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Trivial Extensions of Local Rings and a Conjecture of Costa

S. Kabbaj[1,*] and N. Mahdou[2]

[1]Department of Mathematics, KFUPM, P.O. Box 5046, Dhahran 31261, Saudi Arabia
[2]Department of Mathematics, FST Fez-Saïss, B.P. 2202, Univ. of Fez, Fez, Morocco

Abstract. This paper partly settles a conjecture of Costa on (n, d)-rings, i.e., rings in which n-presented modules have projective dimension at most d. For this purpose, a theorem studies the transfer of the (n, d)-property to trivial extensions of local rings by their residue fields. It concludes with a brief discussion -backed by original examples- of the scopes and limits of our results.

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0. Introduction

All rings considered in this paper are commutative with identity elements and all modules are unital. For a nonnegative integer n, an R-module E is n-presented if there is an exact sequence $F_n \rightarrow F_{n-1} \rightarrow ... \rightarrow F_0 \rightarrow E \rightarrow 0$ in which each F_i is a finitely generated free R-module (In [1], such E is said to have an n-presentation). In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Also, $pd_R E$ will denote the projective dimension of E as an R-module.

In 1994, Costa [2] introduced a doubly filtered set of classes of rings throwing a brighter light on the structures of non-Noetherian rings. Namely, for nonnegative integers n and d, a ring R is an (n, d)-ring if every n-presented R-module has projective dimension at most d. The Noetherianness deflates the (n, d)-property to the notion of regular ring. However, outside Noetherian settings, the richness of this classification resides in its ability to unify classic concepts such as von Neumann regular, hereditary/Dedekind, and semi-hereditary/Prüfer rings. Costa was motivated by the sake of a deeper understanding of what makes a Prüfer domain Prüfer. In this context, he asked "what happens if we assume only that every finitely presented (instead of generated) sub-module of a finitely generated free module is projective?" It turned out that a non-Prüfer domain having this property exists, i.e., (In the (n, d)-jargon) a $(2, 1)$-domain which is not a $(1, 1)$-domain. This gave rise to the theory of (n, d)-rings. Throughout, we assume familiarity with n-presentation, coherence, and basics of the (n, d)-theory as in [1, 2, 3, 6, 7, 8, 10].

Costa's paper [2] concludes with a number of open problems and conjectures, including the existence of (n, d)-rings, specifically whether: "*There are examples of (n, d)-rings which are neither $(n, d-1)$-rings nor $(n-1, d)$-rings, for all nonnegative integers n and d*". Some limitations are immediate; for instance, there are no (n, 0)-domains which are not fields. Also, for $d = 0$ or $n = 0$ the conjecture reduces to "$(n, 0)$-ring not $(n-1, 0)$-ring" or "$(0, d)$-ring not $(0, d-1)$-ring", respectively.

Let's summarize the current situation. So far, solely the cases $n \leq 2$ and d arbitrary were gradually solved in [2], [3], and [14]. These partial results were obtained using various pullbacks. For obvious reasons, these were no longer useful for the specific case $d = 0$. Therefore, in [14], the author appealed to trivial extensions of fields k by infinite-dimensional k-vector spaces, and hence constructed a $(2, 0)$-ring (also called 2-von Neumann regular ring) which is not a $(1, 0)$-ring (i.e., not von Neumann regular). This encouraged further work for other trivial extension contexts.

Let A be a ring and E an A-module. The trivial ring extension of A by E is the ring $R = A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. An ideal J of R has the form $J = I \propto E'$, where I is an ideal of A and E' is an A-submodule of E such that $IE \subseteq E'$. Considerable work, part of it summarized in Glaz's book [10] and Huckaba's book [11], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. See for instance [4, 5, 9, 12, 13, 15, 16, 17].

