On the Dimension Theory of Polynomial Rings over Pullbacks

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1 Introduction

Since Seidenberg's (1953-54) papers [35, 36] and Jaffard's (1960) pamphlet [28] on the dimension theory of commutative rings, the literature abounds in works exploring the prime ideal structure of polynomial rings, including four pioneering articles by Arnold and Gilmer on dimension sequences [3, 4, 5, 6]. Of particular interest is Bastida-Gilmer's (1973) precursory article [8] which established a formula for the Krull dimension of a polynomial ring over a $D + M$ issued from a valuation domain. During the last three decades, numerous papers provided in-depth treatments of dimension theory and other related notions (such as the S-property, strong S-property, and catenarity) in polynomial rings over various pullback constructions. All rings considered in this paper are assumed to be integral domains.

A polynomial ring over an arbitrary domain R is subject to Seidenberg's inequalities: $n + \dim(R) \leq \dim(R[X_1, ..., X_n]) \leq n + (n+1)\dim(R)$, $\forall\ n \geq 1$. A finite-dimensional domain R is said to be Jaffard if $\dim(R[X_1, ..., X_n]) = n + \dim(R)$ for all $n \geq 1$; equivalently, if $\dim(R) = \dim_v(R)$, where $\dim(R)$ denotes the Krull dimension of R and $\dim_v(R)$ its valuative dimension (i.e., the supremum of dimensions of the valuation overrings of R). The study of this class was initiated by Jaffard [28]. For the convenience of the reader, recall that, in general, for a domain R with $\dim_v(R) < \infty$ we have: $\dim(R) \leq \dim_v(R)$, $\dim_v(R[X_1, ..., X_n]) = n + \dim_v(R)$ for all $n \geq 1$, and $\dim(R[X_1, ..., X_n]) = n + \dim_v(R)$ for all $n \geq \dim_v(R) - 1$ (Cf. [2, 11, 18, 26, 28]).

As the Jaffard property does not carry over to localizations (see Example 1 below), R is said to be locally Jaffard if R_p is a Jaffard domain for each prime ideal p of R; equivalently, $S^{-1}R$ is a Jaffard domain for each multiplicative subset S of R. A locally Jaffard domain is Jaffard [2]. The class of (locally) Jaffard domains contains most classes involved in dimension theory, including Noetherian domains [31], Prüfer domains [26], and universally catenarian domains [10].

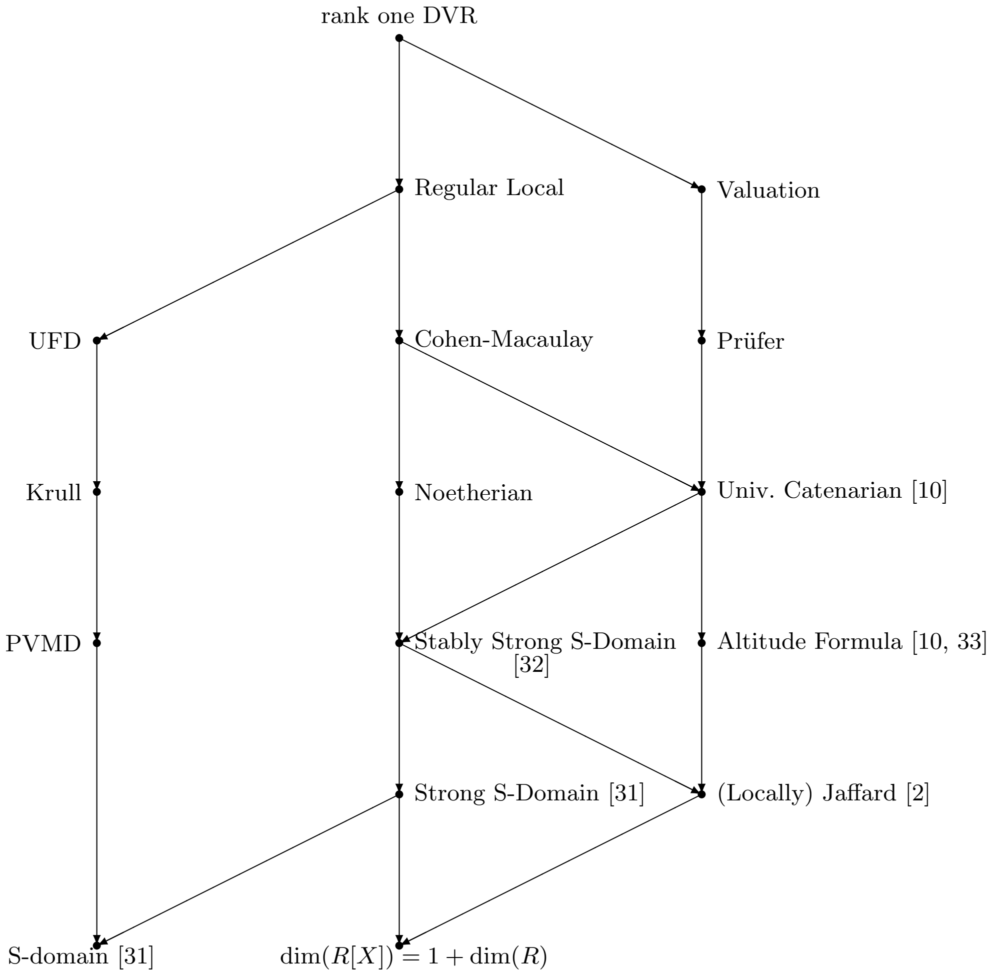

Fig. 1. Diagram of implications

In order to treat Noetherian domains and Prüfer domains in a unified manner, Kaplansky [31] introduced the following concepts: A domain R is called an S-domain if, for each height-one prime ideal p of R, the extension $pR[X]$ in $R[X]$ has height 1 too; and R is said to be a strong S-domain if $\frac{R}{p}$ is an S-domain for each prime ideal p of R. A strong S-domain R satisfies $\dim(R[X]) = \dim(R) + 1$. Notice that while $R[X]$ is always an S-domain for any domain R [24], $R[X]$ need not be a strong S-domain even when R is a strong S-domain [12]. Thus R is called a stably strong S-domain (also called a universally strong S-domain) if the polynomial ring $R[X_1, ..., X_n]$ is a strong S-domain for each positive integer n. A stably strong S-domain is locally Jaffard [2, 29, 32].

