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Trace Properties and Integral Domains

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INTRODUCTION

Throughout this paper, R will denote an integral domain with quotient field K. For a pair of fractional ideals I and J of a domain R we let $(J : I)$ denote the set $\{t \in K \mid tI \subseteq J\}$. Often, we shall use I^{-1} in place of $(R : I)$. Recall that the "v" of a fractional ideal I is the set $I_v = (R : (R : I))$ and the "t" of I is the set $I_t = \bigcup J_v$ with the union taken over all finitely generated fractional ideals contained in I. An ideal I is divisorial if $I = I_v$, and I is a t-ideal if $I = I_t$.

Let R be an integral domain and let M be an R-module. Then the trace of M is the ideal generated by the set $\{fm \mid f \in Hom(M, R) \text{ and } m \in M\}$. For a fractional ideal I of R, the trace is simply the product of I and I^{-1}. We call an ideal of R a *trace ideal* of R if it is the trace of some R-module. An elementary result due to H. Bass is that if J is a trace ideal of R, then $JJ^{-1} = J$; i.e., $J^{-1} = (J : J)$ [5, Proposition 7.2]. It follows that J is a trace ideal if and only if $J^{-1} = (J : J)$. (Such ideals are also referred to as being "strong"; see, for example, [3].) In 1987, D.D. Anderson, J. Huckaba and I. Papick proved that if I is a noninvertible ideal of a valuation domain V, then $I(V : I)$ is prime [1, Theorem 2.8]. Later in the same year, M. Fontana, Huckaba and Papick began the study of the "trace property" and "TP domains". A domain R is said to satisfy the *trace property* (or to be a *TP domain*) if for each R-module M, the trace of M is equal to either R or a prime ideal of R [8, page 169]. Among other things, they showed that each valuation domain satisfies the trace property [8, Proposition 2.1], and that if R satisfies the trace property, then it has at most one noninvertible maximal ideal [8, Corollary 2.11]. For Noetherian domains they proved that if R is a Noetherian domain, then it is a TP domain if and only if it is one-dimensional, has at most one noninvertible maximal ideal M, and if such a maximal ideal exists, then M^{-1} equals the integral closure of R (or, equivalently, $M^{-1} = (M : M)$ is a Dedekind domain) [8, Theorem 3.5]. In Section 2 of [10], S. Gabelli showed that by replacing "integral closure" with "complete integral closure", the same list of conditions characterizes the class of Mori domains which satisfy the trace property. Recall that a Mori domain is an integral domain which satisfies the ascending chain condition on divisorial ideals.

In 1988, W. Heinzer and Papick introduced the "radical trace property" declaring that an integral domain R satisfies the *radical trace property* (or is an *RTP domain*) if for each noninvertible ideal I, II^{-1} is a radical ideal. For Noetherian domains, they proved that if R is a Noetherian domain, then it satisfies the radical trace property if and only if R_P is a TP domain for each prime ideal P [12, Proposition 2.1]. Gabelli extended this result to Mori domains [10, Theorem 2.14].

For Prüfer domains there are results concerning the trace property in [6], [8] and [16] and the radical trace property in [12] and [16]. For a Prüfer domain R, Theorem 23 of [16] gives the following equivalent conditions:

(1) R satisfies the radical trace property.

(2) For each primary ideal Q, either Q is invertible or QQ^{-1} is prime.

(3) For each primary ideal Q, if Q^{-1} is a ring, then Q is prime.

(4) Each branched prime is the radical of a finitely generated ideal.

(A prime ideal P is said to be branched if there is a P-primary ideal Q such that $Q \neq P$ [11, page 189].)

In Theorem 10, we will show that the following statement can be added to this list:

(5) For each trace ideal I, $IR_P = PR_P$ for each prime P minimal over I.

Moreover, we will give a new proof for the equivalence of (1)–(3).

According to [16], a domain R is said to satisfy the *trace property for primary ideals* (or to be a *TPP domain*), if for each primary ideal Q, either Q is invertible or QQ^{-1} is prime. By Corollary 8 of [16], R is a TPP domain if and only if for each primary ideal Q, either $QQ^{-1} = \sqrt{Q}$, or Q is invertible and $\sqrt{Q}$ is maximal. Also from [16], R is a *PRIP domain* if for each primary ideal Q, Q^{-1} a ring implies Q is prime. We say that a domain is an *LTP domain* if for each trace ideal I, $IR_P = PR_P$ for each prime P minimal over I. It is known that each RTP domain is a TPP domain [16, Theorem 4] and that there are Noetherian domains which satisfy the radical trace property (and even the trace property) but are not PRIP domains (see, for example, [16, Example 30]). We will show that each TPP domain is an LTP domain and that each PRIP domain is an LTP domain (Corollary 3).

It is easy to see that for one-dimensional domains, each LTP domain is also an RTP domain. Also, it is known that for Mori domains, the radical trace property and the trace property for primary ideals are equivalent. In Theorem 18, we show that if R is a Mori domain, then it is an LTP domain if and only if it is an RTP domain. However, in general, we have been unable to determine whether each TPP domain is an RTP domain, or whether each LTP domain is an TPP domain (or RTP domain).

