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Duals of Ideals in Polynomial Rings

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Throughout this paper, R will denote a domain with quotient field K. For a nonzero fractional ideal I of R, the fractional ideal $I^{-1} = (R:I) = \{x \in K \mid xI \subseteq R\}$ is called the *inverse* (or *dual*) of I. In [HP], Huckaba and Papick studied the question of when I^{-1} is a ring, and this question has received further attention in [A], [FHP1], [FHP2], [FHP3], [FHPR], [HeP], and [HKLM].

It is clear that I^{-1} is a ring when $I^{-1} = (I:I)$ $(= \{x \in K \mid xI \subseteq I\})$, and in case I is prime, then I^{-1} is a ring $\Leftrightarrow$ $I^{-1} = (I:I)$ [HP, Proposition 2.3]. The question as to whether I^{-1} a ring implies $I^{-1} = (I:I)$ in general was answered in the negative by D.F. Anderson [A]. In this work, we study when I^{-1} is a ring--and when this forces $I^{-1} = (I:I)$--when I is an ideal of $R[\{X_\alpha\}]$, where $\{X_\alpha\}$ is a set of indeterminates over R. Among other things, we completely characterize when P^{-1} is a ring for prime ideals P of $R[\{X_\alpha\}]$.

For an ideal J of R, we denote by $Z(R,J)$ the set $\{x \in R \mid xy \in J$ for some element $y \in R \setminus J\}$.

Lemma 1. *Let J be a nonzero ideal of $R[\{X_\alpha\}]$, and let S denote the complement in R of the set $Z(R[\{X_\alpha\}],J) \cap R$. (Thus S is a multiplicatively closed subset of R.) If $J^{-1} \subseteq R_S[\{X_\alpha\}]$, then $J^{-1} = (J:J)$.*

Proof: Let $f \in J$ and $g \in J^{-1}$. By hypothesis we may choose $s \in S$ with $sg \in R[\{X_\alpha\}]$. Then $sfg \in J$, and since $s \notin Z(R[\{X_\alpha\}],J)$, we have $fg \in J$. It follows that $J^{-1} = (J:J)$. □

Theorem 2. *Let J be a nonzero ideal of $R[\{X_\alpha\}]$ for which $Z(R[\{X_\alpha\}],J) \cap R = 0$. Then:*

(1) The following statements are equivalent.

(a) J^{-1} is a ring.

(b) $J^{-1} = (J:J)$.

(c) $J^{-1} \subseteq K[\{X_\alpha\}]$.

(2) If $\operatorname{ht} J \geq 2$, then the equivalent conditions of (1) hold.

Proof: (1) It is clear that (b) $\Rightarrow$ (a), and the implication (c) $\Rightarrow$ (b) follows from the lemma. Using * to denote complete integral closure, we have by [A, Proposition 2.3] that if J^{-1} is a ring, then $J^{-1} \subseteq R[\{X_\alpha\}]^* = R^*[\{X_\alpha\}] \subseteq K[\{X_\alpha\}]$. Hence (a) $\Rightarrow$ (c).

(2) Let S denote the set of nonzero elements of R. We first note that $J^{-1} \subseteq J^{-1}R[\{X_\alpha\}]_S \subseteq (JR[\{X_\alpha\}]_S)^{-1} = (JK[\{X_\alpha\}])^{-1}$. To complete the proof, it suffices to show that $(JK[\{X_\alpha\}])^{-1} = K[\{X_\alpha\}]$. This is clear if $JK[\{X_\alpha\}] = K[\{X_\alpha\}]$. On the other hand, if $JK[\{X_\alpha\}]$ is a proper ideal of $K[\{X_\alpha\}] = R[\{X_\alpha\}]_S$, then $\operatorname{ht}(JK[\{X_\alpha\}]) \geq 2$, and since $K[\{X_\alpha\}]$ is a Krull domain, this implies that $(JK[\{X_\alpha\}])^{-1} = K[\{X_\alpha\}]$ [G, Corollary 44.8]. □

Remark.

