, where $M = YV$. Clearly, R is integrally closed and, by Theorem 2.7, R is Boole t-regular. However, R is not a PVMD by [17], Theorem 4.1.

3. Clifford *t*-regularity

Recall from [24] that a Krull-type domain is a PVMD with finite t-character (i.e., each nonzero nonunit is contained in only finitely many t-maximal ideals). Also a domain R is said to be pseudo-integrally closed if $R = \tilde{R} = \bigcup (I_t : I_t)$, where I ranges over the set of finitely generated ideals of R [4]. This section seeks a t-analogue for Bazzoni's theorem that "an integrally closed domain R is Clifford regular if and only if R is a Prüfer domain of finite character" [10], Theorem 4.5. In view of Example 2.8, one has to elevate the "integrally closed" assumption to "pseudo-integrally closed." Accordingly, we claim the following:

Conjecture 3.1. *A pseudo-integrally closed domain R is Clifford t-regular if and only if R is a Krull-type domain.*

This is still elusively open. Yet, our main result (Theorem 3.2) of this section recovers Bazzoni's theorem and validates this conjecture in large classes of (pseudo-integrally closed) domains.

Theorem 3.2. *A* PVMD *is Clifford t-regular if and only if it is a Krull-type domain* (*i.e., in a* PVMD, *Clifford t-regularity coincides with the finite t-character condition*).

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The proof of the theorem involves several preliminary results, some of which are of independent interest. Experts of t-operation may skip the proofs of Lemmas 3.8, 3.9 and 3.10 which are similar in form to their respective analogues for the trivial operation.

The following notation, connected with the t-ideal structure of a PVMD, will be of use in the sequel. Assume R is a PVMD and let I be a t-ideal of R and x a nonzero nonunit element of R. We shall use $\mathrm{Max}_t(R)$ to denote the set of maximal t-ideals of R. Set $\mathrm{Max}_t(R, I) = \{M \in \mathrm{Max}_t(R) \mid I \subseteq M\}$, $\mathrm{Max}_t(R, x) = \mathrm{Max}_t(R, xR)$, and $\mathcal{T}_t(R) = \{M \in \mathrm{Max}_t(R) \mid R_M \not\supseteq \bigcap_{M \neq N} R_N, N \in \mathrm{Max}_t(R)\}$. Finally, given M and N two t-maximal ideals of R, we denote by $M \wedge N$ the largest prime ideal of R contained in $M \cap N$. Note that prime ideals of R contained in a t-maximal ideal are necessarily t-ideals and form a chain.

Lemma 3.3. *Let R be a* PVMD *and I a fractional ideal of R. Then for every t-prime ideal P of R, $I_tR_P = IR_P$.*

Proof. Here R_P is a valuation domain (where the t- and trivial operations coincide), so Lemma 2.6 leads to the conclusion. □

Lemma 3.4. *Let R be a* PVMD *which is Clifford t-regular and I a nonzero fractional ideal of R. Then I is t-invertible if and only if I is t-locally principal.*

Proof. Suppose I is t-locally principal and set $J = (II^{-1})_t$. Let $M \in \mathrm{Max}_t(R)$. Then $IR_M = aR_M$ for some nonzero $a \in I$. By Lemma 3.3,

$$(I_t : I_t) \subseteq (I_tR_M : I_tR_M) = (IR_M : IR_M) = (aR_M : aR_M) = R_M.$$

Therefore $R \subseteq (I_t : I_t) \subseteq \bigcap_{M \in \mathrm{Max}_t(R)} R_M = R$. So $(I_t : I_t^2) = \big((I_t : I_t) : I_t\big) = (R : I_t) = I^{-1}$. Since R is Clifford t-regular, then $I_t = \big(I_t^2(I_t : I_t^2)\big)_t = (I_t^2I^{-1})_t = (IJ)_t$. By Lemma 3.3, $aR_M = IR_M = I_tR_M = (IJ)_tR_M = IJR_M = aJR_M$. It follows that $R_M = JR_M$ for every $M \in \mathrm{Max}_t(R)$, which forces J to equal R, as desired.

Conversely, assume that I is t-invertible. Then there is a finitely generated ideal J of R such that $J \subseteq I$ and $I_t = J_t$. Hence for each $M \in \mathrm{Max}_t(R)$, $IR_M = I_tR_M = J_tR_M = JR_M = aR_M$ for some $a \in J$, since R_M is a valuation domain. □

Lemma 3.5. *Let R be a* PVMD *which is Clifford t-regular and let $P \subsetneqq Q$ be two t-prime ideals of R. Then there exists a finitely generated ideal I of R such that $P \subsetneqq I_t \subseteq Q$.*

Proof. Let $x \in Q \backslash P$ and set $J = xR + P$. We claim that J is t-invertible. Indeed, let M be a t-maximal ideal of R. If $J \nsubseteq M$, then $JR_M = R_M$. If $J \subseteq M$, then $PR_M \subsetneqq xR_M$ since R_M is a valuation domain, whence $JR_M = xR_M$. So J is t-locally principal and hence t-invertible by Lemma 3.4. Therefore $J_t = I_t$ for some finitely generated subideal I of J. It follows that $P \subsetneqq I_t \subseteq Q$. □

Lemma 3.6. *Let R be a* PVMD *which is Clifford t-regular and P a t-prime ideal of R. Then $E(P) = (P : P)$ is a* PVMD *which is Clifford t-regular and P is a t-maximal ideal of $E(P)$.*