This review paper deals with dimension theory of polynomial rings over certain families of pullbacks. While the literature is plentiful, this field is still developing and many contexts are yet to be explored. I will thus restrict the scope of the present survey, mainly, to topics I have worked on over the last decade. The set of pullback constructions studied includes $D+M$, $D+(X_1,...,X_n)D_S[X_1,...,X_n]$, $A+XB[X]$, and $D+I$.

Any unreferenced material is standard, as in [9, 26, 28, 31, 33]. In Figure 1, a diagram of implications summarizes the relations between some spectral notions and well-known classes of integral domains (some of which should be either finite-dimensional or locally finite dimensional).

2 Preliminaries on Pullbacks

Pullbacks have proven to be useful for the construction of original examples and counter-examples in Commutative Ring Theory. The oldest in date is due to Krull (Cf. [8, page 1]). However, the first systematic investigation of a particular family of pullbacks; namely, $D+M$ issued from valuation domains, was carried out by Gilmer [25, Appendix 2] and [26]. Later, during the 1970s, six ground-breaking papers [8, 27, 19, 16, 13, 20] provided further development in various pullback contexts and paved the path for most subsequent works on these constructions. In Figure 2, a diagram provides more details on the contexts studied in these works.

Let's recall some results on the classical $D+M$ constructions (i.e., those issued from valuation domains). We shall use $\mathrm{qf}(R)$ to denote the quotient field of a domain R.

Theorem 1 ([25] and [19]). *Let V be a valuation domain of the form $K+M$, where K is a field and M is the maximal ideal of V. Let D be a proper subring of K with $k:=\mathrm{qf}(D)$. Set $R:=D+M$. Then:*
(1) $\dim(R)=\dim(V)+\dim(D)$.
(2) $\dim_v(R)=\dim(V)+\max\{\dim(W)|\ W$ *is valuation on* K *containing* $D\}$.
(3) *The integral closure of R is $D'+M$, where D' is the integral closure of D.*
(4) *R is a valuation domain $\Leftrightarrow$ D is a valuation domain and $k=K$.*
(5) *R is Prüfer $\Leftrightarrow$ D is Prüfer and $k=K$.*
(6) *R is Bezout $\Leftrightarrow$ D is Bezout and $k=K$.*
(7) *R is Noetherian $\Leftrightarrow$ V is a DVR, $D=k$, and $[K\colon k]<\infty$.*
(8) *R is coherent $\Leftrightarrow$ either "$k=K$ and D is coherent" or "M is a finitely generated ideal of R." The latter condition yields $D=k$ and $[K\colon k]<\infty$.*

In [16], the authors established several results, similar to the statements (1-6) and (8) above, for rings of the form $D+XK[X]$ where $K:=qf(D)$; particularly, $\dim(D+XK[X])=1+\dim(D)$ and $\dim_v(D+XK[X])=1+\dim_v(D)$. The next result handles the general context of $D+XD_S[X]$ rings.