(1) In the case of one indeterminate X, we have that $Z(R[X],J) \cap R = 0 \Leftrightarrow J = fK[X] \cap R[X]$ for some $f \in J$. In particular, $\operatorname{ht} J = 1$.

(2) For more than one indeternimate, examples of ideals J which satisfy condition (2) and for which $J^{-1} \underset{\neq}{\supset} R[\{X_\alpha\}]$ are easy to construct. For example, if s,t,u are indeterminates over $\mathbb{Q}(\sqrt{2})$, then $R[X,Y]$, where $R = \mathbb{Q} + (s,t,u)\mathbb{Q}(\sqrt{2})[s,t,u]$, contains a prime ideal J for which $J \cap R = 0$, $\operatorname{ht} J = 2$, and $J \subseteq M[X,Y]$, where M is the maximal ideal $(s,t,u)\mathbb{Q}(\sqrt{2})[s,t,u]$ of R. Since $M^{-1} = \mathbb{Q}(\sqrt{2})[s,t,u]$, $J^{-1} \supseteq \mathbb{Q}(\sqrt{2})[s,t,u][X,Y] \underset{\neq}{\supset} R[X,Y]$.

The following example shows the necessity of the zero divisor assumption in Theorem 2.

Example 3. An example of an ideal J of $R[X_1,X_2]$ for which

(1) $J \cap R = 0$,

(2) $Z(R[X_1,X_2],J) \cap R \neq 0$,

(3) $\operatorname{ht} J \geq 2$,

(4) $J^{-1} \subseteq K[X_1,X_2]$, and

(5) J^{-1} is not a ring.

Let k be a field, and let Y and Z be indeterminates over k. Let $T = k(Y)[Z] = k(Y) + M$, where $M = ZT$, and let $R = k[Y] + M$. Let J denote the ideal of $R[X_1,X_2]$ given by $J = YX_1k[Y,X_1,X_2] + X_1M[X_1,X_2] + X_2M[X_1,X_2]$. It is clear that (1) holds. Also, since $YX_1 \in J$ and $X_1 \notin J$, we have $Y \in Z(R[X_1,X_2],J)$, and (2) holds. Now let P be a minimal prime of J. If $M[X_1,X_2] \subseteq P$, then, since $YX_1 \in P$ but $YX_1 \notin M[X_1,X_2]$, the containment is proper. If $M[X_1,X_2] \not\subseteq P$, then, since $X_2M[X_1,X_2] \subseteq P$, we have $X_2R[X_1,X_2] \subseteq P$; again, the containment is proper, since $YX_1 \notin X_2R[X_1,X_2]$. Thus in both cases, we have $\operatorname{ht} P \geq 2$, so (3) holds. For (4), note that

$X_1, X_2Z \in JT[X_1, X_2]$. It follows that $J^{-1} \subseteq (T[X_1, X_2]:JT[X_1, X_2]) \subseteq (T[X_1, X_2]:(X_1, X_2Z)T[X_1, X_2]) = T[X_1, X_2] \subseteq K[X_1, X_2]$, where $K = k(Y, Z)$. Finally, it is clear that $Y^{-1} \in J^{-1}$, but since $Y^{-2}YX_1 = Y^{-1}X_1 \notin R[X_1, X_2]$, we have $Y^{-2} \notin J^{-1}$, and J^{-1} is not a ring. □

Before stating the next theorem, we recall a concept from [HHJ]. Let U be an upper to zero in $R[X]$; that is, let $U = fK[X] \cap R[X]$ for some irreducible polynomial $f \in K[X]$. Then U is said to be *almost principal* if there is a nonzero element $a \in R$ with $aU \subseteq fR[X]$. We generalize this as follows. Let Q denote a height 1 prime of $R[\{X_\alpha\}]$ for which $Q \cap R = 0$. Then we may write $Q = fK[\{X_\alpha\}] \cap R[\{X_\alpha\}]$ for some $f \in R[\{X_\alpha\}]$ with f prime in $K[\{X_\alpha\}]$. For a proper subset Y of $\{X_\alpha\}$ with $Q \cap R[Y] = 0$, we say that Q is *almost Y-principal* if $\mu Q \subseteq fR[\{X_\alpha\}]$ for some nonzero element $\mu \in R[Y]$. Note that if Y is the empty set, then "almost Y-principal" is the same thing as "almost principal." In [HHJ, Proposition 1.15] it was shown that for an upper to zero Q, Q^{-1} is a ring $\Leftrightarrow$ Q is not almost principal. This is generalized to the case of arbitrarily many indeterminates in (part (3) of) our next result.