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Proof. If $P \in \mathrm{Max}_t(R)$, then $E(P) = R$. We may then assume that $P \notin \mathrm{Max}_t(R)$. By [26], Proposition 1.2 and Lemma 2.4, $E(P) = P^{-1}$ and P is a t-prime ideal of $E(P)$. Further $E(P) = P^{-1}$ is t-linked over R, so $E(P)$ is a PVMD [33]. Let t_1 and v_1 denote the t- and v-operations with respect to $E(P)$ and let J be a nonzero fractional ideal of $E(P)$. Clearly J is a fractional ideal of R and we claim that $J_t \subseteqq J_{t_1}$. Indeed, let $x \in J_t$. Then there is a finitely generated subideal I of J such that $x \in I_v$. So $xI^{-1} \subseteqq R$. Let $z \in \big(E(P) : IE(P)\big) = (P^{-1} : IP^{-1})$. Then $zIP^{-1} \subseteqq P^{-1}$. So $zIP \subseteqq P \subseteqq R$. Then $zP \subseteqq I^{-1}$. So $xzP \subseteqq xI^{-1} \subseteqq R$. Hence $xz \in P^{-1} = E(P)$. So $x\big(E(P) : IE(P)\big) \subseteqq E(P)$ and therefore $x \in \big(IE(P)\big)_{v_1} \subseteqq J_{t_1}$. Now, let J be a t-ideal of $E(P)$. By the above claim J is a t-ideal of R. Since R is Clifford t-regular, then $J = \big(J^2(J : J^2)\big)_t \subseteqq \big(J^2(J : J^2)\big)_{t_1} \subseteqq J$ and therefore $J = \big(J^2(J : J^2)\big)_{t_1}$. It follows that $E(P)$ is Clifford t-regular. To complete the proof, we need to show that P is a t-maximal ideal of $E(P)$. Deny. Then there is a t-maximal ideal Q of $E(P)$ such that $P \subsetneqq Q$. By Lemma 3.5, there is a finitely generated ideal J of $E(P)$ such that $P \subsetneqq J_{t_1} \subseteqq Q$. On the other hand, since P is a non-t-maximal t-prime ideal of $E(P)$, by [26], Proposition 1.2, $\big(E(P) : P\big) = (P : P) = E(P)$. It follows that $E(P) = P_{v_1} \subseteqq (J_{t_1})_{v_1} = J_{v_1} = J_{t_1} \subseteqq Q$, the desired contradiction. □

Lemma 3.7. *Let R be a* PVMD *which is Clifford t-regular and Q a t-prime ideal of R. Suppose there is a nonzero prime ideal P of R such that $P \subsetneqq Q$ and* $\mathrm{ht}(Q/P) = 1$. *Then there exists a finitely generated subideal I of Q such that* $\mathrm{Max}_t(R, I) = \mathrm{Max}_t(R, Q)$.

Proof. By [33], Corollary 2.47, P is a t-prime ideal of R. By [26], Proposition 1.2 and Lemma 2.4, $E(P) = P^{-1}$ and P is a t-prime ideal of $E(P)$. Therefore $E(Q) = (Q : Q) \subseteqq Q^{-1} \subseteqq P^{-1} = E(P)$, and hence P is a prime ideal of $E(Q)$. By Lemma 3.6, $E(Q)$ is a PVMD which is Clifford t-regular and Q is a t-maximal ideal of $E(Q)$. Thus P is a t-prime ideal of $E(Q)$. By Lemma 3.5, there is a finitely generated subideal $J = \sum_{1 \leqq i \leqq n} a_i E(Q)$ of Q such that $P \subsetneqq J_{t_1} \subseteqq Q$. We claim that $\mathrm{Max}_t\big(E(Q), J\big) = \{Q\}$. Indeed, if there is a t-maximal ideal N of $E(Q)$ such that $J \subseteqq N$, then $P \subsetneqq J_{t_1} \subseteqq N$. So $P \subsetneqq J_{t_1} \subseteqq Q \wedge N \subsetneqq Q$, a contradiction since $\mathrm{ht}(Q/P) = 1$. Now set $I = \sum_{1 \leqq i \leqq n} a_i R$. Clearly $I \subseteqq Q$ and $IE(Q) = J$. We claim that $\mathrm{Max}_t(R, I) = \mathrm{Max}_t(R, Q)$. Let $N \in \mathrm{Max}_t(R, I)$. If $Q \nsubseteqq N$, then $R \subseteqq E(Q) \subseteqq R_N$. Hence $E(Q)_{R \setminus N} = R_N$. So R_N is t-linked over $E(Q)$. Since R_N is a valuation domain, then NR_N is a t-prime ideal of R_N. Hence $M = NR_N \cap E(Q)$ is a t-prime ideal of $E(Q)$. Since $I \subseteqq N$, then $I \subseteqq M$. Hence $J = IE(Q) \subseteqq M$. Necessarily $M \subseteqq Q$ since $\mathrm{Max}_t\big(E(Q), J\big) = \{Q\}$. So

$$N = NR_N \cap R = NR_N \cap E(Q) \cap R = M \cap R \subseteqq Q \cap R = Q,$$

which is absurd. Hence $Q \subseteqq N$ and therefore $\mathrm{Max}_t(R, I) \subseteqq \mathrm{Max}_t(R, Q)$. The reverse inclusion is trivial. □

Lemma 3.8. *Let R be a* PVMD *which is Clifford t-regular and M a t-maximal ideal of R. If $M \in \mathscr{T}_t(R)$, then there exists a finitely generated subideal I of M with* $\mathrm{Max}_t(R, I) = \{M\}$.

