Trivial Extensions Defined by Coherent-like Conditions

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Abstract. This paper investigates coherent-like conditions and related properties that a trivial extension $R := A \propto E$ might inherit from the ring A for some classes of modules E. It captures previous results dealing primarily with coherence, and also establishes satisfactory analogues of well-known coherence-like results on pullback constructions. Our results generate new families of examples of rings (with zerodivisors) subject to a given coherent-like condition.

1 INTRODUCTION

All rings considered in this paper are commutative with identity elements and all modules are unital. Let A be a ring and E an A-module. The trivial ring extension of A by E is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. Considerable work, part of it summarized in Glaz's book [20] and Huckaba's book (where R is called the idealization of E in A) [21], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. See for instance [10, 15, 22, 23, 24, 25, 28, 29, 30].

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R [20]. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, valuation rings, and Prüfer/semihereditary rings. The concept of coherence first sprang up from the study of coherent sheaves in algebraic geometry, and

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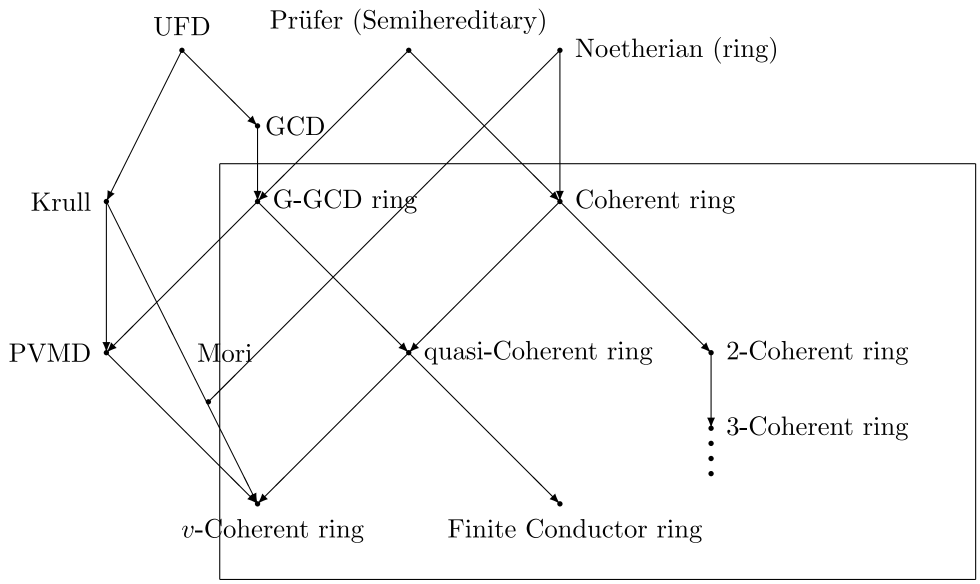
then developed, under the influence of Noetherian ring theory and homology, towards a full-fledged topic in algebra. During the past 30 years, several (commutative) coherent-like notions grew out of coherence such as finite conductor, quasi-coherent, v-coherent, n-coherent, and -to some extent- GCD and G-GCD rings (see the respective definitions in the beginning of Sections 2 and 3). Noteworthy is that both the ring-theoretic and homological aspects of coherence run through most of these generalizations (see for instance [19]).

This paper investigates coherent-like conditions and related properties that a trivial extension $R := A \propto E$ might inherit from the ring A for some classes of modules E. It captures previous results dealing primarily with coherence [20, 30], and also establishes satisfactory analogues of well-known coherence-like results on pullback constructions [17] (see also [7, 13, 11, 12, 14]). Our results generate new families of examples of rings (with zerodivisors) subject to a given coherent-like condition.

The second section provides a ring-theoretic approach. We first extend the definition of a v-coherent domain to rings with zerodivisors and develop a theory of these rings parallel to Glaz's study of finite conductor, quasi-coherent, and G-GCD rings [19]. Afterwards, we study the possible transfer of all these notions for various trivial extension contexts. Thereby, new examples are provided which, particularly, enrich the current literature with new classes of coherent-like rings with zerodivisors.

The third section treats the homological aspect. We first study conditions under which trivial extensions yield (strong) n-coherent rings [8, 9, 11, 12]. Due to reciprocal effects [8, Section 2], we also deal with the (n, d)-rings of Costa, i.e., those in which n-presented modules [6] have projective dimension at most d. In particular, the second part of this section is devoted to Costa's second conjecture that one may characterize the (n, d)-property intrinsically by ideal-theoretic-conditions [8]. We explore the scope of the validity of this conjecture in various trivial extension non-coherent contexts. Recall at this point that Costa's second conjecture is valid in the class of coherent rings [9]. This fact was behind our motivation for studying large classes of coherent-like rings. The paper closes with an independent result showing that this conjecture holds in the class of finite conductor domains (resp., rings) for $n \leq 2$ and $d = 1$ (resp., $d = 0$). The general case is still elusively open.

The following diagram of commutative rings summarizes the relations between the coherent-like notions involved in this paper:



2 RING-THEORETIC APPROACH

A ring R is quasi-coherent (resp., finite conductor) if $(0 : a)$ and $a_1R \cap ... \cap a_nR$ (resp., $bR \cap cR$) are finitely generated ideals of R for any finite set of elements a and $a_1, ..., a_n$ (resp., b, c) of R [3, 19, 32]. Also, R is called a G-GCD ring if every principal ideal of R is projective and the intersection of any two finitely generated flat ideals of R is a finitely generated flat ideal of R [1, 19].

