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LINKAGE OF IDEALS IN INTEGRAL DOMAINS

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Abstract. In this paper, we investigate the linkage of ideals, in Noetherian and non-Noetherian settings, with the aim to establish new characterizations of classical notions of domains through linkage theory. Two main results assert that a Noetherian domain is Dedekind if and only if it has the primary linkage property; and a domain is almost Dedekind (resp., Prüfer) if and only if it has the linkage (resp., finite linkage) property. Also, we prove that a finite-dimensional valuation domain is a DVR (i.e., Noetherian) if and only if it has the primary linkage property.

1. Introduction

Throughout this paper, all rings are commutative with identity elements. Linkage theory lies on the border between commutative algebra and algebraic geometry and dates back to the early twentieth century, and was initially used to study algebraic curves in $\mathbb{P}^3$. From [27]: "Linkage allows to pass from a given curve to another curve, related in a geometric way to the original one. Iterating the procedure one obtains a whole series of curves in the same linkage class. The usefulness of this technique is explained by two observations: (a) certain properties of the curve are preserved under linkage, and (b) the resulting curves may be simpler, and thus easier to handle, than the original one. ... Linkage was applied in more general situations, and during the decades straddling the second world war, significant contributions were made by Dubreil [8], Apéry [1, 2], and Gaeta [9]. The breakthrough, however, came with the 1974 paper [18]. Using sheaves, duality, and homological tools, Peskine and Szpiro reduced general linkage to algebraic questions about certain ideals of regular local rings and thus put linkage theory on a sound algebraic footing."

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Since then, linkage has been an active area of research and turned out to be a powerful tool which has been successfully put into use in a number of contexts in commutative algebra and algebraic geometry alike; including, study of special varieties (e.g., [13], [25]) and class groups (e.g., [16]).

Before we define linkage, we first recall the following basic definitions. A Noetherian local ring (R, M) is Cohen-Macaulay if the grade and height of M coincide; R is Gorenstein if its injective dimension is finite when viewed as a module over itself; and R is a complete intersection if its M-completion is the quotient ring of a local regular ring modulo an ideal generated by a regular sequence; R is regular if its Krull dimension and embedding dimensions coincide. All these notions are globalized by carrying over to localizations.

In a Noetherian local ring R, two ideals I and J are linked by a complete intersection ideal A (i.e., generated by a regular sequence) if $A \subseteq I \cap J$ with $I = (A :_R J)$ and $J = (A :_R I)$. The traditional definition of linkage of ideals was set with respect to complete intersection ideals, first, in the context of regular rings and, later, in the more general context of Cohen-Macaulay rings. In 1982, Schenzel extended the notion of linkage to Gorenstein ideals. Namely, in a Gorenstein local ring R, two ideals I and J of pure height h (i.e., all associated primes have height h) are linked by a Gorenstein ideal A (i.e., R/A is Gorenstein) of pure height h if $A \subseteq I \cap J$ with $I = (A :_R J)$ and $J = (A :_R I)$ [35].

Among the prominent works on complete intersection and/or Gorenstein linkage are Artin-Nagata [3], Peskine-Szpiro [33], Rao [34], Huneke [13, 14], Kustin-Miller [24, 25], Kustin-Miller-Ulrich [26], Huneke-Ulrich [16, 17, 18, 19, 20], Bologio-Migliore [4], Migliore [28], Nagel [30], and Migliore-Nagel [29].

In 2004, Martsinkovsky-Strooker extended the definition of Gorenstein linkage of ideals to arbitrary finitely generated modules over Noetherian semi-perfect rings [27]. Their module-theoretic approach brought about more general and more precise results providing, thus, simple proofs for some traditional settings of algebraic geometry and local algebra. In their general definition of Gorenstein linkage of ideals, they have dispensed with the requirement of purity. In 2000, Yoshino-Isogawa [36] introduced and studied linkage in the special case of Cohen-Macaulay modules which agrees with Martsinkovsky-Strooker's definition. Other notable works on linkage of modules are, among others, Nagel [31], Dibaei-Gheibi-Hassanzadeh-Sadeghi [5], Dibaei-Sadeghi [6, 7], and Iima-Takahashi [21].