A field is trivially an RTP domain. While most of the results in this paper are true for fields, the emphasis is on integral domains that are not fields. To avoid having to add the phrase "but not a field" when it would be required, we will simply assume that R is an integral domain which is not a field. We shall also assume that all of the ideals are nonzero.

Notation is standard as in [Gilmer]. In particular, "$\subseteq$" denotes containment and "$\subset$" denotes proper containment.

We shall make use of a number of results concerning consequences of I^{-1} being a ring. We close the Introduction with a theorem where we list several of these results.

THEOREM 0 Let R be an integral domain and let I be an ideal of R such that I^{-1} is a ring. Then

(a) $I^{-1} = I_v^{-1} = (I_v : I_v) = (II^{-1} : II^{-1}) = (II^{-1})^{-1}$ ([14, Proposition 2.2]).

(b) $\sqrt{I}^{-1}$ is a ring ([13, Proposition 2.1]). Moreover, $\sqrt{I}^{-1} = (\sqrt{I} : \sqrt{I})$ ([2, Proposition 3.3]).

(c) P^{-1} is a ring for each prime P minimal over I ([13, Proposition 2.1] and [16, Lemma 13]). Moreover, $P^{-1} = (P : P)$ ([14, Proposition 2.3]).

1 LTP DOMAINS

The first lemma we present is a variation on a result which appears in Fossum's book [9, Lemma 3.7]. (See also, Lemmas 0 and 1 of [16].)

LEMMA 1 Let R be an integral domain and let Q be a primary ideal of R with radical P. If P does not contain QQ^{-1}, then $(R : QQ^{-1}) = (QQ^{-1} : QQ^{-1}) = (Q : Q)$ and so $(R : I) = (Q : Q)$ for each ideal I such that $Q \subset I \subseteq QQ^{-1}$ and $I \not\subseteq P$.

Proof. It is always the case that $(Q : Q) \subseteq (QQ^{-1} : QQ^{-1}) = (R : QQ^{-1})$. Assume P does not contain QQ^{-1} and let I be an ideal such that $Q \subset I \subseteq QQ^{-1}$ and $I \not\subseteq P$. Since I contains Q and is contained in QQ^{-1}, $(QQ^{-1} : QQ^{-1}) = (R : QQ^{-1}) \subseteq (R : I) \subseteq (R : Q)$. Obviously, $QI(R : I) \subseteq Q$. Since Q is P-primary and I is not contained in P, $Q(R : I) \subseteq Q$. Hence $(R : I) \subseteq (Q : Q)$ and it follows that $(R : I) = (Q : Q) = (QQ^{-1} : QQ^{-1}) = (R : QQ^{-1})$. ♦

Our first use of Lemma 1 is to establish a characterization of LTP domains in terms of primary ideals.

THEOREM 2 The following are equivalent for a domain R.

(1) R is an LTP domain.
(2) For each noninvertible primary ideal Q, $Q(R : Q)R_P = PR_P$ where $P = \sqrt{Q}$.
(3) If a primary ideal is also a trace ideal, then it is prime.

Proof. ((1) ⇒ (2)) Assume R is an LTP domain and let Q be a noninvertible P-primary ideal of R. Since R is an LTP domain and QQ^{-1} is a trace ideal, it suffices to show that P contains QQ^{-1}. By way of contradiction assume there is an element $t \in QQ^{-1} \backslash P$ and set $I = t^2R + Q$. Then from Lemma 1, we have $(R : I) = (Q : Q)$. Let $J = I(R : I)$. Then J is also contained in QQ^{-1}. Hence we have $(J : J) = (R : J) = (R : I) = (Q : Q)$.

Let N be a prime minimal over J. Then $JR_N = NR_N$ since R is an LTP domain. In particular, $t \in JR_N$. It follows that there are elements $a \in (R : I) = (J : J)$, $q \in Q$ and $s \in R \backslash N$ such that $st = at^2 + q$. Hence, $q = t(s - at)$. But a^2t^2 is in J since $(R : I) = (J : J)$. Thus $at \in N$ and so $s - at$ is not in P. As neither t nor $s - at$ is in P, we have a contradiction. Hence we must have $QQ^{-1} \subseteq P$.

((2) ⇒ (3)) Obvious.

((3) ⇒ (1)) Assume that if an ideal is both a primary ideal and a trace ideal, then it is prime. Let I be a trace ideal of R and let P be a minimal

prime of I. Then $Q = IR_P \cap R$ is a P-primary ideal which is also a trace ideal. It follows that $Q = P$ and $IR_P = PR_P$. ♦

COROLLARY 3 Let R be an integral domain. If R is an RTP domain, a TPP domain or a PRIP domain, then R is an LTP domain.