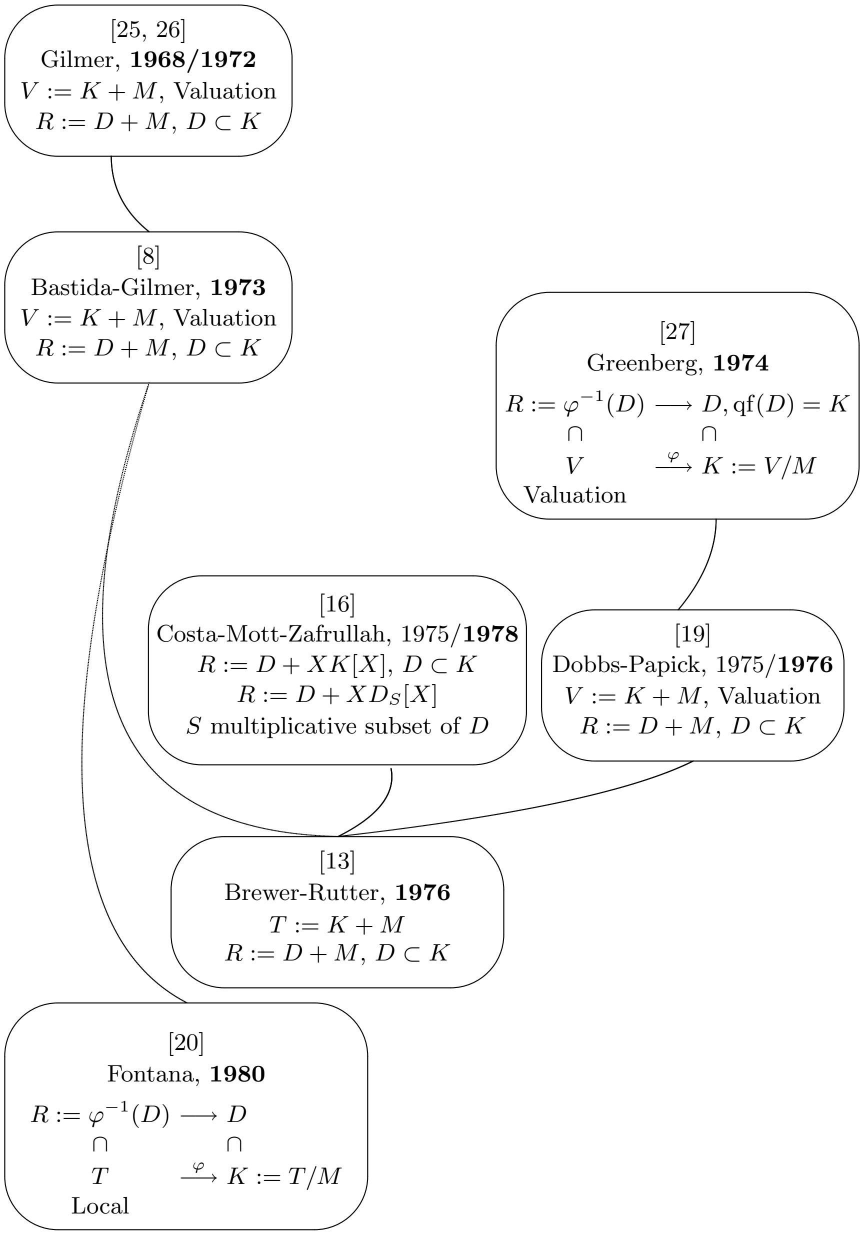

Fig. 2. Diagram of various pullback contexts studied in the 1970s

Theorem 2 ([16]). *Let D be an integral domain and S a multiplicative subset of D. Set $R^{(S)} := D + XD_S[X]$. Then:*
(1) *$R^{(S)}$ is GCD $\Leftrightarrow$ D is GCD and GCD(d, X) exists in $R^{(S)}, \forall\ d \in D^*$.*
(2) $\dim(D_S[X]) \leq \dim(R^{(S)}) \leq \dim(D[X])$.
(3) *If D is a valuation domain, then* $\dim(R^{(S)}) = 1 + \dim(D)$. □

in [13], Brewer and Rutter investigated general $D + M$ constructions (i.e., issued from an integral domain not necessarily valuation) and gave unified proofs of most results known on classical $D+M$ and $D+XK[X]$ rings. Their result on the Krull dimension reads as follows:

Theorem 3 ([13]). *Let T be an integral domain of the form $K + M$, where K is a field and M is a maximal ideal of T. Let D be a proper subring of K with $k := \mathrm{qf}(D)$. Set $R := D + M$.*
If $k = K$, then $\dim(R) = \max\{\mathrm{ht}_T(M) + \dim(D), \dim(T)\}$. □

Later, Fontana [20] used topological methods (particularly, his study of amalgamated sums of two spectral spaces) to extend most of these results to pullbacks (issued from local domains). We close this section by citing some basic facts connected with the prime ideal structure of a pullback. These will be used frequently in the sequel without explicit mention. We shall use Spec(R) to denote the set of prime ideals of a ring R.

Theorem 4 ([20] and [2, Lemma 2.1]). *Let T be an integral domain, M a maximal ideal of T, K its residue field, $\varphi : T \longrightarrow K$ the canonical surjection, D a proper subring of K, and $k := \mathrm{qf}(D)$. Let $R := \varphi^{-1}(D)$ be the pullback issued from the following diagram of canonical homomorphisms:*

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K = T/M \end{array}$$

(1) *$M = (R : T)$ and $R/M \cong D$.*
(2) $\mathrm{Spec}(R) \simeq \mathrm{Spec}(D) \coprod_{\mathrm{Spec}(K)} \mathrm{Spec}(T)$ *(i.e., topological amalgamated sum)*
(3) *Assume T is local. Then M is a divided prime and so every prime ideal of R compares with M under inclusion. If, in addition, $k = K$ then $R_M = T$.*
(4) *Assume T is local. Then* $\dim(R) = \dim(T) + \dim(D)$.
(5) *For each prime ideal P of R such that $M \nsubseteq P$, there exists a unique prime ideal Q of T such that $Q \cap R = P$, and hence $T_Q = R_P$.*
(6) *For each prime ideal P of R such that $M \subseteq P$, there exists a unique prime ideal p of D such that $P = \varphi^{-1}(p)$, and hence R_P can be viewed as the pullback of T_M and D_p over K.*
(7) *T is integral over R $\Leftrightarrow$ $D = k$ and K is algebraic over k.* □

3 Dimension Theory

This section studies the Krull dimension and valuative dimension of polynomial rings over various families of pullbacks. It also examines the transfer of the Jaffard property to these constructions.

In 1969, Arnold established a fundamental theorem, [3, Theorem 5], on the dimension of a polynomial ring over an arbitrary integral domain; namely, for any integral domain R with quotient field K and for any positive integer n, $\dim(R[X_1,...,X_n]) = n + \max\{\dim(R[t_1,...,t_n]) \mid \{t_i\}_{1\leq i\leq n} \subseteq K\}$. In [8], Bastida and Gilmer generalized this result to the case where $\{t_i\}_{1\leq i\leq n}$ is a subset of an extension field of K. It allowed them to establish a formula for the Krull dimension of a polynomial ring over a classical $D+M$ as stated below:

Theorem 5 ([8, Theorem 5.4]). *Let V be a valuation domain of the form $K+M$, where K is a field and M is the maximal ideal of V. Let D be a proper subring of K with $k := \mathrm{qf}(D)$ and let $\mathrm{t.d.}(K\colon k)$ denote the transcendence degree of K over k. Let n be a positive integer. Set $R := D+M$. Then:*

$$\dim(R[X_1,...,X_n]) = \dim(V) + \dim(D[X_1,...,X_n]) + \min\{n, \mathrm{t.d.}(K\colon k)\}. \quad \square$$

In [11], we refined Gilmer's statement on the valuative dimension of a classical $D+M$ in order to build a family of examples of Jaffard domains which are neither Noetherian nor Prüfer domains.

