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REDUCTIONS OF IDEALS IN PULLBACKS

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Abstract. This paper deals with reductions of ideals in various settings of pullback constructions. Precisely, we investigate reductions of several types of ideals in both generic and classical pullbacks. We also characterize pullbacks where reductions of a class of ideals extend to reductions of their respective extended ideals. All results are illustrated with original examples.

1. Introduction

Throughout this paper, all rings are commutative with identity. Let R be a ring and I an ideal of R. An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. An ideal which has no reduction other than itself is called basic. The notion of reduction was introduced by Northcott and Rees with the purpose to contribute to the analytic theory of ideals in Noetherian local rings through reductions.

In [8, 9], Hays extended the study of reductions of ideals to more general contexts of commutative rings; particularly, Prüfer domains and Noetherian rings (not necessarily local). His two main results assert that "a domain is Prüfer if and only if every finitely generated ideal is basic" [8, Theorem 6.1] and "in an integral domain, every ideal is basic if and only if it is a one-dimensional Prüfer domain" [9, Theorem 10]. Moreover, he showed that most results on reductions of ideals do not extend beyond the class of Noetherian rings.

Very recently, a t-analogue of the notion of reduction has been thoroughly studied in various classes of integral domains, including pullback constructions; see [13, 14, 15]. For ample details on reductions of ideals, we refer the reader to Huneke and Swanson's book "Integral closure of ideals, rings, and modules" [11].

Let T be a domain, M a maximal ideal of T, K its residue field, $\varphi : T \longrightarrow K$ the canonical surjection, D a proper subring of K with quotient field k. Let R be the pullback issued from the following diagram of canonical homomorphisms:

$$(\square)\quad \begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K = T/M. \end{array}$$

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So, $R := \varphi^{-1}(D) \subsetneq T$ and $M = (R : T)$ is the conductor of T in R and hence a common ideal to both R and T. Along this paper, we shall refer to the diagram ($\square$) as *generic*, and we say R is a generic pullback issued from (T, M, D). In the special case where $T = V$ is a valuation domain, we refer to the diagram ($\square$) as *classical*, and we say R is a classical pullback issued from (V, M, D). For more details on the ideal structure of pullbacks and their respective ring-theoretic properties, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 10, 12, 16].

This paper deals with reductions of ideals in various settings of pullback constructions. In Section 2, we investigate reductions of several types of ideals in both generic and classical pullbacks. We also characterize pullbacks where reductions of a class of ideals extend to reductions of their respective extended ideals. In Section 3, all results are illustrated with original examples, where we explicitly compute the reductions of given ideals.

2. Reductions of ideals

An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. An ideal which has no proper reduction is called basic [8, 18]. This section investigates the reductions of ideals in pullbacks.

Recall that, given a ring homomorphism $R \longrightarrow S$ and ideals $J \subseteq I$ in R, if J is a reduction of I, then JS is a reduction of IS. The converse holds if the homomorphism is faithfully flat [11, Lemma 8.1.3]. Also, in a Prüfer (and, a fortiori, valuation) domain, J is a reduction of I if and only if $JI = I^2$ [9, Proposition 1].

Throughout, we denote by $\mathrm{Red}_R(I)$ (or, $\mathrm{Red}(I)$, when no confusion is likely) the set of all reductions of I in R, and by $\mathrm{PRed}_R(I)$ the set of all principal reductions of I in R. In particular, $\mathrm{PRed}_R(I) = \emptyset$ means that I has no principal reduction in R. Also, by $A \subsetneq B$, we mean A is strictly contained in B.

For the reader's convenience, we recall a few basic facts about the structure of ideals in a pullback R issued from (T, M, D). Namely, if I is an ideal of R with $M \subsetneq I$ (), then $I = \varphi^{-1}(A)$ for some nonzero ideal A of D; and if T is local, then every ideal of R is comparable to M. Moreover, if $T = V$ is a valuation domain and $I \subsetneq M$, then either I is an ideal of V or $IV = aV$ is a principal ideal of V, for some $0 \neq a \in M$; and in the latter case, if I is not an ideal of V, then $I = a\varphi^{-1}(W)$ for some D-submodule W of K with $D \subseteq W \subsetneq K$. For more details, see [3, 5].