Proof. By Theorem 2 it suffices to show that for RTP domains, TPP domains and PRIP domains, if an ideal is both primary and a trace ideal, then it is prime. Let Q be a trace ideal which is also a primary ideal of R. Then obviously Q^{-1} is a ring. Hence if R is a PRIP domain, then Q is prime. Also, if R is either an RTP domain or a TPP domain, then we have $QQ^{-1} = Q$ is prime.♦

Statement (3) in Theorem 2 is very close to the definition of a PRIP domain. To see that the two are not equivalent consider the ring $R = F[[X^3, X^4, X^5]]$ where F is a field. The ideal $Q = (X^3, X^4)$ is primary but not prime and $Q^{-1} = F[[X]]$ is a ring. Thus R is not a PRIP domain. However, note that $QQ^{-1} = (X^3, X^4, X^5)$ is the maximal ideal of R and $(QQ^{-1})^{-1} = F[[X]]$. That R is an LTP domain now follows from [8, Theorem 3.5] and Corollary 3. At this time we do not know whether each LTP domain is a TPP domain and/or whether each TPP domain is an RTP domain. However in Theorem 10, we prove that if R is a Prüfer domain, then each LTP domain is also a PRIP domain, a TPP domain and an RTP domain.

If R is an RTP domain (a TPP domain), then for each prime ideal P, both R_P and R/P are RTP domains (TPP domains) [16, Theorems 3 and 9]. Next, we establish an analogous result for LTP domains.

THEOREM 4 Let P be a prime ideal of a domain R and let $D = R/P$. If R is an LTP domain, then R_P and D are LTP domains.

Proof. Assume R is an LTP domain.

We first show that D is an LTP domain. Let $\overline{I}$ be a trace ideal of D. Since $\overline{(R : I)} \subseteq (D : \overline{I})$ and $\overline{(I : I)} \subseteq (\overline{I} : \overline{I})$, $(I : I) = (R : I)$. Thus for each prime N minimal over I, $IR_N = NR_N$. It follows that $\overline{I}D_N = \overline{N}D_N$. Hence, D is an LTP domain.

To show R_P is an LTP domain, let IR_P be a trace ideal of R_P. Then $B = IR_P \cap R$ is a trace ideal of R. Hence for each prime N minimal over B, $BR_N = NR_N$. The result follows from the fact that $IR_N = BR_N$ for each $N \subseteq P$. ♦

Our next result collects other useful information concerning the prime ideals of an LTP domain.

THEOREM 5 Let R be an LTP domain. Then

(a) Each maximal ideal is a t-ideal.
(b) Each nonmaximal prime ideal is a divisorial trace ideal.
(c) Each maximal ideal is either idempotent or divisorial.

Proof. To prove (a), it suffices to show that if R is an LTP domain, then for each finitely generated ideal I, $(R : I) \neq R$. By way of contradiction, let I be a finitely generated ideal of R for which $I^{-1} = R$. Then we also have $(I^2)^{-1} = R$. Obviously, both I and I^2 are trace ideals of R. While it may be that $IR_P = PR_P$ for some prime P, the same cannot be true for I^2. Hence if R is an LTP domain, $I^{-1} \neq R$ for each finitely generated ideal I.

For the proof of (b), first note that by statement (2) of Theorem 2, $PP^{-1}R_P = PR_P$. Hence we must have $PP^{-1} = P$.

Since $PP^{-1} = P$, we also have $P_v^{-1} = P^{-1} = (P : P) = (P_v : P_v)$ [14, Proposition 2.2]. Lemma 1 no longer applies, but in its place we simply note that each ideal between P_v and P has inverse equal to $(R : P)$. Starting with an ideal $I = r^2 + P$ for some $r \in P_v\backslash P$, we can repeat the proof given for (1)$\Rightarrow$(2) in Theorem 2 to show that we must have $P = P_v$.

For (c), let M be a maximal ideal which is not idempotent. Since R is an LTP domain, $M \subseteq M^2(R : M^2)$. But $(R : M^2) = ((R : M) : M)$. As M is not idempotent, we cannot have $(R : M) = R$. Hence M is divisorial. ♦

For a TPP domain R, it is known that if R and $(I : I)$ satisfy INC for each trace ideal I, then R is an RTP domain [16, Lemma 33]. We wish to show that the same occurs for LTP domains. Before proving this result, we present a pair of useful lemmas and then prove that if I is a trace ideal of an LTP domain R such that R and $(I : I)$ satisfy INC, then I is a radical ideal of R.

LEMMA 6 Let I be a trace ideal of an integral domain R and let J' be an ideal of $(I : I)$.

(a) If J' contains I, then $J' \bigcap R$ is a trace ideal of R.
(b) If J' is a trace ideal of $(I : I)$, then IJ' is a trace ideal of R.

Proof. For (a), assume J' contains I and set $J = J' \bigcap R$. Since $I \subseteq J'$, $J^{-1} \subseteq I^{-1} = (I : I)$. Hence $JJ^{-1} \subseteq J(I : I) \bigcap R \subseteq J' \bigcap R = J$.

To prove (b), assume J' is a trace ideal of $(I : I) = (R : I)$. Then $IJ'(R : IJ') = IJ'((R : I) : J') = IJ'$; i.e., IJ' is a trace ideal of R. ♦

LEMMA 7 Let I be a trace ideal of an LTP domain R and let $P' \subseteq N'$ be a pair of prime ideals of $(I : I)$ which contain I. Then $P' \bigcap R = N' \bigcap R$.