Proposition 1 ([11, Proposition 2.1]). *Under the same notation of Theorem 5, we have:*
(1) $\dim_v(R) = \dim_v(D) + \dim(V) + \mathrm{t.d.}(K\colon k)$.
(2) *R is a Jaffard domain $\Leftrightarrow$ D is a Jaffard domain and* $\mathrm{t.d.}(K\colon k) = 0$. $\square$

From this result stems a first family of Jaffard domains A_n with dimension $n+3$ which are neither Noetherian nor Prüfer, for every $n \geq 1$. Indeed, the ring $B := \mathbb{Z} + Y\mathbb{Q}(X)[Y]_{(Y)}$ is not a Jaffard domain since $\dim(B) = 2$ and $\dim_v(B) = 3$ by Proposition 1. For each $n \geq 1$, set $A_n := B[X_1,...,X_n]$. For $n = 1$, $A_1 = B[X_1]$ is a 4-dimensional Jaffard domain, since, by Theorem 5, $\dim(B[X_1]) = 4 = \dim_v(B)+1 = \dim_v(B[X_1])$. Clearly, for each $n \geq 2$, A_n is an $(n+3)$-dimensional Jaffard domain. Further, A_1 is not a strong S-domain, otherwise B would be so and hence we would have $5 = \dim(B[X_1,X_2]) = 1+\dim(B[X_1]) = 2+\dim(B) = 4$, which is absurd. Consequently, none of the rings A_n is a strong S-domain (hence it is neither Noetherian nor Prüfer), as desired.

We now proceed to explore a general context. Let T be an integral domain, M a maximal ideal of T, K its residue field, $\varphi : T \longrightarrow K$ the canonical surjection, D a proper subring of K, and $k := \mathrm{qf}(D)$. Let $R := \varphi^{-1}(D)$ be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K = T/M. \end{array}$$

Theorem 6 ([2, Theorem 2.6]). *Assume T is local. Then:*
(1) $\dim_v(R) = \dim_v(D) + \dim_v(T) + \text{t.d.}(K\colon k)$.
(2) *R is Jaffard $\Leftrightarrow$ D and T are Jaffard and* $\text{t.d.}(K\colon k) = 0$. □

The next result generalizes Theorem 1(1), Theorem 4(4), and Theorem 6.

Theorem 7 ([2, Theorem 2.11 and Corollary 2.12]). *Assume T is an arbitrary domain (i.e., not necessarily local). Then:*
(1) $\dim(R) = \max\{\dim(T), \dim(D) + \text{ht}_T(M)\}$.
(2) $\dim_v(R) = \max\{\dim_v(T), \dim_v(D) + \dim_v(T_M) + \text{t.d.}(K\colon k)\}$.
(3) *R is locally Jaffard $\Leftrightarrow$ D and T are locally Jaffard and* $\text{t.d.}(K\colon k) = 0$.
(4) *If T is locally Jaffard with* $\dim_v(T) < \infty$*, D is Jaffard, and* $\text{t.d.}(K\colon k) = 0$*, then R is a Jaffard domain.* □

There are examples which show that none of the hypotheses in Theorem 7(4) is a necessary condition for R to be Jaffard. Indeed, let V and W be two incomparable valuation domains of a suitable field K with $n := \dim(V) \geq 3$ and $\dim(W) = 1$. By [34, Theorem 11.11], $T := V \cap W$ is an n-dimensional Prüfer domain with two maximal ideals, say M_1 and M, $T_{M_1} = V$, and $T_M = W$. Let $\varphi : T \longrightarrow T/M \cong K$ be the canonical surjection. We further require that K has a subfield k and a subring D such that $\dim(D) = \dim_v(D) = 1$, $\text{qf}(D) = k$, and $\text{t.d.}(K\colon k) = 1$. Set $R := \varphi^{-1}(D)$. By Theorem 7(1) & (2), $\dim(R) = \dim_v(R) = n$. So that R is Jaffard though K is not algebraic over k. Now, alter the above construction by taking $n \geq 4$ and $\dim_v(D) = 2$, so that D is not Jaffard anymore, but one can easily check that R is Jaffard.

Next we proceed to the construction of the first example of a Jaffard domain which is not locally Jaffard.

Example 1 ([2, Example 3.2]). Let k be a field and X_1, X_2, Y indeterminates over k. Set $V_1 := k(X_1, X_2)[Y]_{(Y)} = k(X_1, X_2) + M_1$ and $A := k(X_1) + M_1$, where $M_1 = YV_1$. Let (V, M) be a one-dimensional valuation domain of the form $V = k(Y) + M$ such that $k(Y)[X_1, X_2] \subset V \subset k(X_1, X_2, Y)$ (In order to build such a ring, consider the valuation $v\colon k(Y)[X_1, X_2] \longrightarrow \mathbb{Z}^2$ defined by $v(X_1) = (1, 0)$ and $v(X_2) = (0, 1)$, where $\mathbb{Z}^2$ is endowed with the order induced by the group isomorphism $i\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}[\sqrt{2}]$ defined by $i(a, b) = a + b\sqrt{2}$). Consider the two-dimensional valuation ring $V_2 := k[Y]_{(Y)} + M = k + M_2$ with maximal ideal $M_2 = Yk[Y]_{(Y)} + M$. One can easily check that V_1 and V_2 are incomparable. By [34, Theorem 11.11], $B := V_1 \cap V_2$ is a 2-dimensional Prüfer domain with two maximal ideals, say N_1 and N_2, $B_{N_1} = V_1$, and $B_{N_2} = V_2$. Finally, put $R := A \cap V_2$. One can show that R is semi-local with two maximal ideals $\mathcal{M}_1 = N_1 \cap R$ and

$\mathcal{M}_2 = N_2 \cap R$ with $R_{\mathcal{M}_1} = A$ and $R_{\mathcal{M}_2} = V_2$ (Cf. [17, Example 2.5]). Via Theorem 7, we obtain $\dim(R) = \max\{\dim(R_{\mathcal{M}_1}), \dim(R_{\mathcal{M}_2})\} = 2$ and $\dim_v(R) = \max\{\dim_v(R_{\mathcal{M}_1}), \dim_v(R_{\mathcal{M}_2})\} = 2$. Thus R is Jaffard but not locally Jaffard, since $\dim(R_{\mathcal{M}_1}) = \dim(A) = 1 \neq \dim_v(R_{\mathcal{M}_1}) = \dim_v(A) = 2$. □

The next result examines the possibility of extending Bastida-Gilmer's result (Theorem 5) on the classical $D + M$ ring to a general context.