In this work, we extend the definition of linkage of ideals to a much more general and fundamental formalism which deals with linking ideals of arbitrary (commutative) rings by arbitrary ideals (i.e., not necessarily subject to the complete intersection or Gorenstein conditions): *In a ring R, two ideals I and J are linked if there is an ideal $A \subseteq I \cap J$ such that $I = (A :_R J)$ and $J = (A :_R I)$.*

As a first phase, in this paper, we restrict our study of linkage of ideals to integral domains (i.e., commutative rings without zero-divisors). We examine both Noetherian and non-Noetherian settings. Our main goal is to characterize the notion of linkage of ideals in some classical classes of integral domains. Two main results assert that a Noetherian domain is Dedekind if and only if it has the primary linkage property (Theorem 3.6); and a domain is almost Dedekind (resp., Prüfer) if and only if it has the linkage (resp., finite linkage) property (Theorem 3.3). Also, we prove that a finite-dimensional valuation domain is a DVR (i.e., Noetherian) if and only if it has the primary linkage property (Proposition 3.7).

Throughout, given a ring R and two ideals I and J of R, we let

$$(I :_R J) = \{x \in R \mid xJ \subseteq I\}$$

and, if R is a domain with quotient field $\mathrm{qf}(R)$, we let

$$(I : J) = \{x \in \mathrm{qf}(R) \mid xJ \subseteq I\}$$

and

$$I^{-1} = (R : I) = \{x \in \mathrm{qf}(R) \mid xI \subseteq R\}.$$

2. Basic properties and examples

This sections provides basic properties of linkage of ideals in an arbitrary (commutative) ring along with natural examples of linked and non-linked ideals. We first recall the definition.

Definition 2.1. In a ring R, two ideals I and J are linked if there is an ideal $A \subseteq I \cap J$ such that $I = (A :_R J)$ and $J = (A :_R I)$. We say that *I and J are linked over A.*

Next, we show that two ideals are linked if and only if they are linked over their product.

Lemma 2.2. *In a ring R, two ideals I and J are linked if and only if they are linked over IJ.*

Proof. Assume that I and J are linked. Then, there is an ideal $A \subseteq I \cap J$ such that $I = (A :_R J)$ and $J = (A :_R I)$. Therefore, $IJ = J(A :_R J) \subseteq A$. Hence $I \subseteq (IJ :_R J) \subseteq (A :_R J) = I$. Whence $I = (IJ :_R J)$. Similarly, $J = (IJ :_R I)$ and so I and J are linked over IJ. □

Observe that we always have $I \subseteq (IJ :_R J)$ and $J \subseteq (IJ :_R I)$ and so I and J are linked if and only if the two reverse inclusions hold. Also, an ideal I is *self-linked* (i.e., linked to itself) means $I = (I^2 :_R I)$. Clearly, invertible ideals are self-linked, whereas proper idempotent ideals are not self-linked.

Next, we provide families of pairs of ideals which are always linked in any arbitrary (commutative) ring.

Lemma 2.3. *Let R be a ring. Then:*

(1) *Any two comaximal ideals are linked.*
(2) *Any two incomparable prime ideals are linked.*
(3) *Any two primary ideals with incomparable radicals are linked.*
(4) *Any two prime ideals $P \subseteq M$, with M maximal, are linked if and only if $PM \neq P$.*

Proof. (1) Let I and J be two ideals of R with $I + J = R$. Then, we have
$$(IJ :_R J) = I(IJ :_R J) + J(IJ :_R J) \subseteq I + IJ \subseteq I$$
and, likewise, $(IJ :_R I) \subseteq J$.

(2) This is a particular case of (3).

(3) Let I and J be two primary ideals of R such that $P := \sqrt{I}$ and $Q := \sqrt{J}$ are incomparable. Then, $J(IJ :_R J) \subseteq IJ \subseteq I$ with $J \nsubseteq P$. It follows that $(IJ :_R J) \subseteq I$ and, similarly, $(IJ :_R I) \subseteq J$.