Theorem 4. *Let Q be a nonzero prime ideal of $R[\{X_\alpha\}]$, and let $q = Q \cap R$.*

(1) If $q \neq 0$, then Q^{-1} is a ring $\Leftrightarrow$ q^{-1} is a ring or $Q \neq q[\{X_\alpha\}]$.

(2) If $q = 0$ and $htQ \geq 2$, then Q^{-1} is a ring.

(3) If $q = 0$ and $htQ = 1$, then the following statements are equivalent.

(a) Q^{-1} is a ring.

(b) Q is not of the form $(g):h = \{f \in R[\{X_\alpha\}] \mid fh \in gR[\{X_\alpha\}]$.

(c) For each proper subset Y of $\{X_\alpha\}$ for which $Q \cap R[Y] = 0$, Q is not almost Y-principal.

(d) For each finite proper subset Y of $\{X_\alpha\}$ for which $Q \cap R[Y] = 0$, Q is not almost Y-principal.

Proof: (1) If $Q = q[\{X_\alpha\}]$, then (an easy extension of the argument in) [HH, Proposition 4.3] shows that $Q^{-1} = q^{-1}[\{X_\alpha\}]$, so that Q^{-1} is a ring $\Leftrightarrow$ q^{-1} is a ring. Thus it suffices to prove that if $Q \supsetneqq q[\{X_\alpha\}]$, then Q^{-1} is a ring. For the case of one indeterminate, [BH, Corollary 8] shows that Q is not of the form $(f):g$, and by [HZ, Lemma 1.2], this implies that $Q^{-1} = (Q:Q)$ (so that Q^{-1} is a ring). Now suppose that $\{X_\alpha\}$ is the finite set $\{X_1, \ldots, X_n\}$. Let $Q' = Q \cap R[X_1, \ldots, X_{n-1}]$. If $Q \supsetneqq Q'[X_n]$, then Q^{-1} is a ring by the case of one indeterminate. If $Q = Q'[X_n]$, then $Q' \supsetneqq q[X_1, \ldots, X_{n-1}]$, and we may assume by induction that Q'^{-1} is a ring. It follows easily that Q^{-1} is also a ring. For the general case, choose $X_1, \ldots, X_r \in \{X_\alpha\}$ with $p = Q \cap R[X_1, \ldots, X_r] \supsetneqq q[X_1, \ldots, X_r]$. Then p^{-1} is a ring by the finite case. Note that $Q^{-1} \subseteq K[\{X_\alpha\}]$, since $q \neq 0$. Let $f \in Q$ and $g \in Q^{-1}$; it suffices to show that $gf \in Q$. Choose $X_{r+1}, \ldots X_s \in \{X_\alpha\}$ with $g, f \in K[X_1, \ldots, X_s]$. Let $P = Q \cap R[X_1, \ldots, X_s]$, and note that $P \cap R[X_1, \ldots, X_r] = p$. Then $gP \subseteq gQ \cap K[X_1, \ldots, X_s] \subseteq R[\{X_\alpha\}] \cap K[X_1, \ldots, X_s] = R[X_1, \ldots, X_s]$. Thus $g \in P^{-1}$. By what has already been proved, P^{-1} is a ring, so that $P^{-1} = (P:P)$, and $gf \in P \subseteq Q$, as desired.

(2) Clearly, $Z(R[\{X_\alpha\}], Q) = Q$. Hence (2) follows easily from Theorem 2.