Proof. Assume that $M \in \mathscr{T}_t(R)$. Let $x \in \Big(\bigcap_{M \neq N \in \mathrm{Max}_t(R)} R_N\Big) \setminus R_M$. Since R_M is a valuation domain, then $x^{-1} \in MR_M$. Set $I = x^{-1}R \cap R$. We claim that I is t-invertible.

By Lemma 3.4, it suffices to check that I is t-locally principal. Let Q be a t-maximal ideal of R. If $Q \neq M$, then $I \nsubseteq Q$. Indeed, since $Q \neq M$, then $x \in R_Q$. Hence $IR_Q = (x^{-1}R \cap R)R_Q = x^{-1}R_Q \cap R_Q = R_Q$. So $I \nsubseteq Q$. Then M is the unique t-maximal ideal of R that contains I and $IR_M = x^{-1}R_M$, as desired. Hence I is t-invertible. So there is a finitely generated subideal J of I such that $I_t = J_t$ and clearly $\mathrm{Max}_t(R, J) = \{M\}$. □

Lemma 3.9. *Let R be a* PVMD *which is Clifford t-regular. Then every nonzero nonunit element of R belongs to a finite number of t-maximal ideals in $\mathcal{T}_t(R)$.*

Proof. Let x be a nonzero nonunit element of R and let $\{M_\alpha\}_{\alpha\in\Omega}$ be the set of all t-maximal ideals in $\mathcal{T}_t(R)$ that contain x. For each $\alpha \in \Omega$, let A_α be a finitely generated subideal of M_α such that $\mathrm{Max}_t(R, A_\alpha) = \{M_\alpha\}$ (Lemma 3.8). Without loss of generality, we may assume that $x \in A_\alpha$ (otherwise, we consider $B_\alpha = xR + A_\alpha$). Let $B = \sum_{\alpha\in\Omega} (A_\alpha)^{-1}$. Clearly, $x(A_\alpha)^{-1} \subseteq R$, for each $\alpha \in \Omega$. Then $xB \subseteq R$ so that B is a fractional ideal of R. We claim that B is t-locally principal. Indeed, let N be a t-maximal ideal of R. We envisage two cases.

Case 1. $N \neq M_\alpha$ for each $\alpha \in \Omega$. Since A_α is finitely generated, $(A_\alpha)^{-1}R_N = (A_\alpha R_N)^{-1} = R_N$. So $BR_N = R_N$.

Case 2. $N = M_\alpha$ for some $\alpha \in \Omega$. Then $(A_\beta)^{-1}R_N = (A_\beta R_N)^{-1} = R_N$, for each $\beta \neq \alpha$ in Ω. Hence $BR_N = (A_\alpha)^{-1}R_N = (A_\alpha R_N)^{-1} = a^{-1}R_N$ where $A_\alpha R_N = aR_N$ (since R_N is a valuation domain). It follows that B is t-invertible (Lemma 3.4) and hence there is a finitely generated subideal J of B such that $J_v = J_t = B_t = B_v$. So $B^{-1} = J^{-1}$. Since J is finitely generated, then there are $\alpha_1, \dots, \alpha_r$ such that $J \subseteq \sum_{1\leq i\leq r} (A_{\alpha_i})^{-1} \subseteq B$. Therefore $B^{-1} = J^{-1} = \left(\sum_{1\leq i\leq r} (A_{\alpha_i})^{-1}\right)^{-1} = \bigcap_{1\leq i\leq r} (A_{\alpha_i})_v = \bigcap_{1\leq i\leq r} (A_{\alpha_i})_t$. Consequently, for each $\alpha \in \Omega$, we have $\bigcap_{1\leq i\leq r} (A_{\alpha_i})_t = B^{-1} \subseteq (A_\alpha)_v = (A_\alpha)_t \subseteq M_\alpha$. So there is α_i such that $(A_{\alpha_i})_t \subseteq M_\alpha$, hence $M_\alpha = M_{\alpha_i}$, whence $\alpha = \alpha_i$. Therefore $\Omega = \{\alpha_1, \dots, \alpha_r\}$, as desired. □

Lemma 3.10. *Let R be a* PVMD *which is Clifford t-regular and M a t-maximal ideal of R. Then $M \in \mathcal{T}_t(R)$ if and only if $M \supsetneq \bigcup_N M \wedge N$ where N ranges over* $\mathrm{Max}_t(R)\backslash\{M\}$.

Proof. Let $M \in \mathcal{T}_t(R)$ and let $A = \sum_{1\leq i\leq r} a_iR$ be a finitely generated subideal of M such that $\mathrm{Max}_t(R, A) = \{M\}$ (Lemma 3.8). Suppose that $M = \bigcup_N M \wedge N$, where N ranges over $\mathrm{Max}_t(R)\backslash\{M\}$. Then for each $a_i \in A$, there is a t-maximal ideal $N_i \neq M$ such that $a_i \in M \wedge N_i$. Since $\{M \wedge N_i \mid i = 1, \dots, r\}$ is a chain, let $M \wedge N_j$ be the largest one for some $j \in \{1, \dots, r\}$. So $A \subseteq N_j$ and then $N_j \in \mathrm{Max}_t(R, A) = \{M\}$, absurd.

Conversely, let $x \in M \backslash \bigcup_N M \wedge N$. Then, for each t-maximal ideal $N \neq M$, $x^{-1} \in R_N$ (since R_N is a valuation domain), hence $x^{-1} \in \bigcap_{M\neq N\in\mathrm{Max}_t(R)} R_N$. Since $x^{-1} \notin R_M$, then $M \in \mathcal{T}_t(R)$, as desired. □

The following basic facts provide some background to the theorem and will be of use in its proof.