The first result of this section investigates reductions of ideals in the general setting of generic pullbacks, where three types of ideals are addressed; namely, $I = M$, $I \supsetneq M$, and $I + M = R$.

Theorem 2.1. *Let R be a generic pullback issued from (T, M, D) and I an ideal of R.*

(1) *If $I = M$, then:*
$$\mathrm{Red}_R(M) = \mathrm{Red}_T(M) \Longleftrightarrow M \text{ is basic in } R \Longleftrightarrow M = M^2.$$
(2) *If $M \subsetneqq I$; i.e., $I = \varphi^{-1}(A)$, for some nonzero ideal A of D, then:*
$$\mathrm{Red}_R(I) = \left\{\varphi^{-1}(B) \mid B \in \mathrm{Red}_D(A)\right\}.$$
(3) *If $I + M = R$, then:*
$$\begin{array}{rcl}\mathrm{Red}_R(I) & = & \left\{J \subseteq I \ \mid\ JT \in \mathrm{Red}_T(IT) \text{ with } J + M = R\right\} \\ & = & \left\{H \cap R \ \mid\ H \in \mathrm{Red}_T(IT)\right\}.\end{array}$$

Proof. (1) Suppose that $\mathrm{Red}_R(M) = \mathrm{Red}_T(M)$ and let J be a reduction of M in R. Then, $J \in \mathrm{Red}_T(M)$ and hence $M^{n+1} \subseteq J$ for some positive integer n. So J is an M-primary ideal of T. We claim that $J = M$. Deny and let $m \in M \setminus J$ and $J_o := mR + J$. Then J_o is a reduction of M in R and so in T by hypothesis. In particular, J_o is an ideal of T. Let $\alpha \in T \setminus R$. Then, $m\alpha = am + b$, for some $a \in R$ and $b \in J$. Therefore, $m(\alpha - a) \in J$ and so $\alpha - a \in M$. Hence, $\alpha \in R$, which is absurd. Consequently, M is basic in R. The converse is obvious since the inclusion $\mathrm{Red}_T(M) \subseteq \mathrm{Red}_R(M)$ always holds (as any ideal $J \subseteq M$ of T is also an ideal of R). For the second equivalence, assume that M is basic in R. We claim that M is idempotent. Otherwise, assume $M^2 \subsetneqq M$ and let $\left\{\overline{m_i}\right\}_{i\in\Delta}$ be a basis for the nonzero vector space M/M^2 over the field T/M. Necessarily, we have
$$M = \sum_{i\in\Delta} m_i T + M^2.$$
Next, consider the following ideal of R
$$J := \sum_{i\in\Delta} m_i R + M^2.$$
Clearly, $JT = M$ and so $JM = M^2$; i.e., J is a reduction of M in R. Therefore, $J = M$. Let $t \in T \setminus R$ and let $i_o \in \Delta$. Then, $tm_{i_o} \in J$ and hence
$$\bar{t}\, \overline{m_{i_o}} = \sum_{1\leq i\leq n} \overline{r_i}\, \overline{m_i}$$
in M/M^2, for some integer $n \geq 1$ and some elements $r_1, ..., r_n$ of R. It follows that $t \in R$, the desired contradiction. For the converse, it is already known that an idempotent ideal is always basic [17, Lemma 2.1].

(2) Suppose that $M \subsetneqq I$; that is,
$$I = \varphi^{-1}(A)$$
for some nonzero ideal A of D. If $JI^n = I^{n+1}$, for some positive integer n, then the assumption $M \subsetneqq I$ forces
$$M \subseteq I^{n+1} \subseteq J$$
and so
$$J = \varphi^{-1}(B)$$

for some nonzero ideal B of D. Now it is easy to check that $BA^n = A^{n+1}$ if and only if $JI^n = I^{n+1}$, for any integer $n \geq 0$, leading to the conclusion.