(3) (a) $\Leftrightarrow$ (b) Write $Q = fK[\{X_\alpha\}] \cap R[\{X_\alpha\}]$ with $f \in Q$. Suppose that $Q = (g):h$ for $g, h \in R[\{X_\alpha\}]$. Then $g^{-1}h \in Q^{-1}$; if Q^{-1} is a ring, we have $g^{-1}h \in K[\{X_\alpha\}]$ by Theorem 2. But then $\exists\ s \in R$ with $sg^{-1}h \in R[\{X_\alpha\}]$, and $s \in (g):h = Q$, contradicting that $Q \cap R[\{X_\alpha\}] = 0$. Conversely, if Q^{-1} is not a ring, then, again by Theorem 2, we may choose $g, h \in R[\{X_\alpha\}]$ with $g^{-1}h \in Q^{-1} \setminus K[\{X_\alpha\}]$. Then $fg^{-1}h \in R[\{X_\alpha\}]$, but $fg^{-1}h \notin fK[\{X_\alpha\}]$, whence $fg^{-1}h \notin Q$. Let $k \in (g):h$. Then $kfg^{-1}h \in fR[\{X_\alpha\}] \subseteq Q$, and so $k \in Q$. Thus $(g):h \subseteq Q$. The reverse inclusion is easy.

(a) $\Rightarrow$ (c) Let Y be a subset of $\{X_\alpha\}$ for which $Q \cap R[Y] = 0$, and suppose $\mu Q \subseteq fR[\{X_\alpha\}]$ with $\mu \in R[Y]$. Then $\mu f^{-1} \in Q^{-1}$, and by Theorem 2, we have $\mu = \mu f^{-1} f \subseteq Q^{-1}Q = Q$. Hence $\mu \in Q \cap R[Y]$, so that $\mu = 0$. Thus Q is not almost Y-principal.

(c) $\Rightarrow$ (d) Trivial.

(d) $\Rightarrow$ (a) Suppose that Q^{-1} is not a ring. Recall that $Q = fK[\{X_\alpha\}] \cap R[\{X_\alpha\}]$ for some $f \in Q$. Choose a finite subset Z of $\{X_\alpha\}$ for which $f \in R[Z]$ but $f \notin R[W]$ for each proper subset W of Z. Set $P = Q \cap R[Z]$. Then $P = fK[Z] \cap R[Z]$, and $Q = PR[\{X_\alpha\}]$ (since $\operatorname{ht} Q = 1$). Note that P^{-1} is not a ring. Now choose $Y \subseteq Z$ with $|Y| = |Z| - 1$ (possibly, Y is empty), and note that $P \cap R[Y] = 0$. By [HHJ, Proposition 1.15], P is almost principal with respect to the ring $R[Y]$, i.e., there is a nonzero element $\mu \in R[Y]$ with $\mu P \subseteq fR[Z]$. It follows that $\mu Q \subseteq fR[\{X_\alpha\}]$, and Q is almost Y-principal. $\square$

The following example shows the necessity of the assumption that Q be prime in Theorem 4 (1).

Example 5. An example of a radical ideal J in $R[X]$ such that $M = J \cap R$ is a prime ideal of R, and M^{-1} is a ring, but J^{-1} is not a ring.
Let k be a field, and let Y, Z be indeterminates over k. Let T and R be as in Example 3, and let $J = YXk[Y, X] + M[X]$. Then $J \cap R = M$. By [HKLM, Theorem 1], $M^{-1} = (M:M)$ $(= T)$, so that M^{-1} is a ring. However, J^{-1} is not a ring, since $Y^{-1} \in J^{-1}$, but $Y^{-2} \notin J^{-1}$. $\square$

For the remainder of the paper, we need the concepts of v- and t-closure of an ideal: For a fractional ideal I of a domain R, the v-closure of I is given by $I_v = (I^{-1})^{-1}$ and the t-closure by $I_t = \bigcup\{A_v \mid A$ is a nonzero finitely generated subideal of $I\}$. The v- and t-operations are examples of star-operations, and the reader is referred to [G] for a discussion of their properties. We note the trivial fact that $I^{-1} = R \Leftrightarrow I_v = R$.