• Fact 1. For each ideal I of R, we have $I_t = \bigcap_{M \in \mathrm{Max}_t(R)} IR_M$ [33], Theorem 2.9.

• Fact 2. Let R be a Prüfer domain, I an ideal of R, and A and B R-submodules of $\mathrm{qf}(R)$. Then $I(A \cap B) = IA \cap IB$ [7], Lemma 2.6.

• Fact 3. For a t-ideal I of a domain R, let $\overline{M}(R,I) = \{M \in \mathrm{Max}_t(R) \mid I \nsubseteq M\}$ and $\mathscr{C}_t(I) = \bigcap_M R_M$ where M ranges over $\overline{M}(R,I)$. Then $(\mathscr{C}_t(I) : I) = \mathscr{C}_t(I)$. Indeed, it is clear that $\mathscr{C}_t(I) \subseteq (\mathscr{C}_t(I) : I)$. Conversely, let $x \in (\mathscr{C}_t(I) : I)$. For each $M \in \overline{M}(R,I)$, let $a \in I \backslash M$. Since $xI \subseteq \mathscr{C}_t(I) \subseteq R_M$, then $xa \in R_M$. So $x = \frac{xa}{a} \in R_M$. Hence $x \in \mathscr{C}_t(I)$ and therefore $(\mathscr{C}_t(I) : I) = \mathscr{C}_t(I)$.

• Fact 4. For each t-ideal I of a domain R with finite t-character, there exists a nonzero finitely generated subideal J of I such that $\mathrm{Max}_t(R,I) = \mathrm{Max}_t(R,J)$. The proof apes that of [7], Lemma 2.13, by replacing "maximal ideals" with "t-maximal ideals."

Proof of Theorem 3.2. Assume R is a PVMD which is Clifford t-regular and let $0 \neq x \in R$. We must show that $\mathrm{Max}_t(R,x)$ is finite. Suppose by way of contradiction that $\mathrm{Max}_t(R,x)$ is infinite. By Lemma 3.9, there is $M \in \mathrm{Max}_t(R,x) \backslash \mathscr{T}_t(R)$. By Lemma 3.10, $M = \bigcup_N M \wedge N$ where N ranges over $\mathrm{Max}_t(R) \backslash \{M\}$. Since R_M is a valuation domain, N may range over $\mathrm{Max}_t(R,x) \backslash \{M\}$, so that $\{P_\alpha\}_{\alpha \in \Omega} = \{M \wedge N\}_{M \neq N \in \mathrm{Max}_t(R,x)}$ is an infinite totally ordered set. For each $\alpha \in \Omega$, we have $0 \subsetneqq (x) \subseteq P_\alpha = M \wedge N_\alpha \subsetneqq N_\alpha$, for some $N_\alpha \in \mathrm{Max}_t(R,x)$. By [35], Theorem 11, there exist distinct prime ideals P'_α and Q_α such that $0 \subsetneqq P_\alpha \subseteq P'_\alpha \subsetneqq Q_\alpha \subseteq N_\alpha$ with $\mathrm{ht}(Q_\alpha/P'_\alpha) = 1$.

Claim 1. For every $\alpha \neq \beta$, Q_α and Q_β are incomparable.

We may assume $P_\alpha \subsetneqq P_\beta$. Suppose that $Q_\alpha \subseteq Q_\beta$. Then Q_α and P_β are comparable. If $Q_\alpha \subseteq P_\beta$, then $P_\alpha \subsetneqq Q_\alpha \subseteq M \wedge N_\alpha = P_\alpha$, absurd. If $P_\beta \subseteq Q_\alpha$, then $P_\beta \subseteq M \wedge N_\alpha = P_\alpha \subsetneqq P_\beta$, absurd. Now, if $Q_\beta \subseteq Q_\alpha$, then $P_\beta \subseteq M \wedge N_\alpha = P_\alpha$, which is absurd too. This proves the claim.

Since $P_\alpha \subsetneqq Q_\alpha$, then $Q_\alpha \nsubseteq M$. For each α, let $a_\alpha \in Q_\alpha \backslash M$ and consider the ideal $J_\alpha = P_\alpha + a_\alpha R$.

Claim 2. J_α is t-invertible.

By Lemma 3.4, it suffices to check that J_α is t-locally principal. Let N be a t-maximal ideal of R. Assume—without loss of generality—that $J_\alpha \subseteq N$. Since R_N is a valuation domain and $a_\alpha \notin P_\alpha$, then $P_\alpha R_N \subsetneqq a_\alpha R_N$. Hence $J_\alpha R_N = a_\alpha R_N$, as desired. Therefore there is a finitely generated subideal F_α of J_α such that $(F_\alpha)_v = (F_\alpha)_t = (J_\alpha)_t = (J_\alpha)_v$.

Moreover, by Lemma 3.7, there is a finitely generated subideal I_α of Q_α such that $\mathrm{Max}_t(R,I_\alpha) = \mathrm{Max}_t(R,Q_\alpha)$. Consider the finitely generated ideal given by $A_\alpha = F_\alpha + I_\alpha$. Since $I_\alpha \subseteq A_\alpha \subseteq Q_\alpha$, then $\mathrm{Max}_t(R,A_\alpha) = \mathrm{Max}_t(R,Q_\alpha)$. Finally, let $B = \sum_{\alpha \in \Omega} (A_\alpha)^{-1}$.

Claim 3. B is a fractional ideal of R which is t-invertible.