(3) Assume J is a reduction of I. Then, obviously JT is a reduction of IT and, as $I^{n+1} \subseteq J$ for some positive integer n, any maximal ideal containing J contains also I; that is, $J+M=R$. Conversely, assume $JI^nT = I^{n+1}T$, for some positive integer n, and $J+M=R$. Then, we get

$$\begin{array}{rcl} JI^nM^{n+1} & = & JI^nTM^{n+1} \\ & = & I^{n+1}TM^{n+1} \\ & = & I^{n+1}M^{n+1}. \end{array}$$

Moreover, the assumption "$J+M=R$" forces $J+M^{n+1}=R$. It follows that

$$\begin{array}{rcl} I^{n+1} & = & JI^{n+1}+I^{n+1}M^{n+1} \\ & = & JI^{n+1}+JI^nM^{n+1} \\ & \subseteq & JI^n. \end{array}$$

Therefore, J is a reduction of I, completing the proof of the first equality.

Next, let J be a reduction of I in R and let $H := JT$. Then, H is a reduction of IT with $J+M=R$. Therefore, $H+M=T$. It follows that

$$\begin{array}{rcl} H\cap R & = & H\cap (J+M) \\ & = & J+(H\cap M) \\ & = & J+HM \\ & = & J+JM \\ & = & J. \end{array}$$

Conversely, let H be a reduction of IT and let $J := H\cap R$. Clearly, $IT+M=T$ and hence $H+M=T$. Consequently, we get

$$\begin{array}{rcl} J+M & = & (H\cap R)+M \\ & = & (H+M)\cap R \\ & = & R \end{array}$$

with $JT=H$; that is, J is a reduction of I, completing the proof of the theorem. □

The next result investigates the special setting of classical pullbacks, where all types of ideals are addressed; namely, $I=M$, $M \subsetneq I$, $0 \neq I \subsetneq M$ with I not an ideal of V, and $I \subsetneq M$ with I an ideal of V.

Theorem 2.2. *Let R be a classical pullback issued from (V,M,D) and I an ideal of R.*

(1) *The two cases "$I=M$" and "$M \subsetneq I$" are similar to Theorem 2.1(1)&(2).*

(2) *If $0 \neq I \subsetneq M$ and I is an ideal of V, then:*

$$\mathrm{Red}_R(I) = \mathrm{Red}_V(I) \Longleftrightarrow \mathrm{PRed}_R(I) = \emptyset \Longleftrightarrow \mathrm{PRed}_V(I) = \emptyset.$$

(3) *If $I \subsetneq M$ and I is not an ideal of V; i.e., $I = a\varphi^{-1}(W)$, for some $0 \neq a \in M$ and D-submodule W with $D \subseteq W \subsetneq K$, then:*

$$\mathrm{Red}_R(I) = \Big\{a\varphi^{-1}(H) \mid H \textit{ submodule of } W \textit{ with } HW^n = W^{n+1} \textit{ for some } n \geq 0\Big\}.$$

Proof. (2) Suppose that $0 \neq I \subsetneq M$ and I is an ideal of V. First, it is worthwhile noting that I is not principal in R in this case (since $R \subsetneq V$). Clearly, aR is a principal reduction of I in R if and only if aV is a principal reduction of I in V. This proves the second equivalence. Next, assume that $\mathrm{PRed}_V(I) = \emptyset$ and let $J \in \mathrm{Red}_R(I)$. Then, JV is a reduction of I in V and hence JV is not a principal ideal of V (by hypothesis). It follows that J is necessarily an ideal of V and so $J \in \mathrm{Red}_V(I)$. This proves that $\mathrm{Red}_R(I) = \mathrm{Red}_V(I)$. Conversely, assume the latter equality holds. If $0 \neq aR$ is a principal reduction of I, then $aR \in \mathrm{Red}_V(I)$, hence aR is an ideal of V. This is absurd since $R \subsetneq V$. So, $\mathrm{PRed}_R(I) = \emptyset$, as desired.