2.1 v-Coherent Rings with Zerodivisors

In view of Glaz's recent work on finite conductor, quasi-coherent, and G-GCD rings [19], we first extend the definition of a v-coherent domain [14, 17, 26, 27] to rings with zerodivisors. For this purpose, we review some terminology related to basic operations on fractional ideals in an arbitrary ring (i.e., not necessarily a domain). Let R be a commutative ring and let $\mathrm{Q}(R)$ denote the total ring of quotients of R. By an ideal of R we mean an integral ideal of R. Let I and J be two nonzero fractional ideals of R. We define the fractional ideal $(I : J) = \{x \in \mathrm{Q}(R) \mid xJ \subset I\}$. We denote $(R : I)$ by I^{-1} and $(I^{-1})^{-1}$ by I_v (called the v-closure of I). A nonzero fractional ideal I is said to be invertible if $II^{-1} = R$, divisorial (or a v-ideal) if $I_v = I$, and v-finite if $I_v = J_v$ (or, equivalently, if $I^{-1} = J^{-1}$) for some finitely generated fractional ideal J of A. The v-operation on R is not necessarily a $*$-operation, since, in general, $(a)_v \neq (a)$ when a is a zerodivisor of R. However, the other basic properties of the v-operation on integral domains [18, (32.1)(2)&(3), (32.2)(a)&(b)] carry over to arbitrary rings.

Definition 2.1 A ring R is v-coherent if $(0 : a)$ and $\bigcap_{1\leq i\leq n} Ra_i$ are v-finite ideals of R for any finite set of elements a and $a_1, ..., a_n$ of R.

Proposition 2.2 *Let R be a ring and let's consider the following assertions:*
(1) *I^{-1} is v-finite for any finitely generated ideal I of R.*
(2) *$I_v \cap J_v$ is v-finite for any two finitely generated ideals I and J of R.*
(3) *$\bigcap_{1\leq i\leq n} Ra_i$ is v-finite for any finite set of elements $a_1, ..., a_n$ of R.*
Then (1)$\Longrightarrow$(2). *Moreover, if R is an integral domain, then the three assertions are equivalent, each of which characterizes v-coherence.* $\square$

Proof. Assume that (1) is true and let I and J be any finitely generated ideals of R. Then there exist two finitely generated ideals I_1 and J_1 such that $I_v = I_1^{-1}$ and $J_v = J_1^{-1}$. So, $I^{-1} = (I_1)_v$ and $J^{-1} = (J_1)_v$, hence $I_v \cap J_v = (I^{-1} + J^{-1})^{-1} = ((I_1)_v + (J_1)_v)^{-1} = ((I_1)_v)^{-1} \cap ((J_1)_v)^{-1} = (I_1)^{-1} \cap (J_1)^{-1} = (I_1 + J_1)^{-1}$ which is v-finite by hypothesis since $I_1 + J_1$ is a finitely generated ideal of R.

Now, assume that R is an integral domain. Then (1)$\Longleftrightarrow$(2) is handled by [14, Proposition 3.6], and (1)$\Longleftrightarrow$(3) always holds since $(\sum_{i=1}^{n} Ra_i)^{-1} = \bigcap_{1\leq i\leq n} Ra_i^{-1}$ for each $a_i \in R$ and any integer $n \geq 1$. $\square$

Clearly, quasi-coherent rings are v-coherent, and if R is a domain, the above definition matches the definition of a v-coherent domain. It is worth recalling that v-coherent domains offer a large context of validity for the so-called Nagata's theorem for the class group [16]. Also, recall from [26] that PVMDs [18, 32] and Mori domains [4] are v-coherent. Moreover, non-Krull integrally closed Mori domains [2] are (v-coherent but) not finite conductor [32].

Let $(R_j)_{1\leq j\leq m}$ be a family of rings and $R = \prod_{j=1}^{m} R_j$. For $C = (c_j)$ and $A_1 = (a_{1j}), ..., A_n = (a_{nj}) \in R$, we have $(0 : C) = \prod_{j=1}^{m}(0 : c_j)$ and $\bigcap_{1\leq i\leq n} RA_i = \prod_{j=1}^{m}\left(\bigcap_{1\leq i\leq n} Ra_i\right)$. Further, for any ideals $I = \prod_{j=1}^{m} I_j$ and $J = \prod_{j=1}^{m} J_j$ of R, we have $I \cap J = \prod_{j=1}^{m}(I_j \cap J_j)$ and $I^{-1} = \prod_{j=1}^{m} I_j^{-1}$. Then $\prod_{j=1}^{m} R_j$ is v-coherent if and only if so is R_j for each $j = 1, \ldots, m$. Thus, finite products (for instance, of Mori domains) may provide us with original examples of v-coherent rings with zerodivisors.

Let's now examine v-coherence for rings of small weak dimension. Recall first that rings of weak dimension 0 are precisely the von Neumann regular rings. Moreover, Glaz showed that for a ring R of weak dimension 1 the finite conductor property, quasi-coherence, and coherence deflate to the mere fact that $(0 : c)$ is finitely generated for every $c \in R$ [19, Proposition 2.2]. She also proved that the finite conductor and quasi-coherence properties coincide for rings of weak dimension 2 [19, Theorem 2.3]. In contrast with these results, the next example denies any similar effect to weak dimension on v-coherence.

Example 2.3 Let E be a countable direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$ with addition and multiplication defined component wise, where $\mathbb{Z}$ is the ring of integers. Let $R = \mathbb{Z} \times E$ with multiplication defined by $(a, e)(b, f) = (ab, af + be + ef)$. Then:
(1) $\text{w.dim}(R) = 1$.
(2) R is not coherent.
(3) R is a v-coherent ring.