Costa's conjecture is still elusively outstanding. A complete solution (i.e., for all nonnegative integers n and d) would very likely appeal to new techniques and constructions. Our aim in this paper is much more modest. We shall resolve the case "$n = 3$ and d arbitrary". For this purpose, Section 1 investigates the transfer of the (n, d)-property to trivial extensions of local (not necessarily Noetherian) rings by their residue fields. A surprising result establishes such a transfer and hence enables us to construct a class of $(3, d)$-rings which are neither $(3, d-1)$-rings nor $(2, d)$-rings, for d arbitrary. Section 2 is merely an attempt to show that Theorem 1.1 and hence Example 1.4 are the best results one can get out of trivial extensions of local rings by their residue fields.

1 Result and Example

This section develops a result on the transfer of the (n, d)-property for a particular context of trivial ring extensions, namely, those issued from local (not necessarily Noetherian) rings by their residue fields. This will enable us to construct a class of $(3, d)$-rings which are neither $(3, d-1)$-rings nor $(2, d)$-rings, for d arbitrary.

The next theorem not only serves as a prelude to the construction of examples, but also contributes to the study of the homological algebra of trivial ring extensions.

Theorem 1.1 *Let (A, M) be a local ring and let $R = A \propto A/M$ be the trivial ring extension of A by A/M. Then*
1) R is a $(3,0)$-ring provided M is not finitely generated.
2) R is not a $(2,d)$-ring, for each integer $d \geq 0$, provided M contains a regular element.

The proof of this theorem requires the next preliminary.

Lemma 1.2 *Let A be a ring, I a proper ideal of A, and R the trivial ring extension of A by A/I. Then $pd_R(I \propto A/I)$ and hence $pd_R(0 \propto A/I)$ are infinite.*

Proof. Consider the exact sequence of R-modules

$$0 \rightarrow I \propto A/I \rightarrow R \rightarrow R/(I \propto A/I) \rightarrow 0$$

We claim that $R/(I \propto A/I)$ is not projective. Deny. Then the sequence splits. Hence, $I \propto A/I$ is generated by an idempotent element $(a, e) = (a, e)(a, e) = (a^2, 0)$. So $I \propto A/I = R(a, 0) = Aa \propto 0$, the desired contradiction (since $A/I \neq 0$). It follows from the above sequence that

$$pd_R(R/(I \propto A/I)) = 1 + pd_R(I \propto A/I). \tag{1}$$

Let $(x_i)_{i \in \Delta}$ be a set of generators of I and let $R^{(\Delta)}$ be a free R-module. Consider the exact sequence of R-modules

$$0 \rightarrow Ker(u) \rightarrow R^{(\Delta)} \oplus R \xrightarrow{u} I \propto A/I \rightarrow 0$$

where

$$u((a_i, e_i)_{i \in \Delta}, (a_0, e_0)) = \sum_{i \in \Delta} (a_i, e_i)(x_i, 0) + (a_0, e_0)(0, 1) = (\sum_{i \in \Delta} a_i x_i, a_0)$$

since $x_i \in I$ for each $i \in \Delta$. Hence,

$$Ker(u) = (U \propto (A/I)^{(\Delta)}) \oplus (I \propto A/I)$$

where $U = \{(a_i)_{i\in\Delta} \in A^{(\Delta)} / \sum_{i\in\Delta} a_i x_i = 0\}$. Therefore, we have the isomorphism of R-modules $I \propto A/I \cong (R^{(\Delta)}/(U \propto (A/I)^{(\Delta)})) \oplus (R/(I \propto A/I))$. It follows that

$$pd_R(R/(I \propto A/I)) \leq pd_R(I \propto A/I). \tag{2}$$

Clearly, (1) and (2) force $pd_R(I \propto A/I)$ to be infinite.