(3) Suppose that $I \subsetneq M$ and I is not an ideal of V; that is,

$$I = a\varphi^{-1}(W)$$

for some $0 \neq a \in M$ and D-submodule W with $D \subseteq W \subsetneq K$. Let J be a reduction of I and let n be a positive integer such that $JI^n = I^{n+1}$. Then

$$a^n JT = JI^n T = I^{n+1}T = a^{n+1}T.$$

We obtain

$$JT = aT = IT.$$

Next, let

$$H := \varphi\left(\varphi^{-1}(W) \cap a^{-1}J\right) \subseteq W.$$

Clearly, H is an additive subgroup of W and, since φ is a surjective ring homomorphism, H is stable under scalar multiplication by D. That is, H is a D-submodule of W. Further, we claim that

$$J = a\varphi^{-1}(H).$$

Indeed, let $x \in J \subseteq I$ and let $y \in \varphi^{-1}(W)$ such that $x = ay$. Then, $\varphi(xa^{-1}) = \varphi(y) \in W$ and, as $axa^{-1} = x \in J$, $\varphi(xa^{-1}) \in H$. Hence $x \in a\varphi^{-1}(H)$. Therefore, $J \subseteq a\varphi^{-1}(H)$. The reverse inclusion is straight, proving the claim. Finally, it is easy to check that $JI^n = I^{n+1}$ if and only if $HW^n = W^{n+1}$, for any integer $n \geq 0$, leading to the conclusion and completing the proof of the theorem. $\square$

Recall that a ring R is called a pseudo-valuation domain [10] if it is local and shares its maximal ideal with a valuation overring V or, equivalently, if R is a classical pullback issued from the following diagram

$$\begin{array}{ccc} R = \varphi^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & K := V/M \end{array}$$

We say that R is a pseudo-valuation domain issued from (V,M,k).

The next result investigates the setting of classical pullbacks with the property that "*reductions of an ideal I (in R) extend to reductions of IV (in V), for all ideals $I \subseteq M$.*"

It turns out that this property characterizes a special class of pseudo-valuation domains.

Theorem 2.3. *Let R be a classical pullback issued from (V,M,D). Then, the following assertions are equivalent:*

(1) *For every ideal $I \subseteq M$ of R, $\mathrm{Red}_R(I) = \Big\{J \subseteq I \mid JV \in \mathrm{Red}_V(IV)\Big\}$;*
(2) *R is a pseudo-valuation domain issued from (V,M,k) (i.e., $D = k$), where K is an algebraic extension of k and, for every k-vector subspace W of K containing k, W^n is a field for some positive integer n.*

The following lemma proves the implication (1) $\Rightarrow$ (2) for the general setting of generic pullbacks.

Lemma 2.4. *Let R be a generic pullback issued from (T,M,D) such that, for every ideal $I \subseteq M$, $\mathrm{Red}_R(I) = \Big\{J \subseteq I \mid JT \in \mathrm{Red}_T(IT)\Big\}$. Then, $D = k$, K is algebraic over k and, for every k-vector subspace W of K containing k, W^n is a field for some integer $n \geq 0$.*

Proof. Notice that if J is a reduction of I in R, then obviously JT is a reduction of IT in T. Next, let $\lambda \in K \setminus D$ and let

$$W_o := D + \lambda D.$$

Let I be the ideal of R given by $I = a\varphi^{-1}(W_o)$, for some $0 \neq a \in M$, and let $J := aR = a\varphi^{-1}(D)$. Clearly,

$$JT = aT = IT.$$

By (1), J is a reduction of I in R; that is, $JI^n = I^{n+1}$ for some integer $n \geq 0$. It follows that

$$DW_o^n = W_o^n = W_o^{n+1}$$

and so there is a monic polynomial $f \in D[X]$ of degree $n+1$ such that $f(\lambda) = 0$. Thus, $\lambda \in \overline{D}$, where $\overline{D}$ denotes the integral closure of D in K. Consequently, $K = \overline{D}$ and therefore $D = k$ is a field. Next, let W be a k-subspace of K and consider the ideal of R given by $I := a\varphi^{-1}(W)$, for some $0 \neq a \in M$, and let $J := aR$. Same arguments as above lead to

$$kW^n = W^n = W^{n+1}$$

for some positive integer n. So, we get

$$W^n = W^{2n} = (W^n)^2$$

proving that W^n is a ring. But, we have

$$k \subseteq W \subseteq W^n \subseteq K = \overline{k}$$

which forces W^n to be a field, completing the proof of the lemma. $\square$

Proof of Theorem 2.3. (1) $\Longrightarrow$ (2) is handled by Lemma 2.4.