(4) Clearly, $M = (M^2 :_R M)$ if and only if $(M^2 :_R M) \subsetneqq R$ if and only if $M \neq M^2$. So, we may assume $P \subsetneqq M$. Then, $M(PM :_R M) \subseteq PM \subseteq P$ forces $(PM :_R M) \subseteq P$. Moreover, maximality yields $(PM :_R P) = M$ if and only if $(PM :_R P) \subsetneqq R$ if and only if $PM \neq P$, completing the proof of the lemma. □

The following example features two non-linked primary ideals (a fortiori, with comparable radicals) and two linked comparable prime ideals in a Noetherian domain.

Example 2.4. Let k be a field and let x, y, z be indeterminates over k. Let $R := k[x, y, z]$, $I := (x^2, y^2)$, and $J := (x^3, y^3, z^3)$. Observe that
$$\sqrt{I} = (x, y) \subset \sqrt{J} = (x, y, z).$$
Clearly, we have
$$IJ = (x^5, x^2y^3, x^2z^3, y^2x^3, y^5, y^2z^3)$$
and
$$x^2y^2z^2I = (x^4y^2z^2, x^2y^4z^2) \subseteq IJ.$$
It follows that $x^2y^2z^2 \in (IJ :_R I) \setminus J$. Hence $J \subsetneqq (IJ :_R: I)$ and so I and J are not linked. Moreover, by Lemma 2.3(4), $P := (x, y)$ and $M := (x, y, z)$ are linked since, obviously, $PM \subsetneqq P$.

Given a nonzero ideal A in a domain, an ideal I is said to be *A-divisorial* if $(A : (A : I)) = I$. Next, we show that A-divisoriality is a necessary condition for linkage over A.

Lemma 2.5. *Let R be a domain and A a nonzero ideal of R with $(A : A) = R$. Then, any two linked ideals over A are A-divisorial.*

Proof. Let I and J be two ideals of R with $I = (A :_R J)$ and $J = (A :_R I)$. Then, $I = (A :_R (A :_R I)$ and $J = (A :_R (A :_R J)$. Since $A(A : I) \subseteq I(A : I) \subseteq A$, $(A : I) \subseteq (A : A) = R$; that is, $(A : I) = (A :_R I)$. Hence

$$I = (A :_R (A :_R I) = (A :_R (A : I)).$$

Now, let $x \in (A : (A : I))$. Then $xA \subseteq x(A : I) \subseteq A$ and so $x \in (A : A) = R$. Therefore, $(A : (A : I)) \subseteq R$ and hence $(A : (A : I) = (A :_R (A : I))$. It follows that $I = (A : (A : I))$. Likewise, we have $J = (A : (A : J))$, completing the proof of the proposition. □

The converse of the above result is not true in general, as shown below.

Example 2.6. Let k be a field and let x, y be indeterminates over k. Let $R := k[[x, y]]$, $M := (x, y)$, $I := xM$, $J := yM$ and $A := xyM$. Then

$$IJ = xyM^2 \subsetneqq A \subsetneqq xyR = I \cap J$$

and $(A : A) = R$. Moreover,

$$(A : I) = (xyM : xM) = y(M : M) = yR$$

and so

$$(A : (A : I)) = (xyM : yR) = (I : R) = I.$$

Likewise, we have

$$(A : (A : J)) = (xyM : xR) = (J : R) = J.$$

Hence I and J are A-divisorial. But $(A :_R I) = (A : I) = yR \supsetneqq J$. Thus, I and J are not linked over A.

Next, we provide another necessary condition for linkage related to the dual I^{-1} and endomorphism ring $(I : I)$ of an ideal I.

Lemma 2.7. *Let R be a domain and let I, J be two nonzero ideals with $I^{-1} = J^{-1}$. Then, I and J are linked only if $(I : I) = (J : J)$.*

Proof. Let $x \in (I : I) \subseteq I^{-1} = J^{-1}$. Then $xI \subseteq I$ and $xJ \subseteq R$. Hence $xIJ \subseteq IJ$ and so $xJ \subseteq (IJ :_R I) = J$; that is, $x \in (J : J)$ and so $(I : I) \subseteq (J : J)$. Likewise, we get $(J : J) \subseteq (I : I)$, completing the proof of the lemma. □

Here is an illustrative example.