Proof. Set $T = (I : I)$ and let Q' be a primary ideal of T that contains I. Then $Q = Q' \bigcap R$ is a primary ideal of R which is also a trace ideal by Lemma 6. Since R is an LTP domain, Q must be prime. If $P \neq N$, then there is an element $r \in N\backslash P$. Without loss of generality, we may assume N' is minimal over $J' = r^2T + P'$ and that $Q' = J'T_{N'} \bigcap T$. As the corresponding ideal $Q = Q' \bigcap R$ is a prime ideal of R, we must have $Q = N$. But as in the proof of Theorem 2, Q contains r^2 but not r. Hence it must be that $P = N$. ♦

THEOREM 8 Let I be a trace ideal of an LTP domain R. If the pair R and $(I : I)$ satisfy INC, then I is a radical ideal of R.

Proof. Set $T = (I : I)$ and assume R and $(I : I)$ satisfy INC. Let $r \in \sqrt{I}$ and let P' be a prime of T that is minimal over I. By Lemma 7, if N' is a maximal ideal of T that contains P', then $P' \bigcap R = N' \bigcap R$. But since R and T satisfy INC, we must then have that $P' = N'$; i.e., each prime of T that is minimal over I is also a maximal ideal of T. Let $J = \{t \in T \mid tr \in I\}$. Let $Q' = IT_{P'} \bigcap T$ and $Q = Q' \bigcap R$. By Lemma 6, Q is a trace ideal of R. But it is also a primary ideal of R, so Q must be prime. In particular, Q' must contain r. Hence P' cannot contain J. Since J obviously contains I, we must have $J = T$ and, therefore, $I = \sqrt{I}$. ♦

COROLLARY 9 Let R be an LTP domain. If the pair R and $(I : I)$ satisfy INC for each trace ideal I, then R is an RTP domain.

We are now in a position to show that if R is simultaneously a Prüfer domain and an LTP domain, then it is also an RTP domain and a PRIP domain.

THEOREM 10 Let R be a Prüfer domain. Then the following are equivalent

(1) R is an RTP domain.
(2) R is a TPP domain.
(3) R is an LTP domain.
(4) R is a PRIP domain.

Proof. For the equivalence of (1)–(4), Corollary 3 handles the implications of (1)$\Rightarrow$(3), (2)$\Rightarrow$(3) and (4)$\Rightarrow$(3). Furthermore, as each RTP domain is also a TPP domain, all we need prove is that if R is a Prüfer LTP domain, then it is also an RTP domain and a PRIP domain.

Assume R is an LTP domain. Since R is a Prüfer domain, if T is an overring of R, then the primes of T are all extended from primes of R 11, Theorem 26.2].. Hence the pair R and T satisfy INC. That R is an RTP domain now follows from Corollary 9.

Let Q be a primary ideal of R. Since R is Prüfer, if Q^{-1} is a ring, then $Q^{-1} = (Q : Q)$ by Lemma 4.4 of [8]; i.e., Q is a trace ideal. Hence, Q^{-1} a ring implies Q is prime and, therefore, R is a PRIP domain. ♦

In [8], it was noted that if R is an almost Dedekind domain which is not Dedekind, then R is not a TP domain since it contains a maximal ideal M for which $(R : M) = R$. As R_M is a discrete rank one valuation domain, $M^2 \neq M$ yet $(R : M^2) = ((R : M) : M) = (R : M) = R$ so $M^2(R : M^2) = M^2 \neq M$. This same proof shows that R is not an LTP domain. A different way to establish this result is to use Theorem 5 and the fact that the only divisorial maximal ideals of a Prüfer domain are the invertible ones (see, for example, [14, Corollary 3.4]).

COROLLARY 11 Let R be an almost Dedekind domain. Then the following are equivalent

(1) R is a TP domain.
(2) R is an RTP domain.
(3) R is a TPP domain.
(4) R is an LTP domain.
(5) R is Dedekind.

Another corollary to Theorem 5 concerns Prüfer v-multiplication domains. (A domain R is a Prüfer v-multiplication domain (or PVMD for short) if R_P is a valuation domain for each maximal t-ideal P.) For a Prüfer domain, each maximal ideal is also a maximal t-ideal since each finitely generated ideal is invertible. Thus an integral domain is a Prüfer domain if and only if it is a PVMD where each maximal ideal is a maximal t-ideal.

COROLLARY 12 Let R be a PVMD. If R is an LTP domain, then it is a Prüfer domain and also an RTP domain.

Heinzer and Papick proved that the only Krull domains which satisfy the radical trace property are the Dedekind domains [12, page 112]. Since each Krull domain is a PVMD, Corollary 12 gives a different proof of their result.

COROLLARY 13 Let R be an almost Krull domain. Then the following are equivalent

(1) R is a TP domain.
(2) R is an RTP domain.
(3) R is a TPP domain.
(4) R is an LTP domain.
(5) R is a Dedekind domain.
(6) R is a PRIP domain.