Now the exact sequence of R-modules

$$0 \to I \propto A/I \to R \xrightarrow{v} 0 \propto A/I \to 0,$$

where $v(a,e) = (a,e)(0,1) = (0,a)$, easily yields $pd_R(0 \propto A/I) = \infty$, completing the proof of Lemma 1.2.2. □

Proof of Theorem 1.2.1. 1) Suppose M is not finitely generated. Let $H_0 (\neq 0)$ be a 3-presented R-module and let $(z_i)_{i=1,\dots,n}$ be a minimal set of generators of H_0 (for some positive integer n). Consider the exact sequence of R-modules

$$0 \to H_1 := Ker(u_0) \to R^n \xrightarrow{u_0} H_0 \to 0$$

where $u_0((r_i)_{i=1,\dots,n}) = \sum_{i=1}^{n} r_i z_i$. Throughout this proof we identify R^n with $A^n \propto (A/M)^n$. Our aim is to prove that $H_1 = 0$. Deny. By the above exact sequence, H_1 is a 2-presented R-module. Let $(x_i, y_i)_{i=1,\dots,m}$ be a minimal set of generators of H_1 (for some positive integer m). The minimality of $(z_i)_{i=1,\dots,n}$ implies that $H_1 \subseteq M^n \propto (A/M)^n$, whence $x_i \in M^n$ (and $y_i \in (A/M)^n$) for $i = 1, \dots, m$. Consider the exact sequence of R-modules

$$0 \to H_2 := Ker(u_1) \to R^m \xrightarrow{u_1} H_1 \to 0$$

where $u_1((a_i, e_i)_i) = \sum_{i=1}^{m} (a_i, e_i)(x_i, y_i) = \sum_{i=1}^{m} (a_i x_i, a_i y_i)$, since $x_i \in M^n$ for each i. Then, $H_2 = U \propto (A/M)^m$, where $U = \{(a_i)_{i=1,\dots,m} \in A^m / \sum_{i=1}^{m} a_i x_i = 0$ and $\sum_{i=1}^{m} a_i y_i = 0\}$. By the above exact sequence, H_2 is a finitely presented (hence generated) R-module, so that (via [11, Theorem 25.1]) U is a finitely generated

A-module. Further, the minimality of $(x_i, y_i)_{i=1,\dots,m}$ yields $U \subseteq M^m$. Let $(t_i)_{i=1,\dots,p}$ be a set of generators of U and let $(f_i)_{i=p+1,\dots,p+m}$ be a basis of the (A/M)-vector space $(A/M)^m$. Consider the exact sequence of R-modules

$$0 \to H_3 := Ker(u_2) \to R^{p+m} \xrightarrow{u_2} H_2 \to 0$$

where

$$u_2((a_i, e_i)_i) = \sum_{i=1}^{p}(a_i, e_i)(t_i, 0) + \sum_{i=p+1}^{p+m}(a_i, e_i)(0, f_i) = (\sum_{i=1}^{p} a_i t_i, \sum_{i=p+1}^{p+m} a_i f_i),$$

since $t_i \in M^m$ for each $i = 1, \dots, p$ and $(f_i)_i$ is a basis of the (A/M)-vector space $(A/M)^m$. It follows that $H_3 \cong (V \propto (A/M)^p) \oplus (M^m \propto (A/M)^m)$, where $V = \{(a_i)_{i=1,\dots,p} \in A^p / \sum_{i=1}^{p} a_i t_i = 0\}$. By the above sequence, H_3 is a finitely generated R-module. Hence $M \propto A/M$ is a finitely generated ideal of R, so M is a finitely generated ideal of A by [11, Theorem 25.1], the desired contradiction.

Consequently, $H_1 = 0$, forcing H_0 to be a free R-module. Therefore, every 3-presented R-module is projective (i.e., R is a $(3, 0)$-ring).

2) Assume that M contains a regular element m. We must show that R is not a $(2, d)$-ring, for each integer $d \geq 0$. Let $J = R(m, 0)$ and consider the exact sequence of R-modules

$$0 \to Ker(v) \to R \xrightarrow{v} J \to 0$$

where $v(a, e) = (a, e)(m, 0) = (am, 0)$. Clearly, $Ker(v) = 0 \propto (A/M) = R(0, 1)$, since m is a regular element. Therefore, $Ker(v)$ is a finitely generated ideal of R and hence J is a finitely presented ideal of R. On the other hand, $pd_R(Ker(v)) = pd_R(0 \propto A/M) = \infty$ by Lemma 1.2.2, so $pd_R(J) = \infty$. Finally, the exact sequence

$$0 \to J \to R \to R/J \to 0$$

yields a 2-presented R-module, namely R/J, with infinite projective dimension (i.e., R is not a $(2, d)$-ring, for each $d \geq 0$), completing the proof. □

We are now able to construct a class of $(3,d)$-rings which are neither $(3,d-1)$-rings nor $(2,d)$-rings, for d arbitrary. In order to do this, we first recall from [14] an interesting result establishing the transfer of the (n,d)-property to finite direct sums.