Proof. (1) That $\text{w.dim}(R) = 1$ this is handled in [31, Example 1.3, page 10].
(2) Let $x = (2, 0) \in R$. Then $(0 : x) = \{(a, e) \in R | (a, e)(2, 0) = 0\} = \{(a, e) \in R | (2a, 0) = 0\} = 0 \times E$ which is not a finitely generated ideal of R. Therefore, R is not a coherent ring.
(3) Notice first that an element $s \in R$ is regular if and only if $s = (a, 0)$ with $a \in \mathbb{Z} \setminus 2\mathbb{Z}$. This easily follows from the four basic facts: E is Boolean; $2E = 0$; $ae = e$ for any $a \in \mathbb{Z} \setminus 2\mathbb{Z}$ and $e \in E$; and for any $e \neq 0 \in E$, there exists $f \neq 0 \in E$ such that $ef = 0$.

Next, we wish to show that each ideal of R is v-finite which implies that R is v-coherent. Let J be an ideal of R and let $I = \{a \in \mathbb{Z} / (a, e) \in J \text{ for some } e \in E\}$. Assume $I = 0$. Let s be any regular element of R. Clearly, $(0, e) = s(0, e)$ for any $e \in E$. It follows that $sJ = J$ and hence $J^{-1} = Q(R) = (R(0, e))^{-1}$ for any $e \neq 0 \in E$. Now, assume $I = x\mathbb{Z}$, where x is a nonzero integer. We claim that $J^{-1} = (R(x, 0))^{-1}$. Indeed, let $y/s \in Q(R)$, where $y = (a, e) \in R$ and $s = (b, 0)$ is a regular element. It can easily be seen that $sR = b\mathbb{Z} \times E$. Then $y/s \in J^{-1} \Leftrightarrow yJ \subseteq sR \Leftrightarrow (a, e)J \subseteq b\mathbb{Z} \times E \Leftrightarrow aI \subseteq b\mathbb{Z} \Leftrightarrow ax \in b\mathbb{Z} \Leftrightarrow (a, e)(R(x, 0)) \subseteq sR \Leftrightarrow y/s \in (R(x, 0))^{-1}$. Thus, in both cases, J is v-finite, as asserted. □

While a ring R which is a total ring of quotients trivially is v-coherent, R need not be finite conductor [19, Example 3.5]. The following construction may offer new contexts that illustrate this fact.

Example 2.4 Let (R, M) be any local ring with $M^2 = 0$. Then:
(1) R is a v-coherent ring that is not G-GCD.
(2) The following conditions are equivalent:
(i) R is a coherent ring;
(ii) R is a quasi-coherent ring;
(iii) R is a finite conductor ring;
(iv) $(0 : c)$ is finitely generated for every $c \in R$;
(v) M is finitely generated.

Proof. (1) That R is v-coherent this is straightforward since $R = Q(R)$ is a total ring of quotients. Let $c \neq 0 \in M$. Then $\mathrm{Ann}(c) = (0 : c) = M$. Hence Rc is not projective (since not free), so that R is not a G-GCD ring [19].
(2) Clearly, we only need prove $(v) \Longrightarrow (i)$. Assume that M is finitely generated and let I be a finitely generated proper ideal of R. Let $\{x_1, \ldots, x_n\}$ be a minimal generating set of I and consider the exact sequence of R-modules:

$$0 \to Ker(u) \to R^n \xrightarrow{u} I \to 0$$

where $u(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i x_i$. We claim that $Ker(u) = \prod M =: M^n$. Indeed, $M^n \subseteq Ker(u)$ is clear since $M^2 = 0$ and $x_i \in M$ for each $i = 1, \ldots, n$. On the other hand, $Ker(u) \subseteq M^n$ since $\{x_1, \ldots, x_n\}$ is minimal. Therefore, $Ker(u) = M^n$ is a finitely generated R-module (since M is). Hence, I is finitely presented and thus R is coherent. □

2.2 Results of Transfer and Examples

This subsection investigates the possible transfer of the coherence properties for various trivial extension contexts. Our results generate new families of examples subject to a given coherent-like condition.

For the convenience of the reader, we next discuss some basic facts connected to trivial ring extensions. These will be used frequently in the sequel without explicit mention. Let A be a ring and E an A-module and let $R := A \propto E$ be the trivial ring extension of A by E. An ideal of R of the form $I \propto IE$, where I is an ideal of A, is finitely generated if and only if I is finitely generated [20, page 141]. Also recall that R has always its Krull dimension equal to the Krull dimension of A [21, Theorem 25.1(3)].

For a general description of modules over a trivial ring extension, we refer the reader to [20, pages 140 & 141]. Here, we describe a specific type of R-modules that play a crucial role within the R-module structure, namely, finitely generated free R-modules and their R-submodules. Let n be a positive integer. Define the "multiplication" on E by elements of A^n within E^n through the natural A-bilinear map $\varphi : A^n \times E \longrightarrow E^n$ defined by $ae = \varphi(a, e) := (a_i e)_{1 \leq i \leq n}$, for any $a = (a_i)_{1 \leq i \leq n} \in A^n$ and $e \in E$. Now let U be an A-submodule of A^n and E' an A-submodule of E^n such that $UE \subseteq E'$. Let $U \propto E'$ denote the set $U \times E'$ with natural addition and scalar multiplication defined by $(a, e)(u, e') = (au, ae' + ue)$. Clearly, $U \propto E'$ is an R-module; and, under this notation, the finitely generated free R-module R^n identifies with $A^n \propto E^n$. Also, $U \propto E'$ is a finitely generated R-module only if U is a finitely generated A-module. Conversely, let M be an R-submodule of R^n. Set $U := \{u \in A^n | (u, e') \in M$ for some $e' \in E^n\}$ and $E' := \{e' \in E^n | (u, e') \in M$ for some $u \in A^n\}$. It is easily seen that U and E' are A-modules such that $M \subseteq U \propto E'$. The next example illustrates the fact that equality does not hold in general.