Proof. It suffices to show (4) implies (5). Assume R is an LTP domain. Since R is an almost Krull domain, R_P is a Krull domain for each prime ideal P. By Theorem 4, each R_P is also an LTP domain. Hence from Corollary 12, each R_P is a Dedekind domain. It follows that R is an almost Dedekind domain. From Corollary 11, we have that R is Dedekind.♦

In [15], J. Lipman considered ideals of one-dimensional semi-local Macaulay rings. He defined an open ideal I of such a ring to be "stable" if $\bigcup(I^n : I^n) = (I : I)$. Building on Lipman's work, J. Sally and W. Vasconcelos developed a more general notion of stability by declaring an ideal to be stable if it was projective over its endomorphism ring [19, page 323]. For a nonzero ideal I of an integral domain (or just an ideal which contains an element which is not a zero divisor of a ring), their condition is equivalent to saying that I is invertible as an ideal of $(I : I)$ (since for such an ideal, being projective is equivalent to being invertible). In general, an ideal I of a domain R can be such that $\bigcup(I^n : I^n) = (I : I)$ without being stable in the sense of Sally and Vasconcelos. For example, this will be true for an ideal whose inverse is equal to R. But, if an ideal I is stable in the sense of Sally and Vasconcelos, then it will be true that $(I^n : I^n) = (I : I)$ for each positive integer n. Hence, I will be stable in the sense of Lipman. As in [1], we say that an ideal I is *L-stable* (for Lipman-stable) if $\bigcup(I^n : I^n) = (I : I)$ and *SV-stable* (for Sally-Vasconcelos-stable) if I is invertible as an ideal of $(I : I)$.

Heinzer and Papick showed that if R is an RPT domain and I is an integrally closed ideal of R, then I is L-stable [12, Remark 2.13a]. They also observed that in an RTP domain, each ideal J is such that $JJ^{-1} = J^nJ^{-n}$ (where J^{-n} denotes the inverse of J^n) [12, Remark 2.13b]. Our next result considers the radical ideals of an RTP domain. In [13], E. Houston and the three authors of this paper proved that if a radical ideal I can be realized as an intersection of divisorial radical ideals which are also trace ideals, then I is a trace ideal [13, Proposition 3.15]. We shall make use of this result in the proof below.

THEOREM 14 Let R be an RTP domain. Then each radical ideal of R is L-stable.

Proof. Let I be a radical ideal of R. We first consider the two opposite cases of I being invertible and I being a trace ideal. Next we show that IR_M is L-stable for each maximal ideal. This will complete the proof since $\bigcap(BR_M : BR_M) = (B : B)$ for each ideal B of R.

If I is invertible, then so is each power of I. Hence I is L-stable since $(I^n : I^n) = R$ for each positive integer n.

If I is a trace ideal of R, then $I^nI^{-n} = II^{-1} = I$ [12, Remark 2.13b].

Hence $(I^n : I^n) \subseteq (I^n I^{-n} : I^n I^{-n}) = (I : I)$ and it follows that I is L-stable.

Let M be a maximal ideal of R. Then R_M is an RTP domain by Theorem 4. If M does not contain I, then $IR_M = R_M$, so IR_M is trivially L-stable. If M not only contains I, but is also minimal over I, then $MR_M = IR_M$. As MR_M is either invertible or a trace ideal of R_M, IR_M is L-stable. If M contains I but is not minimal over I, then each of the minimal primes of IR_M is a divisorial trace ideal of R_M [Theorem 5]. That IR_M is a trace ideal of R_M now follows from [13, Proposition 3.15]. Hence, we again have that IR_M is L- stable. ♦

Two of the questions raised in [16] concerning RTP domains were whether $(I : I)$ will always be an RTP domain when R is an RTP, and whether INC would always hold between R and $(I : I)$ when R is an RTP domain and I is a trace ideal. Our next example shows that the answer to the first of these questions is NO. Then we prove that the answer to both questions is YES when we restrict to trace ideals which are SV-stable.

EXAMPLE 15 Let V be the power series ring $F(X,Y)[[Z]]$ where F is a field and let $R = F+ZV$. Then V is a valuation domain with maximal ideal $M = ZF(X,Y)[[Z]]$ and R is pseudo-valuation domain. By [16, Theorem 31], R is an RTP domain. Let I be the ideal $Z(F[X,Y] + M)$. Then it is clear that $(I : I) = F[X,Y]+M$. There are a number of ways to verify that $(I : I)$ is not an RTP domain. For example: (a) $(I : I)/M = F[X,Y]$ is a Krull domain which is not an RTP domain since it is not Dedekind (Theorem 4 and [12, page 112]); or (b) the maximal ideal $N = (X,Y)(I : I)$ is neither idempotent nor divisorial (Theorem 5); or (c) the ideal $P = X(I : I)$ is a principal prime ideal which is not maximal (Theorem 5).

The ideal I in the example above is not a trace ideal of R. Thus this example leaves open the possibility that $(I : I)$ may be an RTP domain when I is a trace ideal of R. In our next result we show that if I is SV-stable, then not only will $(I : I)$ be an RTP domain, but the pair R and $(I : I)$ will satisfy INC.