Theorem 8 ([2, Proposition 2.3 and Proposition 2.7]). *Under the same notation as above, the following statements hold.*
(1) *Assume* $k = K$. *Then:* $\dim(R[X_1, ..., X_n]) = \dim(D[X_1, ..., X_n]) + \dim(T[X_1, ..., X_n]) - \dim(K[X_1, ..., X_n])$, *for each positive integer* n.
(2) *Assume* $D = k$ *and set* $d :=$ t.d.$(K\colon k)$. *Then, for each* $n \geq 0$, *we have:* $n + \dim(T) + \min\{n, d\} \leq \dim(R[X_1, ..., X_n]) \leq n + \dim_v(T) + d$. □

Now, one should design an example to show that the above can be strict.

Example 2 ([2, Example 3.9]). Let Y_1, Y_2, U, V, Z, W be indeterminates over a field k. Define $K := k(Y_1, Y_2)$, $S := K(U)[V]_{(V)}$, $R_1 := K(U, V, Z)[W]_{(W)}$, $A := K(U, V) + WR_1$, $B := K + VS$, $R_2 := S + WR_1$, and $T := K + VS + WR_1$. Thus, we have the following pullbacks (with canonical homomorphisms):

$$\begin{array}{ccccc}
T & \longrightarrow & B & \longrightarrow & K \\
\downarrow & & \downarrow & & \downarrow \\
R_2 & \longrightarrow & S & \longrightarrow & K(U) \\
\downarrow & & \downarrow & & \\
A & \longrightarrow & K(U, V) & & \\
\downarrow & & \downarrow & & \\
R_1 & \longrightarrow & K(U, V, Z) & &
\end{array}$$

R_1 and S are discrete valuation rings. Further, by applying Theorem 4(4) and Theorem 6, we obtain:

$\dim(A) = 1$; $\dim_v(A) = 2$
$\dim(R_2) = \dim(S) + \dim(R_1) = 2$; $\dim_v(R_2) = 3$
$\dim(B) = 1$; $\dim_v(B) = 2$
$\dim(T) = \dim(k) + \dim(R_2) = 2$; $\dim_v(T) = 4$.

Let $\varphi : T \longrightarrow K$ be the canonical surjection and $R := \varphi^{-1}(k)$. The pullback R has Krull dimension 2 and valuative dimension 6. Further, $\dim(R[X]) = 5$ by [21, Theorem 2.1]. Set $d :=$ t.d.$(K\colon k) = 2$. The desired strict inequalities follow: $1 + \dim(T) + \min\{1, d\} \lneqq \dim(R[X]) \lneqq 1 + \dim_v(T) + d$. □

Next, we explore Costa-Mott-Zafrullah's $D{+}XD_S[X]$ construction under a slight generalization. Let D be a domain, S a multiplicative subset of D, and r an integer ≥ 1. Put $R^{(S,r)} := D{+}(X_1, ..., X_r)D_S[X_1, ..., X_r]$. Let $p \in \mathrm{Spec}(D)$. The S-coheight of p, denoted S-coht(p), is defined as the supremum of the lengths of all chains $p \subset p_1 \subset p_2 \subset ... \subset p_n$ of prime ideals of D with $p_1 \cap S \neq \emptyset$. Set S-$\dim(D) := \max\{S\text{-coht}(p) \mid p \in \mathrm{Spec}(D)\}$.

Theorem 9 ([16] and [24]). *Under the above notation, the following statements hold.*
(1) $\max\{\dim(D_S[X_1, ..., X_r]), r + \dim(D)\} \leq \dim(R^{(S,r)})$
$\leq \min\{\dim(D[X_1, ..., X_r]), \dim(D_S[X_1, ..., X_r]) + S\text{-}\dim(D)\}$.
(2) $\dim_v(R^{(S,r)}) = r + \dim_v(D)$.
(3) *D is Jaffard $\Leftrightarrow$ $R^{(S,r)}$ is Jaffard and* $\dim(R^{(S,r)}) = r + \dim(D)$.
(4) *$R^{(S,r)}$ is Jaffard $\Leftrightarrow$ so is $D[X_1, ..., X_r]$ with the same dimension as $R^{(S,r)}$.* □

Now, we provide an example to show that the Jaffard property of $R^{(S,r)}$ does not force D to be Jaffard. Here too we appeal to pullbacks. Let k be a field and X, Y two indeterminates over k. Put $V := k(X) + Yk(X)[Y]_{(Y)}$ and $D := k + Yk(X)[Y]_{(Y)}$. Clearly, D is a local domain with maximal ideal $M := Yk(X)[Y]_{(Y)}$, $\dim(D) = 1$, and $\dim_v(D) = 2$ by Theorem 1(1) and Proposition 1. Set $S := D \setminus M$ and $R^{(S,1)} := D + XD_S[X]$. So $R^{(S,1)} \cong D[X]$ since $D_M \cong D$. It follows that $\dim(R^{(S,1)}) = \dim(D[X]) = 1 + \dim_v(D) = 3 = \dim_v(R^{(S,1)})$, as desired.

Next we move to a general context. Let $A \subseteq B$ an extension of integral domains and X an indeterminate over B. Put $R := A + XB[X] = \{f \in B[X] \mid f(0) \in A\}$. This construction was introduced by D.D. Anderson-D.F. Anderson-Zafrullah in [1]. Also, R is a particular case of the constructions B, I, D introduced by P.-J. Cahen [15]. Also, $\text{Int}(A) \cap B[X] = \{f \in B[X] \mid f(A) \subseteq A\}$ is a subring of R and hence a deeper knowledge of $A + XB[X]$ constructions may have some interesting impact on the integer-valued polynomial rings.