Example 2.5 Let (A, M) be a local domain which is not a field, $E := A/M$, and $R := A \propto E$ be the trivial ring extension of A by E. Let $J = R(x, 1)$, where $x \neq 0 \in M$. Set $I = \{a \in A | (a, e) \in J$ for some $e \in E\}$ and $E' = \{e \in E | (a, e) \in J$ for some $a \in A\}$. Then $J \subsetneqq I \propto E'$.

Proof. One may easily check that $I = Ax$ and $E' = E$. Further, we claim that $(x, 0) \in I \propto E' \setminus J$. Deny. We have $(x, 0) = (a, e)(x, 1)$ for some $(a, e) \in R$ so that $x = ax$. Hence $a = 1 \in M$, the desired contradiction. □

Nevertheless, it is easily seen that $M = U \propto E'$ if and only if $0 \propto E' \subseteq M$ if and only if $U \propto 0 \subseteq M$.

Example 2.5 shows that [21, Theorem 25.1(1)] is not true. This was confirmed by the author of [21] through a private e-communication.

THEOREM 2.6 *Let (A, M) be a local ring and E an A-module with $ME = 0$. Let $R := A \propto E$ be the trivial ring extension of A by E. Then:*
(1) *R is a v-coherent ring that is not G-GCD.*
(2) *R is coherent (resp., quasi-coherent, finite conductor) if and only if A is coherent (resp., quasi-coherent, finite conductor), M is finitely generated, and E is an (A/M)-vector space of finite rank.*

Before proving Theorem 2.6, we establish the following Lemma.

Lemma 2.7 *Under the hypotheses of Theorem 2.6,* $(0 : c)$ *is a finitely generated ideal of R for each $c \in R$ if and only if* $(0 : a)$ *is a finitely generated ideal of A for each $a \in A$, M is finitely generated, and E is an (A/M)-vector space of finite rank.*

Proof. Assume that $(0 : c)$ is a finitely generated ideal of R for each $c \in R$. Let $a \in A$ and set $c := (a, 0) \in R$. Then $(0 : c) = (0 : a) \propto E'$, where $E' = \{e \in E | ae = 0\}$. Therefore, $(0 : a)$ is a finitely generated ideal of A. Let $e \neq 0 \in E$ and set $c := (0, e) \in R$. Similar arguments show that M is a finitely generated ideal of A since $(0 : c) = M \propto E$. Further, assume that $M \propto E = \sum_{i=1}^{n} R(x_i, e_i)$, where $x_i \in M$ and $e_i \in E$ for each $i = 1, ..., n$. Then $E \subseteq \sum_{i=1}^{n}(A/M)e_i$, and hence E is an (A/M)-vector space of finite rank.

Conversely, let $c := (a, e) \neq 0 \in R$. If a is invertible in A, then c is invertible in R. Then, without loss of generality, we may assume that $a \in M$. Hence, $(0 : c) = \{(b, f) \in R | (ab, be) =$

$0\} = \{(b,f) \in M \propto E | ab = 0\}$ (since if b is invertible in A, then (b,f) is invertible in R, and so $c = 0$, a contradiction). It can easily be seen that if $a = 0$ then $(0:c) = M \propto E$, and if $a \neq 0$ then $(0:c) = (0:a) \propto E$. In both cases, $(0:c)$ is a finitely generated ideal of R since M and $(0:a)$ are finitely generated ideals of A and E is an (A/M)-vector space of finite rank, completing the proof of Lemma 2.7. □

Proof of Theorem 2.6. (1) One may easily verify that R is local with maximal ideal $M \propto E$ and that each element of R is either a unit or a zerodivisor. Then $R = Q(R)$ is v-coherent. Let $c \neq 0 \in M$ and $e \neq 0 \in E$. Clearly, (c,e) is a zerodivisor. Hence $R(c,e)$ is not projective (since not free), so that R is not a G-GCD ring [19].

(2) Assume that R is a coherent ring. By Lemma 2.7, it remains to show that A is coherent. Let $I = \sum_{i=1}^{n} Aa_i$, where $a_i \in M$ and set $J := \sum_{i=1}^{n} R(a_i, 0)$. Consider the exact sequence of R-modules:

$$0 \to Ker(u) \to R^n = A^n \propto E^n \xrightarrow{u} J \to 0$$

where $u((b_i, e_i)_{1 \leq i \leq n}) = \sum_{i=1}^{n}(b_i, e_i)(a_i, 0) = (\sum_{i=1}^{n} a_i b_i, 0)$ since $a_i \in M$ for each $i = 1, \ldots, n$. On the other hand, consider the exact sequence of A-modules:

$$0 \to Ker(v) \to A^n \xrightarrow{v} I \to 0$$

where $v((b_i)_{1 \leq i \leq n}) = \sum_{i=1}^{n} a_i b_i$. Then, $Ker(u) = Ker(v) \propto E^n$. But J is finitely presented since R is coherent, so $Ker(u)$ is a finitely generated R-module and hence $Ker(v)$ is a finitely generated A-module. Therefore, I is a finitely presented ideal of A, so A is coherent.