THEOREM 16 Let I be a trace ideal of an RTP domain R. If I is SV-stable, then

(a) $(I : I)$ is an RTP domain.
(b) Each ideal of $(I : I)$ that contains I is invertible as an ideal of $(I : I)$.
(c) Each prime of $(I : I)$ that contains I is minimal over I.
(d) The pair R and $(I : I)$ satisfy INC.

Proof. Assume I is SV-stable. To simplify notation, set $T = (I : I)$. Hence $I(T : I) = T$.

We will first show that T is an RTP domain. To this end let J be a trace ideal of T and let $\sqrt[T]{J}$ denote the radical of J in T. Then $\sqrt[T]{J}$ is a trace ideal of T [13, Proposition 2.1]. By Lemma 6, both IJ and $I\sqrt[T]{J}$ are trace ideals of R. Since R is an RTP domain, both are radical ideals. Hence we must have $IJ = I\sqrt[T]{J}$. That $J = \sqrt[T]{J}$ now follows from the assumption that I is an invertible ideal of $T = (I : I)$. Thus $(I : I)$ is an RTP domain.

For part (b), let B be an ideal of $(I : I)$ which contains I. Then $J = B(T : B)$ will be a trace ideal of T that contains I. By part (a), IJ is then a radical ideal of R. It follows that $IJ = I \bigcap J = I = IT$ since $I \subset J$. As I is an invertible ideal of T, $J = T$; i.e., B is an invertible ideal of T.

By (b), each prime of T that contains I is invertible. That each of these primes must then be maximal ideals of T follows from Lemma 1 (see also 11, Theorem 7.6]). Therefore each such prime is also minimal over I. For a pair of distinct primes $P' \subset N'$ of T where P' does not contain I, then $P = P' \bigcap R$ and $N = N' \bigcap R$ will be distinct primes of R (no matter whether I is invertible or not) [7, Theorem 1.4]. It follows that the pair R and $(I : I)$ satisfy INC. ♦

We have not been able to prove an analogous result for either TPP domains or LTP domains. The best we have been able to do is prove that statements (b) and (c) will hold for a prime P' if $P' \bigcap R$ is minimal over I.

THEOREM 17 Let R be an LTP domain and let I be a trace ideal R which is invertible as an ideal of $(I : I)$. Let P' be a prime of $(I : I)$ which contains I and let $P = P' \bigcap R$. Then

(a) P' survives in $(IR_P : IR_P)$.
(b) If P is minimal over I, then P' is both maximal and invertible as an ideal of $(I : I)$.

Proof. Set $T = (I : I)$.

Since $P = P' \bigcap R$, $R_P \subseteq T_{P'}$. Since I is invertible as an ideal of T, $T_{P'} = (IT_{P'} : IT_{P'}) = (IT_{P'} : I) \supseteq (IR_P : I) = (IR_P : IR_P)$. Hence P' survives in $(IR_P : IR_P)$.

Assume P is minimal over I. If P' is not invertible as an ideal of T, then there will be a prime N' which contains P' and is a trace ideal of T. Also $N' \bigcap R = P$ by Lemma 7. So, without loss of generality, we may assume that P' is a trace ideal of T. Hence IP' is a trace ideal of R and P is minimal over IP'. As R is an LTP domain, we have $IP'R_P = PR_P = IR_P$.

Set $T_{(P)} = (IR_P : IR_P)$. Obviously, IR_P is an invertible ideal of $T_{(P)}$. Hence $P'T_{(P)} = P'[IR_P(T_{(P)} : IR_P)] = (P'IR_P)(T_{(P)} : IR_P) = IR_P(T_{(P)} : IR_P) = T_{(P)}$. This contradicts the fact that P' survives in $(IR_P : IR_P)$. Thus it must be that P' is invertible. ♦

2 MORI DOMAINS

If I is an ideal of a Mori domain R, then $I_t = I_v = A_v$ for some finitely generated ideal A contained in I [17, Théorème 1]. This property of a Mori domain makes dealing with the various trace properties much easier. For one thing it guarantees that if R is an LTP domain, then not only is each maximal ideal divisorial, but also that each is the v of a finitely generated ideal. We begin this section by showing that each Mori LTP domain is also a Mori RTP domain. We also give a characterization of Mori LTP domains in terms of SV-stability.

THEOREM 18 Let R be a Mori domain which is not a field. Then the following are equivalent

(1) R is an RTP domain.
(2) R is a TPP domain.
(3) R is an LTP domain.
(4) For each maximal ideal M and each M-primary ideal Q, M is SV-stable and QQ^{-1} contains M.
(5) For each maximal ideal M, M is SV-stable and each maximal ideal of $(M : M)$ that contains M is invertible as an ideal of $(M : M)$.
(6) For each nonzero radical ideal I, I is SV-stable and each maximal ideal of $(I : I)$ that contains I is invertible as an ideal of $(I : I)$.

Proof. Obviously, (6) implies (5). By [16, Theorem 4] and Corollary 3, it suffices to show that (3) implies (4), (4) implies (5), (5) implies (1), and (1) implies (6).