As a consequence of some general properties of the spectrum of a pullback [20], we state the following: First, $XB[X]$ is a prime ideal of $R := A + XB[X]$ with $R/XB[X] \cong A$ and hence we have an order-isomorphism $\text{Spec}(A) \longrightarrow \{P \in \text{Spec}(R) \mid XB[X] \subseteq P\}$, $p \longmapsto p + XB[X]$. Second, $S := \{X^n \mid n \geq 0\}$ is a multiplicatively closed subset of R and $B[X]$ with $S^{-1}R = S^{-1}B[X] = B[X, X^{-1}]$; by contraction, we obtain an order-isomorphism $\{Q \in \text{Spec}(B[X]) \mid X \notin Q\} \longrightarrow \{P \in \text{Spec}(R) \mid X \notin P\}$. Finally, the spectral space $\text{Spec}(R)$ is canonically homeomorphic to the amalgamated sum of $\text{Spec}(A)$ and $\text{Spec}(B[X])$ over $\text{Spec}(B)$.

For the subfamilies $D + XK[X]$ and $D + XD_S[X]$, it is known that $\text{ht}(XK[X]) = \text{ht}(XD_S[X]) = 1$. The next result probes the situation of $XB[X]$ inside $\text{Spec}(R)$.

Theorem 10 ([22, Theorem 1.2]). *Let $R := A + XB[X]$ and $N := A \setminus \{0\}$. Then:*
(1) $\text{ht}_R(XB[X]) = \dim(N^{-1}B[X]) = \dim(B[X] \otimes_A \text{qf}(A))$.
(2) $1 \leq \text{ht}_R(XB[X]) \leq 1 + \text{t.d.}(B\colon A)$. □

Thus, if $\text{qf}(A) \subseteq B$, then $\text{ht}_R(XB[X]) = \dim(B[X])$; and if $A \subseteq B$ is an algebraic extension, then $\text{ht}_R(XB[X]) = 1$. In general, $\text{ht}_R(XB[X])$

can describe all integers between 1 and $1 + \text{t.d.}(B\colon A)$, as shown by the following example: Let d be an integer, $t \in \{1, ..., d+1\}$, K a field, and $X, X_1, ..., X_{d+1}, Y_1, ..., Y_d$ indeterminates over K. Set $A := K$ and $B := K(X_1, ..., X_{d-t+1})[Y_1, ..., Y_{t-1}]$. Hence $\text{t.d.}(B\colon A) = d$ and $\text{ht}_R(XB[X]) = \dim(B[X]) = t$.

The next result studies the Krull and valuative dimensions as well as the transfer of the Jaffard property.

Theorem 11 ([22, Theorems 2.1 & 2.3]). *Let* $R := A + XB[X]$ *and set* $k := \text{qf}(A)$ *and* $d := \text{t.d.}(B\colon A)$*. Then:*
(1) $\max\{\dim(A) + \text{ht}_R(XB[X]), \dim(B[X])\} \leq \dim(R)$
$\leq \dim(A) + \dim(B[X])$.
(2) *If* $k \subseteq B$*, then* $\dim(R) = \dim(A) + \dim(B[X])$.
(3) $\dim_v(R) = \dim_v(A) + d + 1$.
(4) *R is Jaffard and* $\dim(R) = \dim(A) + 1 \Leftrightarrow$ *A is Jaffard and* $d = 0$.
(5) *If* $k \subseteq B$*, then: R is Jaffard* $\Leftrightarrow$ *so is A and* $\dim(B[X]) = 1 + d$. □

Now, one can easily construct new classes of Jaffard domains. For instance, $\mathbb{R} + X\mathbb{C}[X, Y]$ and $\mathbb{Z} + X\overline{\mathbb{Z}}[X]$ both are 2-dimensional Jaffard domains, where $\overline{\mathbb{Z}}$ denotes the integral closure of $\mathbb{Z}$ inside an algebraic extension of $\mathbb{Q}$.

The next result handles the locally Jaffard property.

Theorem 12 ([22, Theorems 2.8]). *Let* $R := A + XB[X]$ *and suppose that A is a locally Jaffard domain. Then R is locally Jaffard* $\Leftrightarrow$ *$B[X]$ is locally Jaffard and* $\text{ht}_R(XB[X]) = 1 + \text{t.d.}(B\colon A)$. □

We cannot knock down the hypothesis "A is locally Jaffard" to "A is Jaffard." For, assume A is Jaffard but not locally Jaffard (Example 1). Set $B := \text{qf}(A)$ and $R := A + XB[X] = A + X\text{qf}(A)[X]$. In this situation $B[X]$ is locally Jaffard and $\text{ht}_R(XB[X]) = 1 = 1 + \text{t.d.}(B\colon A)$; whereas, R is not locally Jaffard by Theorem 7(3). Notice, however, that the hypothesis "A is locally Jaffard" is not necessary as shown below.

While several results concerning $D + XK[X]$ and $D + XD_S[X]$ are recovered, some known results on these rings do not carry over to the general context of $A + XB[X]$ constructions. Next, an example provides some of these pathologies and, also, shows that the double inequality established in Theorem 11(1) can be strict.

Example 3 ([22, Example 3.1]). Let K be a field and let X, X_1, X_2, X_3, X_4 be indeterminates over K. Set:

$L := K(X_1, X_2, X_3)$	;	$V_1 := k + N$
$k := K(X_1, X_2)$	;	$D := K(X_1)[X_2]_{(X_2)} + N$
$M := X_4L[X_4]_{(X_4)}$	;	$A := K[X_1]_{(X_1)} + M$
$N := X_3k[X_3]_{(X_3)}$	;	$B := D + M$
$V := L + M$	;	$R := A + XB[X]$

Then:
(1) $\max\{\dim(A)+\mathrm{ht}_R(XB[X]), \dim(B[X])\} \lneqq \dim(R) \lneqq \dim(A)+\dim(B[X])$.
(2) $\dim(A[X]) \lneqq \dim(R)$ (in contrast with Theorem 9(1)).
(3) R is Jaffard and $A[X]$ is not Jaffard (in contrast with Theorem 9(4)).
(4) R is locally Jaffard and A is not locally Jaffard (in contrast with Theorem 7(3) applied to $D + XK[X]$).