Lemma 6. If J is a nonzero ideal of $R[\{X_\alpha\}]$ for which $J \cap R \neq 0$, then $J_t = R[\{X_\alpha\}] \Leftrightarrow \exists\, f \in J$ with $c(f)^{-1} = R$.

Proof: Suppose that $c(f)^{-1} = R$ for some $f \in J$. If a is any nonzero element of $J \cap R$, then (an easy extension of the argument in) [HH, Lemma 4.4] shows that $(a, f)_v = R[\{X_\alpha\}]$, from which it follows that $J_t = R[\{X_\alpha\}]$. Conversely, if $J_t = R[\{X_\alpha\}]$, then, since $J \subseteq c(J)R[\{X_\alpha\}]$, we have $c(J)_t = R$, and we may choose $g_1, \ldots, g_n \in J$ with $(c(g_1) + \cdots + c(g_n))_v = R$. Let $X \in \{X_\alpha\}$. It is then easy to choose exponents $\alpha_2, \ldots, \alpha_n$ for which $f = g_1 + X^{\alpha_2} g_2 + \cdots + X^{\alpha_n} g_n$ satisfies $c(f) = c(g_1) + \cdots + c(g_n)$. For this f, we then have $c(f)^{-1} = R$. $\square$

Proposition 7. *Let J be a nonzero ideal of $R[\{X_\alpha\}]$, and let $I = J \cap R$.*

(1) If $I \neq 0$ and $rad(J)$ contains an element f for which $c(f)^{-1} = R$, then $J^{-1} = R[\{X_\alpha\}]$ $(= (J:J))$.

(2) If I is either a maximal ideal or a maximal t-ideal of R and $J \supsetneq IR[\{X_\alpha\}]$, then $J^{-1} = R[\{X_\alpha\}]$.

(3) If $Z(R[\{X_\alpha\}], J) \cap R$ is a prime ideal P of R and $J^{-1} \subseteq R_P[\{X_\alpha\}]$, then $J^{-1} = (J:J)$.

Proof: (1) We have $f^n \in J$ for some n, and $c(f^n)^{-1} = R$. By Lemma 6, $J_t = R[\{X_\alpha\}]$, whence $J^{-1} = R[\{X_\alpha\}]$ as well.

(2) Pick $g \in J \setminus IR[\{X_\alpha\}]$. Then $(I + gR[\{X_\alpha\}])_t = R[\{X_\alpha\}]$ by [FGH, Proposition 2.2]. It follows that $J_t = R[\{X_\alpha\}]$, whence $J^{-1} = R[\{X_\alpha\}]$.

(3) This follows easily from Lemma 1. $\square$

We conclude with some examples showing that the converses of statements (1), (2), and (3) in Proposition 7 do not hold.

Example 8. An example of a domain R and a prime ideal Q in $R[X]$ such that

(1) $Q^{-1} = R[X]$,

(2) Q contains no polynomial f with $c(f)^{-1} = R$, and

(3) $Q \cap R$ is neither a maximal ideal nor a maximal t-ideal of R.

First let $D = k[s, \{st^{2^n} \mid n \geq 0\}]$, where k is a field and s, t are indeterminates over