Example 2.8. Let x be an indeterminate over $\mathbb{Q}$. Let $V := \mathbb{Q}[[x]] = \mathbb{Q} + M$, where $M := xV$, $R := \mathbb{Z} + M$, $I := x(\mathbb{Z}_{(p)} + M)$ and $J := x(\mathbb{Z}_{(q)} + M)$, where p and q are distinct prime integers. It is easy to check that $I^{-1} = J^{-1} = V$. However, we have

$$(I : I) = \mathbb{Z}_{(p)} + M \neq \mathbb{Z}_{(q)} + M = (J : J)$$

and hence I and J are not linked.

3. Characterizations through linkage

This section investigates the concept of linkage of ideals, in Noetherian and non-Noetherian settings, with the aim to establish new characterizations of classical notions of integral domains through linkage theory.

Next, we characterize two large classes of (integral) domains through linkage.

Proposition 3.1. *Let R be a domain. The following statements are equivalent:*

(1) *R is integrally closed (resp., completely integrally closed);*
(2) *Each nonzero finitely generated ideal (resp., each nonzero ideal) is linked to a principal subideal.*
(3) *Each nonzero finitely generated ideal (resp., each nonzero ideal) is linked to an invertible subideal.*

Proof. Assume that R is integrally closed (resp., completely integrally closed) and let I be a nonzero finitely generated ideal (resp., nonzero ideal) of R. Then $(I : I) = R$. Let $0 \neq a \in I$ and $J := aR$. So $(IJ :_R J) = (I :_R R) = I$ and

$$\begin{array}{rcl}(IJ :_R I) & = & (aI :_R I) \\ & = & (aI : I) \cap R \\ & = & a(I : I) \cap R \\ & = & aR \\ & = & J.\end{array}$$

Thus, I and J are linked.

Next, let I be a nonzero finitely generated ideal (resp., nonzero ideal) of R and assume that I is linked to an invertible ideal $J \subseteq I$. Let $x \in (I : I)$. Then

$$xJ \subseteq xI \subseteq I \subseteq R$$

and $xIJ \subseteq IJ$. Hence, $xJ \subseteq (IJ :_R I) = J$ and whence $x \in (J : J) = R$. Therefore, $(I : I) = R$, completing the proof of the proposition. ☐

Remark 3.2. The above proof ensures that: *in a domain R, a nonzero ideal I is linked to an invertible (resp., principal) subideal if and only if $(I : I) = R$.*

Next, we use linkage to characterize two important subclasses of integrally closed domains and completely integrally closed domains; namely, Prüfer domains and almost Dedekind domain, respectively. To this purpose, recall that, given an ideal I in a ring R, an ideal J is called a *reduction* of I if $J \subseteq I$ with $JI^n = I^{n+1}$, for some positive integer n; and I is called *basic* if it has no proper reduction [15, 32]. Invertible ideals and idempotent ideals are basic [22, Lemma 2.1]. Of relevance to our study of linkage is Hays' characterization of Prüfer domains through reduction: a domain "*R is Prüfer if and only if every finitely generated ideal of R is basic.* [10, 11]."

Throughout, for the sake of convenience, we say that a ring R has the *linkage property* if any two distinct nonzero ideals of R are linked; and R has the *finite linkage property* if any two distinct nonzero finitely generated ideals of R are linked.

Next, we establish the first main result of this paper.