Conversely, let J be a finitely generated ideal of R and let $S := \{(a_i, e_i)\}_{1 \leq i \leq n}$ be a minimal generating set of J, where $a_i \in M$ and $e_i \in E$. Consider the exact sequence of R-modules:

$$0 \to Ker(u) \to R^n \xrightarrow{u} J \to 0$$

where $u((b_i, f_i)_{1 \leq i \leq n}) = \sum_{i=1}^{n}(a_i, e_i)(b_i, f_i) = (\sum_{i=1}^{n} a_i b_i, \sum_{i=1}^{n} b_i e_i)$ since $a_i \in M$ for each $i = 1, \ldots, n$. Further, the minimality of S yields $Ker(u) = \{(b_i, f_i)_{1 \leq i \leq n} \in R^n | \sum_{i=1}^{n} a_i b_i = 0\}$. Let $I := \sum_{i=1}^{n} Aa_i$ and consider the surjective A-module homomorphism v defined above. Then $Ker(v)$ is a finitely generated A-module since A is coherent. Consequently, $Ker(u) = Ker(v) \propto E^n$ is a finitely generated R-module. Therefore, J is finitely presented and hence R is coherent.

Now, assume that R is quasi-coherent. We only need show that $\bigcap_{1 \leq i \leq n} Ra_i$ is a finitely generated ideal of A for each $a_i \in M$. This is straightforward since $\bigcap_{1 \leq i \leq n} R(a_i, 0) = (\bigcap_{1 \leq i \leq n} Aa_i) \propto 0$ is a finitely generated ideal of R.

Conversely, let $J = \bigcap_{1 \leq i \leq n} R(a_i, e_i)$, where $a_i \in M$ and $e_i \in E$. We may suppose that $J \subsetneqq R(a_i, e_i)$ for each $i = 1, \ldots, n$. Let $(a, e) \in J$. Then, there exist $b_i \in A$ and $f_i \in E$

such that $(a,e) = (b_i, f_i)(a_i, e_i) = (a_ib_i, b_ie_i)$ for each $i = 1, \ldots, n$. We claim that $b_i \in M$ for each $i = 1, \ldots, n$. Deny. Then, there exists j such that b_j is invertible in A and so is (b_j, f_j) in R. Hence $(a_j, e_j) = (b_j, f_j)^{-1}(a, e) \in J$ yielding $J = R(a_j, e_j)$, a contradiction. Therefore, $(a, e) = (a_ib_i, 0)$. It follows that $J = \bigcap_{1\leq i\leq n}(Aa_i \propto 0) = (\bigcap_{1\leq i\leq n} Aa_i) \propto 0$ is finitely generated in R since $\bigcap_{1\leq i\leq n} Aa_i$ is by hypothesis finitely generated in A. Thus R is quasi-coherent.

Finally, similar arguments as above with $n = 2$ lead to the conclusion for the finite conductor property, to complete the proof of Theorem 2.6. □

Next, we explore a different context, namely, the trivial ring extension of a domain by its quotient field.

THEOREM 2.8 *Let A be a domain which is not a field, $K = \mathrm{qf}(A)$, and $R := A \propto K$ be the trivial ring extension of A by K. Then:*
(1) R is not a finite conductor ring. In particular, R is neither quasi-coherent nor coherent.
(2) R is a v-coherent ring if and only if A is v-coherent.

Proof. (1) Let $x := (0, 1) \in R$. Then $(0 : x) = 0 \propto K$ is not a finitely generated ideal of R. Therefore, R is not a finite conductor ring, as asserted.
(2) Observe first that $(a, e) \in R$ is regular if and only if $a \neq 0$, and that $(0 : c) = 0 \propto K$ for any $c := (0, e) \neq 0 \in R$. Further, [21, Theorem 25.1(4)] yields $\bigcap_{1\leq i\leq n} R(a_i, e_i) = \bigcap_{1\leq i\leq n}(Ra_i \propto K) = (\bigcap_{1\leq i\leq n} Ra_i) \propto K$, for every finite set of elements $(a_i, e_i)_{1\leq i\leq n}$ of R (with $a_i \neq 0$ for each i). Now, Let I be any nonzero ideal of A and E any A-submodule of K with $IK \subseteq E$ and let $J := I \propto E$. By [21, Theorem 25.10], we have $J^{-1} = (I \propto E)^{-1} = I^{-1} \propto K = (I \propto IK)^{-1}$. Finally, since $I \propto IK$ is finitely generated if I is, the conclusion is straightforward. □

New examples of original coherent-like rings with zerodivisors with arbitrary Krull dimensions may stem from Theorems 2.6 & 2.8, as shown by the following constructions.

Example 2.9 Let K be any field and $X_1, X_2, ...$ be indeterminates over K. Let n be any integer ≥ 1, $A = K[[X_1, ..., X_n]]$ the power series ring in n variables over K, and $R := A \propto K$. Then, by Theorem 2.6, R is an n-dimensional coherent ring that is not G-GCD. □

Example 2.10 Let A be as in the above example and $R := A \propto K[Y]$, where Y is another indeterminate over K. Then, by Theorem 2.6, R is an n-dimensional v-coherent ring that is not finite conductor. □

Example 2.11 Let $R := \mathbb{Z}[X_1, ..., X_{n-1}] \propto \mathbb{Q}(X_1, ..., X_{n-1})$, where n is any integer ≥ 1, $\mathbb{Z}$ the ring of integers, and $\mathbb{Q}$ the field of rational numbers. Then, by Theorem 2.8, R is an n-dimensional v-coherent ring that is not finite conductor. $\Box$

More examples are provided in the next section, denying any possible interplay between some of these coherent-like conditions and n-coherence.

3 HOMOLOGICAL APPROACH

For a nonnegative integer n, an R-module E is n-presented if there is an exact sequence $F_n \to F_{n-1} \to ... \to F_0 \to E \to 0$ in which each F_i is a finitely generated free R-module [6]. In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Throughout, $\mathrm{pd}_R(E)$ and $\mathrm{fd}_R(E)$ will denote the projective dimension and the flat dimension of E as an R-module, respectively. Also $\mathrm{w.dim}(R)$ will denote the weak dimension of R.