Indeed, by Theorems 1 & 5 & 6, V, V_1, D, and B are valuation domains of dimensions 1, 1, 2, and 3, respectively; moreover, we have:

- $\dim(B[X]) = \dim(B) + 1 = 4$,
- $\dim(A) = \dim(K[X_1]_{(X_1)}) + \dim(V) = 2$,
- $\dim_v(A) = \dim_v(K[X_1]_{(X_1)}) + \dim(V) + \mathrm{t.d.}(L{:}\, K(X_1)) = 4$,
- $\dim(A[X]) = \dim(K[X_1]_{(X_1)}[X]) + \dim(V) + \min\{1, \mathrm{t.d.}(L{:}\, K(X_1))\} = 4$,
- $\mathrm{Spec}(B) = \{(0), M, P_1 := N + M, P_2 := X_2K(X_1)[X_2]_{(X_2)} + P_1\}$,
- $\mathrm{Spec}(A) = \{(0), M, Q := X_1K[X_1]_{(X_1)} + M\}$,
- $M \cap A = P_1 \cap A = P_2 \cap A = M$.

Notice first that $\mathrm{qf}(A) = \mathrm{qf}(B) = \mathrm{qf}(V)$. Now, inside $\mathrm{Spec}(R)$ we have the following chain of prime ideals (in view of the discussion in the paragraph right before Theorem 10):

$$(0) \subsetneqq M[X] \cap R \subsetneqq P_1[X] \cap R \subsetneqq P_2[X] \cap R \subsetneqq M + XB[X] \subsetneqq Q + XB[X].$$

Therefore $\dim(R) \geq 5$, and hence R is a 5-dimensional Jaffard domain since $\dim_v(R) = \dim_v(A) + \mathrm{t.d.}(B{:}\, A) + 1 = 5$ by Theorem 11. Consequently, (1) and (2) hold, and so does (3) since $\dim_v(A[X]) = \dim_v(A) + 1 = 5$. It remains to deal with (4). The domain A is not locally Jaffard (since it is not Jaffard). Let $P \in \mathrm{Spec}(R)$ with $X \notin P$. Then $R_P = B[X, X^{-1}]_{PB[X,X^{-1}]}$ is a universally strong S-domain (Cf. [10, 32]) and hence Jaffard (since B is a valuation domain). So, in order to show that R is locally Jaffard, it suffices to consider the localizations with respect to the prime ideals that contain X. Let $P := p + XB[X] \in \mathrm{Spec}(R)$ with $p \in \mathrm{Spec}(A)$. One can check that $R_P = A_p + XB[X]_P$ and thus $A_p + XB_p[X] \subseteq R_P \subseteq A_p + XL[X]_{(X)}$. We obtain, via Theorems 6 & 11, that $\dim_v(R_P) = \dim_v(A_p + XB_p[X]) = \dim_v(A_p + XL[X]_{(X)}) = \dim_v(A_p) + \mathrm{t.d.}(B{:}\, A) + 1 = \dim_v(A_p) + 1$. We claim that R_P is Jaffard for all $p \in \mathrm{Spec}(A)$:

- Let $p := (0)$. Then $\dim(R_P) = \mathrm{ht}_R(XB[X]) = 1 = \dim_v(A_{(0)}) + 1$.
- Let $p := M$. Then the above maximal chain yields $\mathrm{ht}(P) = 4$. Hence $\dim(R_P) = 4 = \dim_v(K(X_1)) + \dim(V) + \mathrm{t.d.}(L{:}\, K(X_1)) + 1 = \dim_v(A_M) + 1$. Here we view A_M as a pullback of V and $K(X_1)$ over L.
- Let $p := Q$. Then $\dim(R_P) = 5 = \dim_v(A) + 1 = \dim_v(A_Q) + 1 = \dim_v(R_P)$ (since $A_Q = A$). □

Next we move to a more general context. let T be a domain, I an nonzero ideal of T, and D a subring of T such that $D \cap I = (0)$. Throughout, D will be identified with its image in T/I. Also $\mathrm{ht}_T(I)$ will be assumed to be finite (though it's not always indispensable). Let $R := D + I$; it is a pullback determined by the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R := D + I & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I. \end{array}$$

So Spec(R) is canonically homeomorphic to the amalgamated sum of Spec(D) and Spec(T) over Spec(T/I). Precisely, I is a prime ideal of R and we have the order isomorphisms: $\mathrm{Spec}(D) \longrightarrow \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$, $p \longmapsto p + I$; and $\{Q \in \mathrm{Spec}(T) \mid I \nsubseteq Q\} \longrightarrow \{P \in \mathrm{Spec}(R) \mid I \nsubseteq P\}$, $Q \longmapsto Q \cap R$.

This construction was introduced and developed by Cahen [14, 15]. Since its study has proven to be difficult in its generality, the scope was mainly limited to the so-called $(T = B, I, D)$ almost-simple constructions (i.e., every ideal of T containing I is maximal). The following results -due to Cahen- approximate $\mathrm{ht}_R(I)$ and $\dim(R)$ with respect to $\mathrm{ht}_T(I)$, $\dim(D)$, and $\dim(T)$ in the general context.