[(3)⇒(4)] Assume R is an LTP domain and let M be a maximal ideal of R. For each M-primary ideal Q, either Q is invertible or QQ^{-1} is M-primary. Thus since R is an LTP domain, QQ^{-1} must contain M. Also from our assumption that R is an LTP domain, each nonmaximal prime ideal is a divisorial trace ideal [Theorem 5]. Obviously, every invertible ideal is SV-stable, so we need only consider the case where M is not invertible as an ideal of R. From (the proof of) Theorem 5, if A is a finitely generated ideal of R, then $A^{-1} = R$ only if $A = R$. But as R is also a Mori domain, $M^{-1} = A^{-1}$ for some finitely generated ideal $A \subseteq M$ [17, Théorème 1]. It follows that $M = M_v = A_v$. Set $T = (R : M)$. As M is not invertible, T is a ring equal to $(M : M)$. Since R is an LTP domain, each nonmaximal prime of R is divisorial. Thus no such prime can contain A. Also, no other maximal ideal can contain A, since each maximal ideal is divisorial. Hence M is minimal over A, and, therefore A is M-primary. By Theorem 2 we have $M \subseteq A(R : A) = A(R : M) = AT \subseteq MT = M$. So M is a finitely generated ideal of T. As M is not invertible as an ideal of R, neither is M^2. Thus, again by Theorem 2, $M \subseteq M^2(R : M^2) = M[M((R : M) : M)] = M[M(T : M)] \subseteq MT = M$. In particular, we have $M[M(T : M)] = M$. As M is finitely generated as an ideal of T, Nakayama's Lemma implies that $M(T : M) = T$; i.e., M is SV-stable.

[(4)⇒(5)] Assume that for each maximal ideal M and each M-primary ideal Q, M is SV-stable and QQ^{-1} contains M. Let M be a fixed maximal ideal of R. As in the proof of (3) implies (4), there is nothing to prove if M is invertible as an ideal of R. Hence we assume that M is not invertible as an ideal of R. Set $T = (M : M)$ and let N be a maximal ideal of T that contains M. Set $B = N(T : N)$. As B is a trace ideal of T, BM is a trace ideal of R by Lemma 6. Since N contains M, BM is M-primary. Hence, by Theorem 2, $BM = M = MT$. As M is invertible as an ideal of T, we have $B = T$.

[(5)⇒(1)] Assume that for each maximal ideal M, M is SV-stable and each maximal ideal of $(M : M)$ that contains M is invertible as an ideal of $(M : M)$. Let M be a maximal ideal of R and let N be a maximal ideal of $(M : M)$ that contains M. Since R is a Mori domain, so is $(M : M)$ [18, page 11], [3, Corollary 11]. It follows that N has height one [4, Theorem 2.5], and, therefore, M has height one. Hence R must be one-dimensional.

Let Q be an M-primary ideal of R. Since the radical trace property and the trace property for primary ideals are known to be equivalent for one-dimensional domains, we need only show that $Q(R : Q)$ contains M. In $(M : M)$ each maximal ideal that contains Q also contains M and, therefore, each such ideal is invertible and minimal over Q. It follows that no maximal ideal of T can contain $Q(T : Q)$ [13, Proposition 2.1]. Hence $Q(T : Q) = T$. Hence, $M = MT = QM(T : Q) \subseteq Q(R : Q)$, and, therefore, R is an RTP domain.

[(1)⇒(6)] Assume R is an RTP domain and let I be a nonzero radical ideal of R. By Proposition 2.1 of [13], there is nothing to prove if I is invertible. Hence we assume that I is not invertible. From the argument above, R is one-dimensional and each maximal ideal is divisorial (or see [10, Section 2]). It follows that I is divisorial. Since R is a Mori domain, only finitely divisorial prime ideals can contain I. In this case that means that I is a finite intersection of maximal ideals. Let $\{M_1, M_2, \ldots, M_n\}$ denote the set of invertible maximal ideals that contain I and let $\{N_1, N_2, \ldots, N_m\}$ denote the set of noninvertible maximal ideals that contain I. Set $A = \bigcap M_k$ and $B = \bigcap N_k$. Then A and B are comaximal with A an invertible ideal of R [13, Proposition 2.1] and B a trace ideal of R [13, Proposition 3.15]. Hence, $I = A \bigcap B = AB$. Since A is invertible, we have $(I : I) = (AB : AB) = (B : B)$. Thus to show that I is SV-stable, it suffices to show that B is SV-stable. Set $T = (I : I)$. As B is a trace ideal of R, we also have $T = (R : B)$. Since B is divisorial, $B_v = C_v$ for some finitely generated ideal $C \subseteq B$. Moreover $\sqrt{C} = B$. As in the proof of (3) implies (4), $B = BT = CT$. Let J be a trace ideal of T that contains B. Then by Lemma 6, JB is a trace ideal of R. It follows that $JB = J \bigcap B = B$ since R is an RTP domain. But since B is finitely generated as an ideal of T, $JB = BT$ is possible only if $J = T$. Hence each ideal of T that contains B is invertible as an ideal of T. In particular, B is invertible as an ideal of T, as is each maximal ideal of T that contains B.