In 1994, Costa introduced a doubly filtered set of classes of rings, called the (n, d)-rings, with the aim of obtaining a good understanding of the structures of some non-Noetherian rings [8]. The Noetherianity forces the regularity of the (n, d)-rings. However, outside Noetherian settings, the richness of this classification resides in its ability to unify classic concepts such as von Neumann regular, hereditary, Dedekind, semihereditary, and Prüfer rings.

Given nonnegative integers n and d, a ring R is called an (n, d)-ring if every n-presented R-module has projective dimension $\leq d$; and a weak (n, d)-ring if every n-presented cyclic R-module has projective dimension $\leq d$ (equivalently, if every $(n-1)$-presented ideal of R has projective dimension $\leq d - 1$). For instance, the $(0, 1)$-domains are the Dedekind domains, the $(1, 1)$-domains are the Prüfer domains, and the $(1, 0)$-rings are the von Neumann regular rings [8]. Costa's paper concludes with a number of open problems, including his second conjecture that the (n, d)- and weak (n, d)-properties are equivalent. This conjecture is valid in the class of coherent rings [9].

The first part of this section studies the transfer of the $n-$coherence properties (see definitions below) to trivial ring extensions. Due to reciprocal effects [8, Section 2], results of transfer for the (n, d)-property are also provided. Our purpose, in the second part, is mainly to test the validity of Costa's second conjecture for non-coherent contexts.

3.1 n-Coherence and Strong n-Coherence

Recall from [11, 12], for $n \geq 1$, that R is n-coherent if each $(n-1)$-presented ideal of R is n-presented; and that R is strong n-coherent if each n-presented R-module is $(n+1)$-presented (This terminology is not the same as that of [8], where Costa's "n-coherence" is our "strong n-coherence"). In particular, "1-coherence" coincides with coherence, and one may view "0-coherence" as Noetherianity. Any strong n-coherent ring is n-coherent, and the converse holds for $n = 1$ or for coherent rings [12, Proposition 3.3]. Strong n-coherence arose naturally in Costa's study [8] of the (n, d)-rings. As a matter of fact, every (n, d)-ring is strong $\max\{n, d\}$-coherent [8, Theorem 2.2]; and an (n, d)-ring is strong r-coherent $(r < n)$ only if it is an (r, d)-ring [8, Theorem 2.4].

Our main result examines the context of trivial ring extensions of domains by their respective quotient fields.

THEOREM 3.1 *Let A be a domain which is not a field, $K = \mathrm{qf}(A)$, and $R := A \propto K$ be the trivial ring extension of A by K. Let $n \geq 2$ and $d \geq 1$ be integers. Then the following hold:*
(1) *R is not coherent.*
(2) *R is strong n-coherent (resp., n-coherent) if and only if so is A.*
(3) *R is an (n, d)-ring (resp., a weak (n, d)-ring) if and only if so is A.*

The proof of this theorem lies mainly on the following two lemmas which characterize, respectively, finitely generated and n-presented R-submodules of free R-modules.

Let us fix the notation for the next two results. Let R be as in Theorem 3.1 and let H be an R-submodule of R^m, where m is an arbitrary positive integer. Set $U = \{x \in A^m/(x, e) \in H$ for some $e \in K^m\}$ and $E = \{e \in K^m/(x, e) \in H$ for some $x \in A^m\}$.

Lemma 3.2 *Under the above notation, the following statements are equivalent:*
(i) *H is finitely generated and E is a K-vector space;*
(ii) *U is finitely generated and $H = U \propto KU$.*

Proof. i) $\Longrightarrow$ ii) Let $H = \sum_{i=1}^{p} R(x_i, e_i)$ $(\subseteq U \propto E)$, where p is a positive integer, $x_i \in A^m$, and $e_i \in K^m$ for each $i = 1, \ldots, p$. Necessarily, $U = \sum_{i=1}^{p} Ax_i$ and $E = \sum_{i=1}^{p} Ae_i + KU$. Next assume $KU \subsetneqq E$. Then there exists a nonzero K-vector space F with finite rank such that $F \oplus KU = E$. Write $e_i = y_i + z_i$, where $y_i \in F$ and $z_i \in KU$ for each $i = 1, \ldots, p$. From above, it follows that $E = \sum_{i=1}^{p} Ay_i \oplus KU$ and thus $F = \sum_{i=1}^{p} Ay_i$. Consequently, F (and hence K) is a finitely generated A-module, the desired contradiction. Hence, $E = KU$. Now let $y \in E = KU$. Then $y = \sum_{i=1}^{p} b_i x_i$, where $b_i \in K$ for each $i = 1, \ldots, p$. So

$(0,y)=\sum_{i=1}^{p}(0,b_i)(x_i,e_i)\in H$. It follows that $0\propto E\subseteq H$; equivalently, $H=U\propto KU$.
$ii)\Longrightarrow i)$ Straightforward. $\square$

Lemma 3.3 *Let n be an integer ≥ 1. Under the above notation, the following statements are equivalent:*
(i) *H is n-presented;*
(ii) *U is n-presented and $H=U\propto KU$.*