(2) $\Longrightarrow$ (1) Let $I \subseteq M$ be an ideal of R and let $J \subseteq I$ be an ideal of R such that JV is a reduction of IV in V; i.e., $JIV = I^2V$. Therefore, we have

$$JI \subseteq I^2 \subseteq I^2V = JIV.$$

Hence, if either of I or J is an ideal of V, then $JI = I^2$ and we are done. Next, assume that neither of I and J is an ideal of V. Then, both JV and IV are principal in V and hence the equality $JIV = I^2V$ yields

$$JV = IV = aV$$

for some $0 \neq a \in M$. It follows that

$$I = a\varphi^{-1}(W) \text{ and } J = a\varphi^{-1}(H)$$

for some k-vector subspaces H, W of K containing k; and a fortiori $H \subseteq W$, since φ is surjective. By (2), there is a positive integer n such that W^n is a field. Then, we get

$$HW^n = W^n = W^{n+1}$$

and so $JI^n = I^{n+1}$. Thus, J is a reduction of I, completing the proof of the theorem. □

3. Examples

In this section, the theorems obtained in the previous section are backed with original examples, where we explicitly compute the reductions of ideals. All our examples arise as special pullbacks of the form $R = D + M$ issued from $T = K + M$.

Follow an illustrative example for Theorem 2.1.

Example 3.1. Let $\mathbb{Q}$ and $\mathbb{R}$ be the fields of rational and real numbers, respectively, and X, Y two indeterminates over $\mathbb{R}$. Let

$$\begin{array}{rcl} T & := & \mathbb{R}[X,Y] = \mathbb{R} + M, \text{ where } M := (X,Y)T \\ R & := & \mathbb{Q} + M. \end{array}$$

Notice that T is a two-dimensional Noetherian domain and R is a two-dimensional non-Noetherian domain by [4, Theorem 4]. Consider the ideal of R given by

$$I := (X - 1, Y)R.$$

Clearly, M is a maximal ideal of T with

$$I + M = R.$$

Further, $IT = (X - 1, Y)T$ is a two-generated height-two ideal in the Noetherian domain T. Therefore, IT is of the principal class and hence a basic ideal of T by [8, Theorem 2.3]. This argument cannot apply to I since R is not Noetherian. However, by Theorem 2.1(3), we have

$$\begin{array}{rcl} \mathrm{Red}_R(I) & = & \left\{H\cap R \mid H \in \mathrm{Red}_T(IT)\right\} \\ & = & \left\{IT\cap R\right\} \\ & = & \left\{I\right\}. \end{array}$$

That is, I is a basic ideal of R. □

The following example shows that Theorem 2.2(2) does not carry up, in general, to generic pullbacks.

Example 3.2. Let X, Y be two indeterminates over $\mathbb{Q}$ and let

$$\begin{array}{rcl} T_1 & := & \mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)[[X,Y]] = \mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right) + M, \text{ where } M := (X,Y)T_1 \\ T & := & \mathbb{Q}\left(\sqrt{2}\right) + M \\ R & := & \mathbb{Q} + M. \end{array}$$

Notice that T_1, T, and R are two-dimensional local Noetherian domains sharing the same maximal ideal M. Let $0 \neq a \in M$ and consider the ideal of R given by

$$I := aM.$$

Clearly, M is not basic in T since $J = (X,Y)T$ is a proper reduction of M in T; and however M is basic in T_1 by the principal class property [8, Theorem 2.3]. Further, $a^{-1} \notin T_1 = (M : M)$ and so

$$I \subsetneqq M.$$

We claim that $\mathrm{PRed}_R(I) = \emptyset$. Suppose for contradiction that I has a principal reduction in R, say bR, for some $0 \neq b \in R$. Then, $a^{-1}bT_1 \subsetneqq M$ and $bI^n = I^{n+1}$ for some integer $n \geq 1$. It follows that $a^{-1}bM^n = M^{n+1}$ and hence $a^{-1}bT_1$ is a proper reduction of M in T_1, the desired contradiction.