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Indeed, for each α, we have $(x) \subseteqq P_\alpha = (P_\alpha)_t \subseteqq (J_\alpha)_t = (F_\alpha)_t \subseteqq (A_\alpha)_t$. So $x(A_\alpha)^{-1} \subseteqq (A_\alpha)_t(A_\alpha)^{-1} = (A_\alpha)_t((A_\alpha)_t)^{-1} \subseteqq R$. Hence $xB \subseteqq R$ and therefore B is a fractional ideal of R. Now let N be a t-maximal ideal of R.

Case 1. $A_\alpha \nsubseteq N$ for each $\alpha \in \Omega$. Since A_α is finitely generated, then $(A_\alpha)^{-1}R_N = (A_\alpha R_N)^{-1} = R_N$. Hence $BR_N = R_N$.

Case 2. $A_\alpha \subseteqq N$ for some $\alpha \in \Omega$. Since $\mathrm{Max}_t(R, A_\alpha) = \mathrm{Max}_t(R, Q_\alpha)$, then for each $\beta \neq \alpha$, $A_\beta \nsubseteq N$. Otherwise, $N \in \mathrm{Max}_t(R, A_\beta) = \mathrm{Max}_t(R, Q_\beta)$. Then Q_α and Q_β are comparable since both are included in N, absurd by the first claim. Thus N contains exactly one A_α. So $BR_N = (A_\alpha)^{-1}R_N = (A_\alpha R_N)^{-1} = a^{-1}R_N$ where $A_\alpha R_N = aR_N$ since $A_\alpha R_N$ is a finitely generated ideal of the valuation domain R_N. It follows that B is t-locally principal and therefore t-invertible (Lemma 3.4).

Consequently, there is a finitely generated subideal L of B such that $L_v = L_t = B_t = B_v$. There exist $\alpha_1, \dots, \alpha_r$ such that $L \subseteqq \sum_{1 \leqq i \leqq r} (A_{\alpha_i})^{-1} \subseteqq B$. Therefore $B^{-1} = L^{-1} = \left(\sum_{1 \leqq i \leqq r} (A_{\alpha_i})^{-1}\right)^{-1} = \bigcap_{1 \leqq i \leqq r} (A_{\alpha_i})_v = \bigcap_{1 \leqq i \leqq r} (A_{\alpha_i})_t$. Now, let $\alpha \in \Omega \backslash \{\alpha_1, \dots, \alpha_r\}$. Then $\bigcap_{1 \leqq i \leqq r} (A_{\alpha_i})_t = B^{-1} \subseteqq (A_\alpha)_v = (A_\alpha)_t \subseteqq Q_\alpha \subseteqq N_\alpha$. So there is $i \in \{1, \dots, r\}$ such that $(A_{\alpha_i})_t \subseteqq N_\alpha$. Hence $N_\alpha \in \mathrm{Max}_t(R, A_{\alpha_i}) = \mathrm{Max}_t(R, Q_{\alpha_i})$ and then $Q_{\alpha_i} \subseteqq N_\alpha$. This forces Q_α and Q_{α_i} to be comparable, the desired contradiction. Thus $\mathrm{Max}_t(R, x)$ is finite.

Next, we prove the converse of the theorem. Assume R is a Krull-type domain. Let I be a t-ideal of R, $\mathrm{Max}_t(R, I) = \{M_1, \dots, M_n\}$ and $J = I^2(I : I^2)$. We wish to show that $I = J_t$. By Fact 1, it suffices to show that $IR_M = JR_M$ for each t-maximal ideal of R. Let $M \in \mathrm{Max}_t(R)$. If $I \nsubseteq M$, then $J \nsubseteq M$ (since $I^2 \subseteqq J$). So $IR_M = JR_M = R_M$. Assume $I \subseteqq M$. Mutatis mutandis, we may assume that $M = M_1$. One can easily check via Fact 1 that $(I : I) = \left(\bigcap_{i=1}^{n} (IR_{M_i} : IR_{M_i})\right) \cap \mathscr{C}_t(I)$. By Fact 3,

$$(I : I^2) = \left(\bigcap_{i=1}^{n} (IR_{M_i} : I^2R_{M_i})\right) \cap \mathscr{C}_t(I) = (IR_{M_1} : I^2R_{M_1}) \cap \left(\bigcap_{i=2}^{n} (IR_{M_i} : I^2R_{M_i})\right) \cap \mathscr{C}_t(I).$$

Let $A = (IR_{M_1} : I^2R_{M_1})$ and $B = \left(\bigcap_{i=2}^{n} (IR_{M_i} : I^2R_{M_i})\right)$. We have

$$JR_{M_1} = I^2R_{M_1}\big(AR_{M_1} \cap BR_{M_1} \cap \mathscr{C}_t(I)R_{M_1}\big).$$

By applying Fact 2 in the valuation domain R_{M_1}, we obtain

$$JR_{M_1} = (I^2R_{M_1}AR_{M_1}) \cap (I^2R_{M_1}BR_{M_1}) \cap \big(I^2R_{M_1}\mathscr{C}_t(I)R_{M_1}\big).$$

On one hand, $I^2R_{M_1}AR_{M_1} = IR_{M_1}$ since R_{M_1} is Clifford regular [11]. Further, we claim that $I^2R_{M_1}BR_{M_1} \supseteqq IR_{M_1}$. Indeed,

$$\begin{aligned} I^2R_{M_1}BR_{M_1} &= \bigcap_{i=2}^{n} I^2R_{M_1}(IR_{M_i} : I^2R_{M_i})R_{M_1} = \bigcap_{i=2}^{n} \big(I^2(IR_{M_i} : I^2R_{M_i})\big)R_{M_1} \\ &= \bigcap_{i=2}^{n} \big(I^2R_{M_i}(IR_{M_i} : I^2R_{M_i})\big)R_{M_1} = \bigcap_{i=2}^{n} IR_{M_i}R_{M_1} \supseteqq IR_{M_1}, \end{aligned}$$

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as claimed; the first equality is due to Fact 2 and the last equality holds because R_{M_i} is Clifford regular.