Proof. $i)\Longrightarrow ii)$ By induction on n. Assume $n=1$. As above, write $H=\sum_{i=1}^{p}R(x_i,f_i)$, where p is a positive integer, $x_i\in A^m$, and $f_i\in K^m$ for each $i=1,...,p$. We have $U=\sum_{i=1}^{p}Ax_i$ and $E=(\sum_{i=1}^{p}Af_i)+KU$. Let F' be a K-vector space such that $F'\oplus KU=K^m$. Then, $f_i=g_i+h_i$, where $g_i\in F'$ and $h_i\in KU$ for each $i=1,\dots,p$. Hence $E=(\sum_{i=1}^{p}Ag_i)\oplus KU$. Further, it easily can be seen that $H=\sum_{i=1}^{p}R(x_i,g_i)$. Consider the exact sequence of R-modules:

$$0\rightarrow Ker(w)\rightarrow R^p\overset{w}{\rightarrow}H\rightarrow 0$$

where $w((a_i,e_i)_{1,\dots,p})=\sum_{i=1}^{p}(a_i,e_i)(x_i,g_i)=(\sum_{i=1}^{p}a_ix_i,\sum_{i=1}^{p}a_ig_i+\sum_{i=1}^{p}e_ix_i)$. Set $W:=\{(a_i)_{i=1,\dots,p}\in A^p/\sum_{i=1}^{p}a_ix_i=0$ and $\sum_{i=1}^{p}a_ig_i=0\}$ and $E':=\{(e_i)_{i=1,\dots,p}\in K^p/\sum_{i=1}^{p}e_ix_i=0\}$. Clearly, E' is a K-vector space and $Ker(w)$ is a finitely generated R-submodule of R^p (since H is finitely presented). By Lemma 3.2, W is finitely generated and $Ker(w)=W\propto KW$. Moreover, let $(a_i)_{i=1,\dots,p}\neq 0\in A^p$ such that $\sum_{i=1}^{p}a_ix_i=0$. Then, $(a_i)_{i=1,\dots,p}\in E'=KW$. So there exists $z\neq 0\in K$ and $(b_i)_{i=1,\dots,p}\in W$ such that $(a_i)_{i=1,\dots,p}=z(b_i)_{i=1,\dots,p}$. Hence $\sum_{i=1}^{p}a_ig_i=z\sum_{i=1}^{p}b_ig_i=0$, whence $(a_i)_{i=1,\dots,p}\in W$. Consequently, $W=\{(a_i)_{i=1,\dots,p}\in A^p/\sum_{i=1}^{p}a_ix_i=0\}$. Therefore, the exact sequence of A-modules of natural homomorphisms

$$0\rightarrow W\rightarrow A^p\rightarrow U\rightarrow 0,$$

upon tensoring by the flat A-module R, yields the exact sequence of R-modules

$$0\rightarrow W\otimes_A R\cong W\propto KW=Ker(w)\rightarrow R^p\rightarrow U\otimes_A R\cong U\propto KU\rightarrow 0.$$

It follows that U is finitely presented and $H=U\propto KU$.

The inductive step is carried out just as we did for the case $n=1$ above, provided one substitutes the induction hypothesis for Lemma 3.2.
$ii)\Longrightarrow i)$ Straightforward. $\square$

Proof of Theorem 3.1. (1) is already handled by Theorem 2.8(1). Specifically, $R(0,1)=0\propto A$ is a finitely generated ideal of R which is not finitely presented (by Lemma 3.3).

(2) and (3) follow readily from a combination of Lemma 3.3 with the next three facts:
(a) R is a faithfully flat A-module.
(b) For $n \geq 2$, a ring B is n-coherent if and only if every $(n-1)$-presented submodule of a finitely generated free B-module is n-presented.
(c) For $n \geq 2$ and $d \geq 1$, a ring B is an (n, d)-ring if and only if every $(n-1)$-presented submodule of a finitely generated free B-module has projective dimension $\leq d-1$. □

For $n \leq 1$ or $d = 0$, the (n, d)-property may not survive, in general, in the trivial extension R (even under strong assumption on A). This is illustrated by the next example.

Example 3.4 Let A be any arbitrary Prüfer domain (i.e., $(1,1)$-domain) and let R be the trivial ring extension of A by its quotient field. Then R is a $(2,1)$-ring which is neither a semihereditary ring (i.e., $(1,1)$-ring) nor a 2-von Neumann regular ring (i.e., $(2,0)$-ring).

Proof. That R is a $(2,1)$-ring which is not a $(1,1)$-ring is ensured by Theorem 3.1(3)&(1), respectively. Now, let $J :=R(a, 0)$, where a is a non-zero non-invertible element of A. Since $(a, 0)$ is a regular element of R, then the ideal J of R has no non-zero annihilator. By [24, Theorem 2.1], R is not a 2-von Neumann regular ring. □

The Bézout property, however, does transfer reciprocally from A to R, as shown by the next result.

Proposition 3.5 *Let R be as in Theorem 3.1. Then R is a Bézout ring if and only if A is a Bézout domain.*

Proof. Assume R is a Bézout ring and let I be a finitely generated proper ideal of A. Then $J := I \propto IK = I \propto K$ is a finitely generated ideal of R. So $J = R(a, e)$ for some $a \in A$ and $e \in E$. Therefore, $I = Aa$ and hence A is a Bézout domain.

Conversely, let J be a finitely generated proper ideal of R. Set $I := \{a \in A/(a, e) \in J$ for some $e \in K\}$. We consider two cases. Case 1: $I = 0$. Necessarily, $J = 0 \propto (1/b)L$ for some $b \neq 0 \in A$ and some finitely generated proper ideal L of A. Further, $L = Aa$ since A is a Bézout domain. Hence $J = 0 \propto A(a/b) = R(0, a/b)$, as desired. Case 2: $I \neq 0$. Let $(a, e) \in J$ such that $a \neq 0$. Then, $(a, e)(0 \propto K) = 0 \propto K \subseteq J$; equivalently, $J = I \propto IK = I \propto K$. But $I = Aa$ for some $a \in A$ since A is a Bézout domain. Therefore, $J = I \propto K = R(a, 0)$, completing the proof. □

Noteworthy is that new families of examples of non-semihereditary Bézout rings stem from the combination of Example 3.4 and Proposition 3.5.

At this point, for the convenience of the reader, we recall from [22] the main result that establishes the transfer of the (n, d)-property to trivial ring extensions of local rings by their respective residue fields.