Theorem 3.3. *A domain has the finite linkage property (resp., linkage property) if and only if it is Prüfer (resp., almost Dedekind).*

Proof. Let R be a domain satisfying the finite linkage property and let I be a proper finitely generated ideal of R. We prove that I is basic. Deny, for a contradiction, and suppose that J is a proper reduction of I; i.e., $JI^n = I^{n+1}$, for some positive integer n. Without loss of generality, we may assume that J is finitely generated (cf. [12, Lemma 2.7]). If $n = 1$, then $JI = I^2$. Since I and J are linked, we have

$$J \subsetneqq I \subseteq (I^2 :_R I) = (IJ :_R I) = J$$

which is a contradiction. Next, suppose $n \geq 2$. Necessarily, $J \neq I^n$. Otherwise, if $J = I^n$ we obtain

$$I^{n-1}I^{n+1} = JI^n = I^{n+1}$$

which yields, via [23, Theorem 76], $(1 + a)I^{n+1} = 0$ for some $a \in I^{n-1}$, forcing thus $1 = -a \in I^{n-1} \subseteq I$, which is absurd. Therefore, J and I^n are linked and hence

$$J \subsetneqq I \subseteq (I^{n+1} :_R I^n) = (JI^n :_R I^n) = J$$

which is a contradiction. Consequently, I is basic and so R is a Prüfer domain by [10, Theorem 6.5]. The converse is straightforward through the fact that, in a Prüfer domain, every finitely generated ideal is invertible, completing the proof of the first statement.

Now, let R be a domain satisfying the linkage property. Then, R is a Prüfer domain. Moreover, we claim that $\dim(R) = 1$. Deny and suppose, for a contradiction, that $\dim(R) \geq 2$. Let $P \subsetneqq M$ be two prime ideals of R such that M is maximal. Then, necessarily, $PM = P$. On the other hand, P and M are linked and so, by Lemma 2.3(4), $PM \neq P$, which is absurd, proving the claim. Next, let Q be any nonzero prime ideal of R. We claim that Q is not idempotent. Deny and assume $Q = Q^2$. Let $0 \neq a \in Q$. Then, $Q \neq aQ$ since $1 = a^{-1}a \in a^{-1}Q$. Hence Q and aQ are linked and whence

$$\begin{array}{rcl} Q & = & (aQ^2 :_R aQ) \\ & = & (aQ :_R aQ) \\ & = & R \end{array}$$

which is a contradiction. It follows that R is a one-dimensional strongly discrete Prüfer domain and thus an almost Dedekind domain. Conversely, assume that R is almost Dedekind and let I, J be two distinct nonzero ideals of R. Let $x \in (IJ :_R J)$. Then, for each maximal ideal M of R, R_M is a discrete valuation domain (DVR) and so $JR_M(JR_M)^{-1} = R_M$. Thus

$$\begin{array}{rcl} xR_M & = & xJR_M(JR_M)^{-1} \\ & \subseteq & IJR_M(JR_M)^{-1} \\ & = & IR_M. \end{array}$$

Hence $x \in IR_M$. It follows that $x \in \bigcap_M IR_M = I$. Thus $I = (IJ :_R J)$. Similarly, $J = (IJ :_R I)$ and therefore I and J are linked, completing the proof of the theorem. □

Next, we examine linkage in (locally) Noetherian domains.

Proposition 3.4. *In a (locally) Noetherian domain, any two nonzero radical (and, a fortiori, prime) ideals are linked.*

Proof. Let R be a locally Noetherian domain and let I, J be nonzero radical ideals of R. Assume, for a contradiction, that there is $x \in (IJ :_R J) \setminus I$. Then, there is a minimal prime ideal P of I such that $x \notin P$. We claim that, for each $n \geq 1$, we have

$$JR_P \subseteq P^n R_P.$$

We prove this fact by induction on n. Indeed, $xJ \subseteq IJ \subseteq I$ forces $xJR_P \subseteq IR_P = \sqrt{IR_P} = PR_P$ and hence

$$JR_P \subseteq PR_P.$$

Next, suppose that $JR_P \subseteq P^n R_P$. Then, $xJR_P \subseteq IJR_P \subseteq P^{n+1}R_P$. Moreover, $x \notin PR_P$ and $P^{n+1}R_P$ is PR_P-primary. Therefore, we get

$$JR_P \subseteq P^{n+1}R_P$$

proving the claim. Now, since R_P is a Noetherian domain, via [23, Theorem 77], we get

$$JR_P \subseteq \bigcap_{n \geq 1} P^n R_P = 0.$$

That is, $J = 0$, which is absurd. Consequently, $(IJ :_R J) = I$ and, similar arguments yield $(IJ :_R I) = J$, as desired. □

Remark 3.5. When $I = J$, the above result ensures that: *in a (locally) Noetherian domain R, we have $(I^2 :_R I) = I$, for any nonzero radical ideal I of R.*

Follows the second main result of this paper. For this purpose, we say a ring R has the *primary linkage property* if any two distinct nonzero primary ideals of R are linked.