On the other hand, $\mathscr{C}_t(I)R_{M_1}$ is an overring of R_{M_1} and hence $\mathscr{C}_t(I)R_{M_1} = R_P$ for some t-prime ideal P of R contained in M_1. We claim that $I \nsubseteq P$. Indeed, by Fact 4, there exists a nonzero finitely generated ideal L with $L \subseteq J_t \subseteq I$ and $\mathrm{Max}_t(R, L) = \mathrm{Max}_t(R, J_t) = \mathrm{Max}_t(R, I)$. So $\mathscr{C}_t(I) = \mathscr{C}_t(J) = \mathscr{C}_t(L)$. Since R is integrally closed, $(L : L^2) = \big((L : L) : L\big) = (R : L) = L^{-1}$. Furthermore it is easily seen that $L^{-1} \subseteq \mathscr{C}_t(L)$. So $L^2(L : L^2) \subseteq L^2\mathscr{C}_t(L) = L^2\mathscr{C}_t(I)$. Since R_{M_1} is Clifford regular, we get $LR_{M_1} = L^2R_{M_1}(LR_{M_1} : L^2R_{M_1}) = \big(L^2(L : L^2)\big)R_{M_1} \subseteq L^2\mathscr{C}_t(I)R_{M_1} = L^2R_P$. It results that $LR_P \subseteq L^2R_P$ and hence $LR_P = L^2R_P$. By [35], Theorem 76, $LR_P = R_P$. Hence $L \nsubseteq P$ and thus $I \nsubseteq P$. This proves our claim.

Now, using the above claims, we obtain

$$\begin{aligned} JR_{M_1} &= I^2R_{M_1}AR_{M_1} \cap I^2R_{M_1}BR_{M_1} \cap I^2R_{M_1}\mathscr{C}_t(I)R_{M_1} \\ &= IR_{M_1} \cap I^2R_P = IR_{M_1} \cap R_P = IR_{M_1}. \end{aligned}$$

Consequently, $I = J_t$, as desired. This completes the proof of the theorem. □

Since in a Prüfer domain the t-operation coincides with the trivial operation, we recover Bazzoni's theorem (mentioned above) as a consequence of Theorem 3.2. Recall at this point Zanardo-Zannier's result that "an integrally closed Clifford regular domain is Prüfer" [45]. Also it is worthwhile noticing that during the proof of Theorem 3.2 we made use of Bazzoni-Salce result that "a valuation domain is Clifford regular" [11].

Corollary 3.11 (Bazzoni [10], Theorem 4.5). *An integrally closed domain R is Clifford regular if and only if R is a Prüfer domain of finite character.*

The next result solves Conjecture 3.1 for the context of strongly t-discrete domains.

Corollary 3.12. *Assume R is a strongly t-discrete domain. Then R is a pseudo-integrally closed Clifford t-regular domain if and only if R is a Krull-type domain.*

Proof. In view of Theorem 3.2, we only need to prove the "only if" assertion. Precisely, it remains to show that R is a PVMD. Let I be a finitely generated ideal of R. If $I_t = R$, then $I^{-1} = R$ and therefore $(II^{-1})_t = R$, as desired. Assume that I_t is a proper t-ideal of R. Suppose by way of contradiction that I is not t-invertible. Let M be a t-maximal ideal of R containing $J = (II^{-1})_t$. Since R is pseudo-integrally closed, $(I_t : I_t) = R$. Hence $\big(I_t : (I_t)^2\big) = \big((I_t : I_t) : I_t\big) = (R : I_t) = I^{-1}$. Further if R is Clifford t-regular, then

$$I_t = \big((I_t)^2\big(I_t : (I_t)^2\big)\big)_t = (I_t^2I^{-1})_t = (IJ)_t.$$

Therefore $R \subseteq J^{-1} = (J : J) \subseteq (IJ : IJ) \subseteq \big((IJ)_t : (IJ)_t\big) = (I_t : I_t) = R$. Consequently, $J^{-1} = (J : J) = R$. Hence $R \subseteq M^{-1} \subseteq J^{-1} = R$, whence $M^{-1} = (M : M) = R$. So $(M : M^2) = \big((M : M) : M\big) = (R : M) = R$. Since R is Clifford t-regular, then $M = \big(M^2(M : M^2)\big)_t = (M^2)_t$, and hence M is t-idempotent. This contrasts with the hypothesis that R is strongly t-discrete. It follows that I is t-invertible and thus R is a PVMD. □

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4. Examples

This section is motivated by an attempt to generating original families of integral domains with Clifford t-class semigroup. Next, we announce our first result of this section. It provides necessary and sufficient conditions for a pullback to inherit the Krull type notion.