THEOREM 3.6 [22, Theorem 1.1] *Let (A, M) be a local ring and let $R := A \propto A/M$ be the trivial ring extension of A by A/M. Then:*
(1) *R is a $(3, 0)$-ring provided M is not finitely generated.*
(2) *R is not a $(2, d)$-ring, for each integer $d \geq 0$, provided M contains a regular element.* □

Clearly, Theorems 3.1 and 3.6 generate original examples of n-coherent rings which, moreover, reflect no obvious correlation between (strong) n-coherence and the large class of finite conductor rings.

Example 3.7 Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Q} = \text{qf}(\mathbb{Z})$. Then $R := \mathbb{Z} \propto \mathbb{Q}$ is a strong 2-coherent ring which is not a finite conductor ring.

Proof. Straightforward via Theorem 3.1 and Theorem 2.8. □

Example 3.8 Let (V, M) be a nondiscrete valuation domain. Then $R := V \propto V/M$ is a 3-coherent ring which is neither 2-coherent nor a finite conductor ring.

Proof. R is a $(3, 0)$-ring by Theorem 3.6(1) (since M is not a finitely generated ideal of A). Hence R is 3-coherent. Further R is not a $(2, 0)$-ring by Theorem 3.6(2). So by [8, Theorem 2.4] R is not 2-coherent. Finally, Theorem 2.6(2) ensures that R is not a finite conductor ring. □

3.2 Costa's Second Conjecture

A well-known fact about semihereditary rings is that a ring R is a $(1, 1)$-ring if and only if it is a weak $(1, 1)$-ring. In this vein, Costa's second conjecture is that the (n, d)-property and the weak (n, d)-property are equivalent for any non-negative integers n and d. So far, it has been shown that this conjecture is valid under the coherence assumption [9]. It remains however elusively open, in general.

Our modest objective in this subsection is mainly to test its validity beyond the class of coherent rings. In this line, two results are stated generating two new contexts of validity for this conjecture. The first of these involves trivial ring extensions issued from coherent domains.

Theorem 3.9 *Let A be a non-trivial coherent domain, $K = \mathrm{qf}(A)$, and $R := A \propto K$. Then, for any integers $n \geq 2$ and $d \geq 1$, R is a non-coherent ring such that the following statements are equivalent:*
(i) *R is an (n, d)-ring;*
(ii) *R is a weak (n, d)-ring;*
(iii) $\mathrm{w.dim}(A) \leq d$.

Proof. (i)$\Longleftrightarrow$(ii) is a straightforward application of Theorem 3.1(1)&(3) combined to [9, Proposition 2].
(i)$\Longrightarrow$(iii) Assume R is an (n, d)-ring. By Theorem 3.1(3) and [8, Theorem 2.4], A is a $(1, d)$-domain, hence $\mathrm{w.dim}(A) \leq d$ by [20, Theorem 1.3.9].
(iii)$\Longrightarrow$(i) Assume that $\mathrm{w.dim}(A) \leq d$. Let $M := I \propto K$ be any arbitrary maximal ideal of R, where I is a maximal ideal of A (Cf. [21, Theorem 25.1]). We have $\mathrm{fd}_A(I) \leq d - 1$ and so $\mathrm{fd}_R(M) \leq d - 1$ (since $M \cong I \otimes R$ and R is A-flat). By [8, Theorem 4.1], R is a $(d + 1, d)$-ring. Further, by Theorem 3.1(2), R is strong 2-coherent. It follows that R is a $(2, d)$-ring (by [8, Theorem 2.4]) and hence an (n, d)-ring, as desired. $\Box$

Recall that a ring R is finite conductor if any ideal I of R with $\mu(I) \leq 2$ is finitely presented, where $\mu(I)$ denotes the cardinality of a minimal set of generators of I [19, Proposition 2.1]. Our next (and last) theorem tests Costa's second conjecture in the class of finite conductor rings. As might be expected, the "$\mu(I) \leq 2$" assumption (in the above definition) restricts the scope of this result to $n = 2$ and $d \leq 1$.

Theorem 3.10 *Let R be a finite conductor ring. Then:*
(1) *The following statements are equivalent:*
  (i) *R is a Prüfer domain;*
  (ii) *R is a $(2, 1)$-domain;*
  (iii) *R is a weak $(2, 1)$-domain.*
(2) *The following statements are equivalent:*
  (i) *R is a von Neumann regular ring;*
  (ii) *R is a $(2, 0)$-ring;*
  (iii) *R is a weak $(2, 0)$-ring.*

Proof. (1) We need only prove $(iii) \Longrightarrow (i)$. Suppose that (iii) holds. Let I be an arbitrary 2-generated ideal of R (i.e., $\mu(I) \leq 2$). Then I is finitely presented, and hence projective by (iii). Therefore R is a Prüfer domain by [18, Theorem 22.1].
(2) We need only prove $(iii) \Longrightarrow (i)$. Suppose that (iii) holds. It suffices to show that each principal ideal of R is a direct summand of R. Let I be a principal ideal of R. Then I is

finitely presented by hypothesis, so that R/I is a 2-presented cyclic R-module, and hence projective by (iii). Therefore the following exact sequence splits:

$$0 \to I \to R \to R/I \to 0$$

leading to the conclusion. $\Box$

Remark 3.11 Assertion (1) of Theorem 3.10 cannot extend to rings with zerodivisors. Indeed, let $R := \mathbb{Z} \times E$ as in Example 2.3. Then R is a finite conductor ring which is not a semihereditary ring (since it is not coherent). On the other hand, by [8, Theorem 4.5], R is a $(2, 1)$-ring since $wdim(R) = 1$. $\Box$

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