Theorem 3.6. *A Noetherian domain (resp., one-dimensional locally Noetherian domain) has the primary linkage property if and only if it is Dedekind (resp., almost Dedekind).*

Proof. If R is a Dedekind (resp., an almost Dedekind) domain, then R has the linkage property by Theorem 3.3. So, we only need to prove the "only if" assertion for both statements.

Let R be a Noetherian domain subject to the primary linkage property. We claim that $\dim(R) = 1$. Indeed, suppose for a contradiction, that $\dim(R) \geq 2$ and let M be a prime ideal of R with $\mathrm{ht}(M) = 2$. One can easily check that R_M (and, in fact, any localization of R) satisfies the primary linkage property. So, without loss of generality, we assume that R is local with maximal ideal M and $\mathrm{ht}(M) = \dim(R) = 2$. Let $0 \subsetneqq P \subsetneqq M$ be a chain in the prime spectrum of R and let $a \in M \setminus P$. Consider the two ideals of R given by

$$I := aR + P \quad \text{and} \quad J := a^2R + P^2.$$

Clearly, $a \notin J$ and hence $J \subsetneqq I$. Now, if P_1 is a minimal prime ideal over J, then $a \in P_1$ and hence $P \subsetneqq P_1 \subseteq M$, whence $P_1 = M$. It follows that $\sqrt{J} = \sqrt{I} = M$ and therefore I and J are M-primary. Thus, I and J are linked. So, since

$$aPI = a^2P + aP^2 \subseteq IJ$$

we have

$$aP \subseteq (IJ :_R I) = J.$$

Next, consider the P-primary ideal of R given by

$$Q := P^2R_P \cap R$$

and let $p \in P$. Then, $ap \in J$ and so $ap = ra^2 + d$, for some $r \in R$ and $d \in P^2$. Therefore, $a(p - ra) \in P^2 \subseteq Q$ with $a \notin P$ and so $q := p - ra \in Q$. Hence $ra \in P$ and then $r \in P$. It follows that $p = ar + q \in aP + Q$. Consequently, $P \subseteq aP + Q$ and thus

$$P = aP + Q.$$

Now, let

$$E := aR + Q$$

and, in the Noetherian quotient ring R/Q, let $A := P/Q$ and $B := E/Q$. Then, $BA = A$. By [23, Theorem 76], there is $y \in E$ such that $(1+y)P \subseteq Q$. If $P \nsubseteq Q$, then $1+y \in P \subset M$ (since Q is P-primary), which is absurd since $y \in E \subseteq M$. If $P \subseteq Q$, then $P = Q$ and so

$$PR_P = QR_P = P^2R_P$$

which is absurd, too, since R_P is a Noetherian domain and hence has no nonzero idempotent prime ideals, proving the claim; i.e., $\dim(R) = 1$. Next, let N be a maximal ideal of R and suppose, for a contradiction, that N is not invertible. Necessarily, we have

$$NN^{-1} = N \text{ and so } N^{-1} = (N : N).$$

Moreover, since $\mathrm{ht}(N) = 1$, N is a t-ideal and then $N = N_t = N_v$. This yields

$$R \subsetneqq N^{-1}.$$

Let $0 \neq a \in N$ and let $I := aN^{-1}$ and $J := aR$. Clearly, I and J are N-primary since $\dim(R) = 1$. So I and J are linked. Therefore, we obtain

$$\begin{array}{rcl} J & = & (IJ :_R I) \\ & = & (a^2N^{-1} :_R aN^{-1}) \\ & = & aN^{-1} \cap R \\ & = & I. \end{array}$$

which is absurd. Consequently, N is invertible and thus R is a Dedekind domain, completing the proof of the first statement.