Theorem 13 ([14, Proposition 5, Théorème 1, and Corollaire 1]).
(1) $\mathrm{ht}_T(I) \leq \mathrm{ht}_R(I) \leq \dim(T)$.
(2) $\dim(D) + \mathrm{ht}_R(I) \leq \dim(R) \leq \dim(D) + \dim(T)$.
(3) $\dim(R) \geq \max\{\mathrm{ht}_T(Q) + \dim(R/Q \cap R) \mid Q \in \mathrm{Spec}(T), I \subseteq Q\}$. □

Later, Ayache devoted his paper [7] to the special case where T is either a finitely generated K-algebra or a quotient of a power series ring in a finite number of indeterminates. He established the following results:

Theorem 14 ([7]). *Let K be a field, T a finitely generated K-algebra or a quotient of a power series ring in a finite number of indeterminates, I a proper non-zero ideal of T, D a subring of K with $k := \mathrm{qf}(D)$, and $R := D + I$. Then:*
(1) $\dim(R) = \dim(D) + \dim(T)$.
(2) *Assume either T is a finitely generated K-algebra or* $\mathrm{ht}_T(I) = \dim(T)$. *Then:* $\dim_v(R) = \dim_v(D) + \dim_v(T) + \mathrm{t.d.}(K\colon k)$, *and hence R is Jaffard if and only if D is Jaffard and* $\mathrm{t.d.}(K\colon k) = 0$. □

We return to the general context. The next result shades more light on I within the spectrum of R.

Lemma 1 ([23, Lemme 1.2]). *Set* $\mathcal{X} := \{Q \in \mathrm{Spec}(T) \mid Q \cap R = I\}$ *and* $\mathcal{Y} := \{Q \in \mathrm{Spec}(T) \mid I \nsubseteq Q, \exists\, Q' \in \mathcal{X}, (0) \subset Q \subset Q'\}$. *Then:*
(1) $\mathcal{X} \neq \emptyset$.
(2) $\mathcal{Y} = \emptyset$ *if and only if* $\mathrm{ht}_R(I) = 1$.
(3) $\mathrm{ht}_R(I) = 1 + \max\{\mathrm{ht}_T(Q) \mid Q \in \mathcal{Y}\}$.
(4) *If* $\mathrm{ht}_{R[X]}(I[X]) = 1$, *then* $\mathrm{t.d.}(T/Q\colon D) = 0, \forall\, Q \in \mathcal{X}$. □

Next we show how the S-domain property is reflected on $\mathrm{ht}_R(I)$.

Theorem 15 ([23, Théorème 1.3]). *Assume T is an S-domain. Then R is an S-domain if and only if* $\text{ht}_R(I) > 1$ *or* $\text{t.d.}(\frac{T}{Q}\colon D) = 0$, $\forall\, Q \in \text{Spec}(T)$ *such that* $Q \cap R = I$. □

In the special case where $T := V$ is a valuation domain, one can easily check that $\text{ht}_R(I) = \text{ht}_V(I)$ and $\dim(R) = \dim(D) + \text{ht}_V(I)$. Moreover, we have the following:

Theorem 16 ([23, Théorème 1.13]). *Let V be a valuation domain, I an non-zero ideal of V, D a subring of V with $D \cap I = (0)$, and $R := D + I$. Let P_0 denote the prime ideal of V that is minimal over I and let n be a positive integer. Then:*
(1) $\dim_v(R) = \dim_v(D) + \dim_v(V_{P_0}) + \text{t.d.}(\frac{V}{P_0}\colon D)$.
(2) $\dim(R[X_1, ..., X_n]) = \dim(V_{P_0}) + \dim(D[X_1, ..., X_n]) + \min\{n, \text{t.d.}(\frac{V}{P_0}\colon D)\}$.
(3) *R is a Jaffard domain $\Leftrightarrow$ D is a Jaffard domain and* $\text{t.d.}(\frac{V}{P_0}\colon D) = 0$. □

Another special case is when the $D+I$ ring arises from a polynomial ring. Namely, let B be a domain, X an indeterminate over B, D a subring of B, and I an ideal of $B[X]$ with $I \cap B = 0$. Put $R := D + I$. We have the following pullbacks (with canonical homomorphisms):

$$\begin{array}{ccc} R := D + I & \longrightarrow & D \\ \downarrow & & \downarrow \\ B + I & \longrightarrow & B \\ \downarrow & & \downarrow \\ B[X] & \longrightarrow & B[X]/I. \end{array}$$

Theorem 17 ([23, Théorème 2.1]). *Under the above notation, set* $d := \text{t.d.}(B\colon D)$. *We have:*
(1) $\dim_v(R) = \dim_v(D) + d + 1$.
(2) *R is Jaffard and* $\dim(R) = \dim(D) + 1$ $\Leftrightarrow$ *D is Jaffard and $d = 0$.* □

The above result applies to the particular context of $A + X^n B[X]$ constructions. Specifically, Let $A \subseteq B$ an extension of integral domains, X an indeterminate over B, and n an integer ≥ 1. Put $R_n := A + X^n B[X]$. Then $\dim_v(R_n) = \dim_v(A) + \text{t.d.}(B\colon A) + 1$; and R_n is Jaffard and $\dim(R_n) = \dim(A) + 1$ if and only if A is Jaffard and $\text{t.d.}(B\colon A) = 0$. Here the effect of the S-property appears as follows: R_n is an S-domain if and only if $\text{ht}_{R_1}(XB[X]) > 1$ or $\text{t.d.}(B\colon A) = 0$. (Since B[X] is always an S-domain.)

In this vein, the ring $R := \mathbb{Z}[(XY^i)_{i\geq 0}] = \mathbb{Z} + X\mathbb{Z}[X, Y]$ was shown by Ayache in [7] to be a 3-dimensional totally Jaffard domain [15]. In [23], we improved this result by stating that $R_n := \mathbb{Z}[(X^n Y^i)_{i\geq 0}] = \mathbb{Z} + X^n\mathbb{Z}[X, Y]$ is a universally strong S-domain, for each integer $n \geq 1$.

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