Proposition 4.1. *Let T be an integral domain, M a maximal ideal of T, K its residue field, $\phi : T \to K$ the canonical surjection, and D a proper subring of K. Let $R = \phi^{-1}(D)$ be the pullback issued from the following diagram of canonical homomorphisms:*

$$\begin{array}{ccc} R & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ T & \xrightarrow{\ \phi\ } & K = T/M. \end{array}$$

Then R is a Krull-type domain if and only if D is a semilocal Bézout domain with $\mathrm{qf}(D) = K$ *and T is a Krull-type domain such that T_M is a valuation domain.*

Proof. By [17], Theorem 4.1, R is a PVMD if and only if so are T and D, $\mathrm{qf}(D) = K$, and T_M is a valuation domain. Now notice that $T = S^{-1}R$, where $S = \phi^{-1}(D\backslash\{0\})$. Moreover, by [33], Corollary 2.47, P is a t-prime ideal of R if and only if PT is a t-prime ideal of T, for every prime ideal P of R saturated with respect to S. Also, by [17], Proposition 1.8, q is a t-maximal ideal of D if and only if $\phi^{-1}(q)$ is a t-maximal ideal of T, for every prime ideal q of D. Finally, if A is a domain with only a finite number of maximal t-ideals, then each maximal ideal of A is a t-ideal [44], Proposition 3.5. Using the above four facts, we can easily see that R has finite t-character if and only if D is a semilocal Bézout domain and T has finite t-character. □

The next result investigates the transfer of the finite t-character condition to polynomial rings.

Proposition 4.2. *Let R be an integrally closed domain and X an indeterminate over R. Then R has finite t-character if and only if so does $R[X]$.*

Proof. Assume that R has finite t-character and let f be a nonzero nonunit element of $R[X]$ and $\{Q_\alpha\}_{\alpha\in\Omega}$ the set of all t-maximal ideals of $R[X]$ containing f. Set $\Omega_1 = \{\alpha \in \Omega \mid Q_\alpha \cap R = 0\}$ and $\Omega_2 = \{\alpha \in \Omega \mid q_\alpha = Q_\alpha \cap R \neq 0\}$. Assume $\alpha \in \Omega_1$ and let $K = \mathrm{qf}(R)$ and $S = R\backslash\{0\}$. Then $S^{-1}Q_\alpha$ is a maximal ideal of $K[X]$. Further f is not a unit in $K[X]$ since $Q_\alpha \cap R = 0$. Now $K[X]$ is of finite character (since a PID), then $\{S^{-1}Q_\alpha\}_{\alpha\in\Omega_1}$ is finite (and so is Ω_1). Assume $\alpha \in \Omega_2$. By [33], Lemma 2.32, q_α is a t-prime ideal of R with $Q_\alpha = q_\alpha[X]$. We claim that q_α is t-maximal in R. Deny. Then $q_\alpha \subsetneqq M_\alpha$ for some $M_\alpha \in \mathrm{Max}_t(R)$. So $Q_\alpha = q_\alpha[X] \subsetneqq M_\alpha[X]$, absurd since $M_\alpha[X]$ is a t-prime ideal of $R[X]$. Now let a denote the leading coefficient of f. Clearly, $0 \neq a \in q_\alpha$ (since $Q_\alpha = q_\alpha[X]$). Therefore $\{q_\alpha\}_{\alpha\in\Omega_2}$ is finite (and so is Ω_2) since R has finite t-character. Consequently, Ω is finite, as desired. The converse lies on the fact that the extension of a t-maximal ideal of R is t-maximal in $R[X]$. □

Notice at this point that (as in Example 2.8) one can build numerous examples of non-PVMD Clifford (or Boole) t-regular domains through Propositions 2.4 or 2.5 com-

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bined with Theorem 2.7. Next, we provide new families of Clifford ~~(or Boole)~~ t-regular domains originating from the class of PVMDs via a combination of Theorems 3.2 ~~and 4.1~~ and Propositions 5.1 and 5.2.

Example 4.3. For each integer $n \geqq 2$, there exists a PVMD R_n subject to the following conditions:

(1) $\dim(R_n) = n$.

(2) R_n is Clifford t-regular.

(3) R_n is not Clifford regular.

(4) R_n is not Krull.

Let V_0 be a rank-one valuation domain with $K = \mathrm{qf}(V_0)$. Let $V = K + N$ be a rank-one non strongly discrete valuation domain (cf. [14], Remark 6(b)). We take $R_n = V[X_1, \dots, X_{n-1}]$. For $n \geqq 4$, the classical $D + M$ construction provides more examples. Indeed, consider an increasing sequence of valuation domains $V = V_1 \subset V_2 \subset, \dots, \subset V_{n-2}$ such that, for each $i \in \{2, \dots, n-2\}$, $\dim(V_i) = i$ and $V_i/M_i = V/N = K$, where M_i denotes the maximal ideal of V_i. Set $T = V_{n-2}[X]$ and $M = (M_{n-2}, X)$. Therefore $R_n = V_0 + M$ is the desired example.

References

[1] *D. D. Anderson*, *J. A. Huckaba* and *I. J. Papick*, A note on stable domains, Houston J. Math. **13** (1) (1987), 13–17.

[2] *D. D. Anderson* and *M. Zafrullah*, Almost Bézout domains, J. Algebra **142** (1991), 285–309.

[3] *D. F. Anderson* and *D. E. Dobbs*, Pairs of rings with the same prime ideals, Canad. J. Math. **32** (1980), 362–384.

[4] *D. F. Anderson*, *E. Houston* and *M. Zafrullah*, Pseudo-integrality, Canad. Math. Bull. **34** (1) (1991), 15–22.

[5] *V. Barucci*, Strongly divisorial ideals and complete integral closure of an integral domain, J. Algebra **99** (1986), 132–142.

[6] *E. Bastida* and *R. Gilmer*, Overrings and divisorial ideals of rings of the form $D + M$, Michigan Math. J. **20** (1973), 79–95.

[7] *S. Bazzoni*, Class semigroup of Prüfer domains, J. Algebra **184** (1996), 613–631.