Theorem 1.3 ([14, Theorem 2.4]) *A finite direct sum $\bigoplus_{1\leq i\leq n} A_i$ is an (n,d)-ring if and only if so is each A_i.* □

Example 1.4 Let d be a nonnegative integer and B a Noetherian ring of global dimension d. Let (A_0, M) be a nondiscrete valuation domain and $A = A_0 \ltimes (A_0/M)$ the trivial ring extension of A_0 by A_0/M. Let $R = A \times B$ be the direct product of A and B. Then R is a $(3,d)$-ring which is neither a $(3,d-1)$-ring nor a $(2,d)$-ring, for d arbitrary (The case $d = 0$ reduces to "$(3,0)$-ring not $(2,0)$-ring").

Proof. By Theorem 1.2.1, A is a $(3,0)$-ring (also called 3-Von Neumann regular ring) which is not a $(2,d')$-ring for each nonnegative integer d'. Moreover, R is a $(3,d)$-ring by [14, Theorem 2.4] since both A and B are $(3,d)$-rings (by gnomonic theorems of Costa [2]). Further, R is not a $(2,d)$-ring by [14, Theorem 2.4] (since A is not a $(2,d)$-ring). Finally, we claim that R is not a $(3,d-1)$-ring. Deny. Then B is a $(3,d-1)$-ring by [14, Theorem 2.4]. Hence, by [2, Theorem 2.4] B is a $(0,d-1)$-ring since B is Noetherian (i.e., 0-coherent). So that $gldim(B) \leq d-1$, the desired contradiction. □

2 Discussion

This section consists of a brief discussion of the scopes and limits of our findings. This merely is an attempt to show that Theorem 1.2.1 and hence Example 1.4 are the best results one can get out of trivial extensions of local rings by their residue fields.

Remark 2.1 In Theorem 1.2.1, the (n,d)-property holds for a trivial ring extension of a local ring (A,M) by its residue field sans any (n,d)-hypothesis on the basic ring A. This is the first surprise. The second one resides in the narrow scope revealed by this (strong) result, namely $n = 3$ and $d = 0$.

Thus, the two assertions of Theorem 1.2.1, put together with Costa's gnomonic theorems, restrict the scope of a possible example to $n = 3$ and d arbitrary.

Furthermore, since in Theorem 1.2.1 the upshot is controlled solely by restrictions on M, the next two examples clearly illustrate its failure in case one denies these restrictions, namely, "*M is not finitely generated*" and "*M contains a regular element*", respectively.

Example 2.2 Let K be a field and let $A = K[[X]] = K+M$, where $M = XA$. We claim that the trivial ring extension R of A by $A/M(= K)$ is not an (n, d)-ring, for any integers $n, d \geq 0$.

Proof. Let's first show that R is Noetherian. Let $J = I \propto E$ be a proper ideal of R, where I is a proper ideal of A and E is a submodule of the simple A-module A/M (i.e., $E = 0$ or $E = A/M$). Since A is a Noetherian valuation ring, $I = Aa$ for some $a \in M$. Let $f \in A$ such that $(a, \bar{f}) \in J$. Without loss of generality, suppose $J \neq R(a, \bar{f})$. Let $(c, \bar{g}) \in J \setminus R(a, \bar{f})$, where $c, g \in A$, and let $c = \lambda a$, for some $\lambda \in A$. Then $(0, \bar{g} - \lambda\bar{f}) = (c, \bar{g}) - (a, \bar{f})(\lambda, \bar{0}) \in J \setminus R(a, \bar{f})$, so that we may assume $c = 0$ and $\bar{g} \neq \bar{0}$, i.e., g is invertible in A. It follows that $(0, \bar{1}) = (0, \bar{g})(g^{-1}, \bar{0}) \in J$ (hence $E = A/M$) and $(a, \bar{0}) = (a, \bar{f}) - (0, \bar{g})(g^{-1}f, \bar{0}) \in J$. Consequently, $J = (a, \bar{0})R + (0, \bar{1})R$, whence J is finitely generated, as desired.