Next, let R be a one-dimensional locally Noetherian domain subject to the primary linkage property. Observe, at this point, that the (primary) linkage property does not carry up, necessarily, to localizations in the absence of Noetherianity. Let M be a maximal ideal of R and let IR_M and JR_M be two distinct primary ideals of R_M. Set

$$A := IR_M \cap R \text{ and } B := JR_M \cap R.$$

Since R_M is Noetherian, there exist two finitely generated ideals $A_1 \subseteq A$ and $B_1 \subseteq B$ such that

$$A_1R_M = AR_M = IR_M \text{ and } B_1R_M = BR_M = JR_M.$$

Moreover, A and B are distinct and M-primary as $\dim(R_M) = \dim(R) = 1$. Hence A and B are linked; that is,

$$A = (AB :_R B) \text{ and } B = (AB :_R A).$$

So, the fact that A_1 and B_1 are finitely generated yields

$$\begin{array}{rcl} (IJR_M :_{R_M} JR_M) & = & (A_1B_1R_M :_{R_M} B_1R_M) \\ & = & (A_1B_1 :_R B_1)R_M \end{array}$$

and, likewise

$$(IJR_M :_{R_M} IR_M) = (A_1B_1 :_R A_1)R_M.$$

Next, let $x \in (IJR_M :_{R_M} JR_M)$. Then, $tx \in (A_1B_1 :_R B_1)$, for some $t \in R \setminus M$, and so $txB_1 \subseteq A_1B_1 \subseteq AB$. Now, let $y \in B$. Then, $yr \in B_1$, for some $r \in R \setminus M$, and so $txyr \in AB$. But, AB is M-primary since $\sqrt{AB} = \sqrt{A} \cap \sqrt{B} = M$. Hence, $txy \in AB$, whence $txB \subseteq AB$. Therefore, $tx \in (AB :_R B) = A$ and so $x \in AR_M = IR_M$. It follows that

$$(IJR_M :_{R_M} JR_M) = IR_M$$

and likewise

$$(IJR_M :_{R_M} IR_M) = JR_M.$$

That is, IR_M and JR_M are linked. Consequently, any two distinct primary ideals of R_M are linked. By the first statement, R_M is a Dedekind domain and so R is an almost Dedekind domain, completing the proof of the theorem. □

The next result identifies the primary linkage property as a default condition for a valuation domain to be Noetherian; i.e., a Discrete Valuation Ring (DVR, for short).

Proposition 3.7. *A valuation domain has the primary linkage property if and only if it is Noetherian (i.e., DVR).*

Proof. A DVR has the (primary) linkage property by Theorem 3.6 or Theorem 3.3. So, we only need to prove the "only if" assertion.

Let V be a valuation domain subject to the primary linkage property. We claim that $\dim(V) = 1$. Indeed, suppose for a contradiction, that $\dim(V) \neq 1$ and let $0 \subsetneqq P \subsetneqq Q$ be a chain of prime ideals of V. Therefore, since $PQ = (PV_P)Q = PQV_P = PV_P = P$, we obtain

$$\begin{aligned} Q &= (PQ :_V P) \\ &= (P :_V P) \\ &= V \end{aligned}$$

which is a contradiction. Hence, $\dim(V) = 1$. Finally, we claim that the maximal ideal M of V is not idempotent. Otherwise, assume for a contradiction that $M = M^2$. Let $0 \neq a \in M$ and set

$$I := aM.$$

Clearly, I is M-primary and $I \neq M$. So, by hypothesis, I and M are linked. It follows that

$$\begin{array}{rcl} I & = & (IM :_V M) \\ & = & (aM^2 :_V M) \\ & = & (aM :_V M) \\ & = & (aM : M) \cap V \\ & = & aV \cap V \\ & = & aV \end{array}$$

which is absurd. Hence M is principal and therefore V is a DVR, completing the proof of the proposition. □

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