If M' is a maximal ideal of T that contains I but not B, then $M' \bigcap R = M_k$ for some k [7, Theorem 1.4]. Moreover M' is invertible as an ideal of T since each of the M_js are invertible ideals of R [7, Theorem 1.4]. ♦

By combining Theorems 16 and 18, we have the following.

COROLLARY 19 Let R be a Mori RTP domain. Then for each nonzero radical ideal I, $(I : I)$ is an RTP domain and the pair R and $(I : I)$ satisfy INC.

One of the classic examples of a Mori domain which is not Noetherian is the ring $R = F+XF[X,Y]$ where F is a field. If this ring is localized at its maximal ideal $M = XF[X,Y]$, the resulting ring is a two-dimensional quasi-local Mori domain whose maximal ideal is SV-stable. As R_M is two-dimensional, it cannot be an RTP domain. The corresponding power series ring $F+XF[[X,Y]]$ is also a two-dimensional Mori domain. Unlike R, $F+XF[[X,Y]]$ is quasi-local, but like R, the maximal ideal $XF[[X,Y]]$ is SV-stable. We will do more with this ring in Example 21, but first we give an example of a local one-dimensional Noetherian domain where the maximal ideal is SV-stable yet the ring is not an RTP domain.

EXAMPLE 20 Let $R = F[[X^3,X^5,X^7]]$ where F is a field. Then

(a) R is a local one-dimensional Noetherian domain with maximal ideal $M = (X^3,X^5,X^7)R$.
(b) $(R : M) = (M : M) = F[[X^2,X^3]]$.
(c) $M(M : M) = X^3F[[X^2,X^3]]$ is invertible as an ideal of $(M : M)$.
(d) R is not an RTP domain. For example, $Q = (X^5,X^6,X^7)R$ is an M-primary ideal for which $(Q : Q) = (R : Q) = F[[X]]$.

For the ring $R = F+XF[[X,Y]]$, the ideal $P = YXF[[X,Y]]$ is a height one prime ideal. Since Y^{-1} is in P^{-1} and X^{-1} is not, $PP^{-1} = XF[[X,Y]]$, the maximal ideal of R. In our next example we show that even though R is not an RTP domain, the maximal ideal $XF[[X,Y]]$ is the trace of each primary ideal whose radical has height one.

EXAMPLE 21 Let $R = F+XF[[X,Y]]$ and let $M = XF[[X,Y]]$. Then

(a) M is SV-stable and the ideal $M(X,Y)F[[X,Y]]$ is an M-primary trace ideal.
(b) Each height one prime of R has the form fM for some irreducible $f \in (X,Y)F[[X,Y]] \backslash XF[[X,Y]]$.
(c) If Q is a primary ideal of R whose radical has height one, then $Q = f^nM$ for some irreducible $f \in (X,Y)F[[X,Y]] \backslash XF[[X,Y]]$.
(d) If Q is a primary ideal of R whose radical has height one, then $QQ^{-1} = M$.

Proof. Since M is not invertible as an ideal of R, $(R : M) = (M; M)$ is a ring. Specifically, $(R : M) = F[[X,Y]]$. Obviously, M is invertible as an ideal of $(M : M)$. As $N = (X,Y)F[[X,Y]]$ is a trace ideal of $F[[X,Y]]$, MN is a trace ideal of R by Lemma 6.

Let P be a height one prime ideal of R. Then there is a unique prime ideal P' of $F[[X,Y]]$ which contracts to P, namely $P' = \{g \in F[[X,Y]] \mid gM \subseteq P\}$. Since $F[[X,Y]]$ is a local UFD, P' is a principal prime of $F[[X,Y]]$. Thus $P' = fF[[X,Y]]$ for some irreducible $f \in F[[X,Y]]$. As P has height one, f is not a multiple of X and it follows that $P = MP' = XP' = XfF[[X,Y]]$.

Continuing with the notation above, let Q be a P-primary ideal. Since P does not contain M, $R_P = F[[X,Y]]_{P'}$ is a discrete rank one valuation domain. Hence $QR_P = f^nR_P$ for some integer n. By contracting this ideal to R we see that $Q = Xf^nF[[X,Y]] = f^nM$. Since Q is a principal multiple of M, $Q^{-1} = (1/f^n)M^{-1}$ and from this it follows that $QQ^{-1} = M$.♦

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Pullbacks and Coherent-Like Properties

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INTRODUCTION.

Let I be a nonzero ideal of a domain T, $\varphi: T \to T/I$ the natural projection and D a domain contained in T/I. Let $R = \varphi^{-1}(D)$ be the domain arising from the following pullback of canonical homomorphisms.

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

We explicitly assume that $R \subset T$ and we shall refer to this as a diagram of type (Δ). If $I = P$ is a prime ideal of T, we use $\chi(P)$ to denote the residue field of T_P and $qf(D)$ the quotient field of D. The case where $T = V$ is a valuation domain is of particular

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