Now, by Lemma 1.2.2, $pd_R(0 \propto A/M) = pd_R R(0, 1) = \infty$, hence $gldim(R) = \infty$. Then an application of [2, Theorem 1.3(ix)] completes the proof. □

Example 2.3 Let K be a field and E be a K-vector space with infinite rank. Let $A = K \propto E$ be the trivial ring extension of K by E. The ring A is a local $(2, 0)$-ring by [14, Theorem 3.4]. Clearly, its maximal ideal $M = 0 \propto E$ is not finitely generated and consists entirely of zero-divisors since $(0, e)M = 0$, for each $e \in E$. Let $R = A \propto (A/M)$ be the trivial ring extension of A by $A/M(\cong K)$. Then R is a $(2, 0)$-ring (and hence Theorem 1.2.1(2) fails because of the gnomonic property).

Proof. Let H be a 2-presented R-module and let $(x_1, \ldots, x_n)$ be a minimal set of generators of H. Our aim is to show that H is a projective R-module.

Consider the exact sequence of R-modules

$$0 \to Ker(u) \to R^n \xrightarrow{u} H \to 0$$

where $u((r_i)_{i=1,\ldots,n}) = \sum_{i=1}^{n} r_i x_i$. So, $Ker(u)$ is a finitely presented R-module with $Ker(u) = U \propto E'$, where U is a submodule of A^n and E' is a K-vector subspace of K^n. We claim that $Ker(u) = 0$. Deny. The minimality of $(x_1, \ldots, x_n)$ yields

$$Ker(u) = U \propto E' \subseteq (M \propto A/M)R^n = (M \propto A/M)^n$$

since R is local with maximal ideal $M \propto A/M$. Let $(y_i, f_i)_{i=1,\ldots,p}$ be a minimal set of generators of $Ker(u)$, where $y_i \in M^n$ and $f_i \in K^n$. Consider the exact sequence of R-modules

$$0 \to Ker(v) \to R^p \xrightarrow{v} Ker(u)(= U \propto E') \to 0$$

where $v((a_i, e_i)_{i=1,\ldots,p}) = \sum_{i=1}^{p}(a_i, e_i)(y_i, f_i) = (\sum_{i=1}^{p} a_i y_i, \sum_{i=1}^{p} a_i f_i)$. Here too the minimality of $(y_i, f_i)_{i=1,\ldots,p}$ yields $Ker(v) \subseteq (M \propto A/M)^p$; whence, $Ker(v) = V \propto (A/M)^p$, where $V = \{(a_i)_{i=1,\ldots,p} \in A^p / \sum_{i=1}^{p} a_i y_i = 0\} (\subseteq M^p)$. By the above exact sequence, $Ker(v)$ is a finitely generated R-module, so that V is a finitely generated A-module [11, Theorem 25.1]. Now, by the exact sequence

$$0 \to V \to A^p \xrightarrow{w} U \to 0$$

where $w((a_i)_{i=1,\ldots,p}) = \sum_{i=1}^{p} a_i y_i$, U is a finitely presented A-module (since U is generated by $(y_i)_{i=1,\ldots,p}$). Further, U is an A-submodule of A^n and A is a $(2,0)$-ring, then U is projective. In addition, A is local, it follows that U is a finitely generated free A-module. On the other hand, $U \subseteq M^n = (0 \propto E)^n$, so $(0, e)U = 0$ for each $e \in E$, the desired contradiction (since U has a basis). $\square$

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