[8] *S. Bazzoni*, Idempotents of the class semigroup of a Prüfer domain of finite character, Lect. Notes. Pure Appl. Math., Dekker, **201** (1998), 79–89.

[9] *S. Bazzoni*, Groups in the class semigroup of a Prüfer domain of finite character, Comm. Algebra **28** (11) (2000), 5157–5167.

[10] *S. Bazzoni*, Clifford regular domains, J. Algebra **238** (2001), 703–722.

[11] *S. Bazzoni* and *L. Salce*, Groups in the class semigroups of valuation domains, Israel J. Math. **95** (1996), 135–155.

[12] *J. W. Brewer* and *E. A. Rutter*, $D + M$ constructions with general overrings, Michigan Math. J. **23** (1976), 33–42.

[13] *D. E. Dobbs*, *E. Houston*, *T. Lucas* and *M. Zafrullah*, t-linked overrings and Prüfer v-multiplication domains, Comm. Algebra **17** (11) (1989), 2835–2852.

[14] *D. E. Dobbs* and *I. J. Papick*, When is $D + M$ coherent?, Proc. Amer. Math. Soc. **56** (1976), 51–54.

[15] *S. El Baghdadi*, On a class of Prüfer v-multiplication domains, Comm. Algebra **30** (2002), 3723–3742.

[16] *W. Fangui*, *R. L. McCasland*, On strong Mori domains, J. Pure Appl. Algebra **135** (1999), 155–165.

[17] *M. Fontana* and *S. Gabelli*, On the class group and local class group of a pullback, J. Algebra **181** (1996), 803–835.

[18] *M. Fontana*, *J. A. Huckaba* and *I. J. Papick*, Prüfer Domains, Monogr. Textb. Pure Appl. Math. **203**, Marcel Dekker, Inc., New York 1997.
[19] *M. Fontana*, *J. A. Huckaba*, *I. J. Papick* and *M. Roitman*, Prüfer domains and endomorphism rings of their ideals, J. Algebra **157** (1993), 489–516.
[20] *R. Fossum*, The Divisor Class Group of a Krull Domain, Springer-Verlag, New York-Heidelberg 1973.
[21] *S. Gabelli* and *E. Houston*, Coherentlike Conditions in Pullbacks, Michigan Math. J. **44** (1997), 99–122.
[22] *R. Gilmer*, Multiplicative Ideal Theory, Marcel Dekker, New York 1972.
[23] *M. Griffin*, Some results on v-multiplication rings, Canad. J. Math. **19** (1967), 710–722.
[24] *M. Griffin*, Rings of Krull type, J. reine angew. Math. **229** (1968), 1–27.
[25] *J. Hedstrom* and *E. Houston*, Pseudo-valuation domains, Pacific J. Math. **75** (1978), 137–147.
[26] *E. Houston*, On divisorial prime ideals in Prüfer v-multiplication domains, J. Pure Appl. Algebra **42** (1986), 55–62.
[27] *E. Houston*, *S. Kabbaj*, *T. Lucas* and *A. Mimouni*, Duals of ideals in pullback constructions, Lect. Notes Pure Appl. Math., Dekker, **171** (1995), 263–276.
[28] *E. Houston*, *S. Kabbaj*, *T. Lucas* and *A. Mimouni*, When is the dual of an ideal a ring?, J. Algebra **225** (2000), 429–450.
[29] *E. Houston* and *M. Zafrullah*, Integral domains in which each t-ideal is divisorial, Michigan Math. J. **35** (1988), 291–300.
[30] *J. M. Howie*, Fundamentals of Semigroup Theory, Oxford University Press, Oxford 1995.
[31] *J. A. Huckaba* and *I. J. Papick*, When the dual of an ideal is a ring, Manuscr. Math. **37** (1982), 67–85.
[32] *S. Kabbaj* and *A. Mimouni*, Class semigroups of integral domains, J. Algebra **264** (2003), 620–640.
[33] *B. G. Kang*, $*$-operations in integral domains, Ph.D. thesis, The University of Iowa, Iowa City 1987.
[34] *B. G. Kang*, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, J. Algebra **123** (1989), 151–170.
[35] *I. Kaplansky*, Commutative Rings, The University of Chicago Press, Chicago 1974.
[36] *D. J. Kwak* and *Y. S. Park*, On t-flat overrings, Chinese J. Math. **23** (1) (1995), 17–24.
[37] *J. Lipman*, Stable ideals and Arf rings, Amer. J. Math. **93** (1971), 649–685.
[38] *S. Malik*, *J. Mott* and *M. Zafrullah*, On t-invertibility, Comm. Algebra **16** (1988), 149–170.
[39] *B. Olberding*, Globalizing local properties of Prüfer domains, J. Algebra **205** (1998), 480–504.
[40] *B. Olberding*, On the classification of stable domains, J. Algebra **243** (2001), 177–197.
[41] *B. Olberding*, On the structure of stable domains, Comm. Algebra **30** (2) (2002), 877–895.
[42] *J. Querre*, Idéaux divisoriels d'un anneau de polynômes, J. Algebra **64** (1980), 270–284.
[43] *J. D. Sally* and *W. V. Vasconcelos*, Stable rings and a problem of Bass, Bull. Amer. Math. Soc. **79** (1973), 574–576.
[44] *M. Zafrullah*, What v-coprimality can do for you, in: Multiplicative Ideal Theory in Commutative Algebra, Springer, New York (2006), 387–404.
[45] *P. Zanardo* and *U. Zannier*, The class semigroup of orders in number fields, Math. Proc. Cambridge Phil. Soc. **115** (1994), 379–391.

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