k. This D is from unpublished work of J. Arnold; there he observes that a monomial s^it^j is in $D \Leftrightarrow i \geq \phi(j)$, where $\phi(j)$ is the number of 1's in the binary expansion of j. Let $U = (X-t)F[X] \cap R[X]$, where F is the quotient field of D. Then U is an upper to zero in $D[X]$. It is shown in [H, Proposition 1.2] that U is not almost principal, and by Theorem 2 (1) or [HHJ, Proposition 1.15], $U^{-1} \subseteq F[X]$. We show, in fact, that $U^{-1} = D[X]$. Suppose $h \in U^{-1}$. Write $h = Xg + a$, where $a \in F$ and $g \in F[X]$. It is clear that $ab \in D$ for each element b of D which is the constant term of some element of U. Since $(sX^{2^n} - st^{2^n})/(X-t) \in F[X]$, we have $sX^{2^n} - st^{2^n} \in (X-t)F[X] \cap D[X] = U$. Hence $aI \subseteq D$, where I is the ideal of D generated by $\{st^{2^n} \mid n \geq 0\}$. We claim that $I^{-1} = D$. Granting the claim, we have $a \in D$. Then $Xg = h - a \in U^{-1}$, from which it follows that $g \in U^{-1}$. Continuing in this manner, we see that $c(h) \subseteq D$, whence $U^{-1} = D[X]$. Thus it suffices to show that $I^{-1} = D$, or, equivalently, that $J^{-1} = Dst$, where J is the ideal generated by $\{t^{2^n-1} \mid n \geq 0\}$. It is clear that $Dst \subseteq J^{-1} \subseteq D$, and to complete the proof of the claim, it suffices to show that if $s^it^j \in J^{-1}$, then $s^{i-1}t^{j-1} \in D$. Now $s^it^j \in D$ implies that $i \geq \phi(j)$, and $s^it^j \in J^{-1}$ implies that $i \geq \phi(j + 2^n - 1)$. For large n we have $\phi(j + 2^n - 1) = \phi(j-1) + 1$, whence $i - 1 \geq \phi(j-1)$, and $s^{i-1}t^{j-1} \in D$, as desired.

Now let T be a domain of the form $F + P$, where P is a maximal ideal of T, and let $R = D + P$. Set $Q = U + P[X]$, so that Q is an upper to P in $R[X]$. By [H, Lemma 2.2], $Q^{-1} = R[X]$. Moreover, by [H, Theorem 2.4], Q is a t-prime of $R[X]$, whence by Lemma 6, Q cannot contain an element f with $c(f)^{-1} = R$. Finally, it is easy to see that $P = Q \cap R$ is not a maximal ideal of R, and by taking any maximal t-ideal M of D, [FG, Proposition 1.8] ensures that $M + P$ is a t-ideal of R, so that P is not a maximal t-ideal. □

Example 9. An example of a prime ideal J of $R[\{X_\alpha\}]$ such that $J^{-1} = (J:J)$, but $J^{-1} \not\subseteq R_{J \cap R}[\{X_\alpha\}]$. (Note that since J is prime, we have $Z(R[\{X_\alpha\}], J) = J$.) Let k be a field, and let Y and Z be indeterminates over k. Let $T = k(Y)[[Z]] = k(Y) + M$, where $M = ZT$, and let $R = k + M$. Then R is quasi-local with maximal ideal M. Let $J = M[\{X_\alpha\}]$. Then $J \cap R = M$, and by [HKLM, Theorem 1], $M^{-1} = (M:M)$ $(=T)$. Hence $J^{-1} = M^{-1}[\{X_\alpha\}] = T[\{X_\alpha\}] = (M:M)[\{X_\alpha\}] = (J:J)$, but $J^{-1} \not\subseteq R_M[\{X_\alpha\}] = R[\{X_\alpha\}]$. □

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The Dual of the Socle-Fine Notion and Applications

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1. Introduction

In [9] the authors introduced the notion of socle-fine class; a class of modules is said to be socle-fine if for all M and N in this class, M and N are isomorphic if and only if socle of M and socle of N are isomorphic. And they proved that a ring A is semi-artinian if and only if the class of injective modules is socle-fine.

In [10] we find the next results:

1) A ring A is semi-simple if and only if the class of quasi-injective modules is socle-fine, and if and only if the class of quasi-projective modules is socle-fine.

2) A ring A is Pseudo-Frobenius if and only if A is a left cogenerator and the class of projective modules is socle-fine.

3) A ring A is left noetherian V-ring if and only if the class of quasi-injective modules with large socle is socle-fine.

In [12] A. Kaidi, D.M. Barquero and C.M. Conzalez, proved the following remarkable results:

1) A ring A is left artinian if and only if the class of direct sum of injective A-modules is socle-fine.