On the other hand, let $J := (aX, aY)R$. Since $JT_1 = aM = I$ and I is an ideal of T_1, $JI = I^2$ and so J is a reduction of I in R. But one can easily check that J is not an ideal of T and hence $J \notin \mathrm{Red}_T(I)$. Thus, $\mathrm{Red}_T(I) \subsetneqq \mathrm{Red}_R(I)$, as desired. □

Follow an illustrative example for Theorem 2.2.

Example 3.3. Let $\mathbb{Z}$ denote the ring of integers and let X be an indeterminate over $\mathbb{Q}$. Let

$$\begin{array}{rcl} V & := & \mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)[[X]] = \mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right) + M, \text{ where } M := XV \\ R & := & \mathbb{Z} + M. \end{array}$$

Consider the ideal of R given by

$$I := X(W + M), \text{ where } W := \mathbb{Q}\left(\sqrt{2}\right).$$

Since W is a field, $W^n = W$ for every positive integer n. So, by Theorem 2.2(3), we obtain

$$\mathrm{Red}_R(I) = \Big\{X(H+M) \mid H \ \mathbb{Z}\text{-submodule of } W \text{ with } HW = W\Big\}.$$

For instance, $X(n\mathbb{Z}+M)$, for any integer $n \geq 1$, and $X(\mathbb{Q}+M)$ are proper reductions of I in R. $\square$

Follow an illustrative example for both Theorem 2.2 and Theorem 2.3.

Example 3.4. Let X be an indeterminate over $\mathbb{Q}$ and let

$$\begin{array}{lcl} V & := & \mathbb{Q}\big(\sqrt{2}, \sqrt{3}\big)[[X]] = \mathbb{Q}\big(\sqrt{2}, \sqrt{3}\big) + M, \text{ where } M := XV \\ R & := & \mathbb{Q} + M. \end{array}$$

Consider the ideal of R given by

$$I := X\Big(\mathbb{Q}\big(\sqrt{2}\big) + M\Big).$$

Clearly, $I \subsetneq M$ and I is not an ideal of V. Moreover, since $W := \mathbb{Q}\big(\sqrt{2}\big)$ is a field, $W^n = W$ for every positive integer n. So, by Theorem 2.2(3), we obtain

$$\mathrm{Red}_R(I) = \Big\{X(H+M) \mid H \ \mathbb{Q}\text{-subspace of } W \text{ with } HW = W\Big\}.$$

Let $J := X(H+M) \in \mathrm{Red}_R(I)$. If $\dim_{\mathbb{Q}} H = 1$, then $H = a\mathbb{Q}$, for some $0 \neq a \in W$ and so $J = aXR$. If $\dim_{\mathbb{Q}} H = 2$, then $H = W$ and so $J = I$. On the other hand, by Theorem 2.2(3), we get

$$\mathrm{Red}_R(I) = \Big\{J \subseteq I \mid JV \in \mathrm{Red}_V(IV)\Big\}.$$

Now, $IV = M$ is basic in V and, consequently, we have

$$\begin{array}{lcl} \mathrm{Red}_R(I) & = & \Big\{J \subseteq I \mid JV = M\Big\} \\ & = & \Big\{I\Big\} \cup \mathrm{PRed}_R(I) \\ & = & \Big\{I\Big\} \cup \Big\{aXR \mid 0 \neq a \in \mathbb{Q}\big(\sqrt{2}\big)\Big\}. \end{array}$$

Thus, a combination of both theorems yields the non-trivial fact that the proper subideals of I which extend to M in V are exactly the principal reductions of I; i.e., they have the form aXR, where $0 \neq a$ ranges over $\mathbb{Q}\big(\sqrt{2}\big)$. $\square$

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