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# 20 Duals of Ideals in Pullback Constructions

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## INTRODUCTION

Let $R$ be a domain with quotient field $K$. For a nonzero fractional ideal $I$ of $R$, the fractional ideal $I^{-1} = (R:I) = \{x \in K \mid xI \subseteq R\}$ is called the inverse (or dual) of $I$. In [HuP], Huckaba and Papick studied the question of when $I^{-1}$ is a ring, and this question has received further attention in [A1], [FHP1], [FHP2], [HeP], and [FHPR]. Of course, it is clear that $I^{-1}$ is a ring when $I = (I:I)$, and in case $I$ is prime, $I^{-1}$ is a ring $\Leftrightarrow I^{-1} = (I:I)$ [HuP, Proposition 2.3]. The question as to whether $I^{-1}$ a ring implies $I^{-1} = (I:I)$ in general was left open in [HuP]. This question was answered in the negative by D.F. Anderson [A1]. Anderson gave two counterexamples, one of which [A1, Example 3.2] involved the classical $D+M$ construction (see below for the description of this construction). The purpose of this paper is to study the question of when $I^{-1}$ is a ring in pullback constructions, paying particular attention to the classical $D+M$ construction.

We begin by fixing some notation. Let $T$ be a domain, let $M$ be an ideal of $T$ (which is not necessarily maximal), let $D$ be a domain which is a subring of $T/M$, let $\phi: T \to T/M$ denote the canonical epimorphism, and let $R$ be the pullback of the following diagram.

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & T/M. \end{array}$$

Thus $R = \phi^{-1}(D)$. We explicitly assume that $R \subsetneq T$. We shall refer to this as the <u>generic</u> <u>pullback</u> <u>diagram</u>. It is important to note that $M$ is a prime ideal of R (since $D \simeq R/M$ is a domain).

We shall often be interested in the case where $T$ is a valuation domain with maximal ideal $M$. Although it is possible to formulate results in this generality, in order to simplify notation and take advantage of readily available results, we shall assume (in this case) that $T = V$ is a valuation domain of the form $k+M$, where $k$ is a field and $M$ is the maximal ideal of $V$, and that $R = D+M$. We shall refer to this as the <u>classical</u> <u>$D+M$</u> <u>construction</u>. Our main reference for this construction is [BG].

In what follows, we shall often use the so-called $v$-operation on $R$. For a nonzero fractional ideal $I$ of $R$, $I_v$ is defined by $I_v = (I^{-1})^{-1}$. It will often be the case that an ideal is not only an ideal of the "base" ring $R$ but is also an ideal of a larger ring (say) $T$. In this case, it is understood that inverses and $v$'s are taken with respect to the base ring $R$. For properties of the $v$-operation, the reader is referred to [G, sections 32 and 34] and (especially with respect to the classical $D+M$ construction) to [BG]. Notation is standard as in [G].



**Theorem 1.** In the generic pullback diagram, let $k$ denote the quotient field of $D$. Suppose that either one of the following conditions holds:

(1) $k \subseteq T/M$, or

(2) $k \subsetneq (T/M)_S$, where $S = D \setminus \{0\}$ (equivalently, $R_M \subsetneq T_{R\setminus M}$).

Then $M^{-1} = (M:M)$.

Proof: Assume (1). Let $x \in M^{-1}$, $a \in M$. If $xa \notin M$, then $\phi(xa) \in D$, and $\phi(xa) \neq 0$. Hence by hypothesis, $\phi(xa)$ is a unit of $T/M$, and $\exists\ t_1 \in T$ with $\phi(t_1)\phi(xa) = 1 \in T/M$. Let $t \in T \setminus R$. Then $at_1t \in M$, so that $xat_1t \in R$, and we have $\phi(t) = \phi(t)\phi(t_1)\phi(xa) = \phi(xat_1t) \in D$. However, this implies that $t \in R$, a contradiction. This completes the proof of (1).

If we localize the given diagram, we obtain the following pullback diagram.

$$\begin{array}{ccc} R_M & \longrightarrow & D_S = k \\ \downarrow & & \downarrow \\ T_{R\setminus M} & \xrightarrow{\phi} & (T/M)_S. \end{array}$$

By (1), we have $(R_M : MR_M) = (MR_M : MR_M)$. It follows that if $x \in M^{-1}$ and $a \in M$, then $xa \in MR_M \cap R = M$, as desired. □

The following example shows that the converse of Theorem 1 is false.

**Example 2.** Let $X$ and $Y$ denote indeterminates over the field $\mathbb{Q}$ of rational numbers. Let $T = \mathbb{Q}[Y][X^2, X^3]$, $D = \mathbb{Z}[Y]$, $M = (X^2, X^3)T$, $R = D+M$, and $S = D \setminus \{0\}$. Then $T/M \simeq \mathbb{Q}[Y]$ and $(T/M)_S \simeq \mathbb{Q}(Y)$, the quotient field of $D$. Thus we see that neither condition (1) nor condition (2) is satisfied. However, we shall show that $M^{-1} = (M:M)$.

Let $T_1 = \mathbb{Q}(Y)[X]$, $M_1 = X^2T_1$, $D_1 = \mathbb{Q}(Y)$, and $R_1 = D_1 + M_1$. This yields the following pullback diagram.

$$\begin{array}{ccc} R_1 & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ T_1 & \xrightarrow{\phi} & T_1/M_1. \end{array}$$

By Theorem 1 (1), $M_1^{-1} = (M_1 : M_1) = T_1$. Note that $R_1 = R_S$ and $M_1 = MR_S$; that is, $(R_S : MR_S) = (MR_S : MR_S)$. Thus $MM^{-1} \subseteq M(R_S : MR_S) \subseteq MR_S$. Therefore, $MM^{-1} \subseteq MR_S \cap R = M$, and we have $M^{-1} = M : M$, as desired. □

In the most commonly used pullback construction, one has that $M$ is a maximal ideal of $T$. In this case, condition (1) of Theorem 1 is automatically satisfied. This yields the following corollary.

**Corollary 3.** In the generic pullback diagram, assume that $M$ is a maximal ideal of $T$. Then $M^{-1} = (M : M)$. □

One cannot omit the maximality assumption in Corollary 3, as the following example shows.

**Example 4.** Let $k$ be any field, and let $T = k[X, Y]$, $D = k[X^2, X^3]$, $M = YT$, and $R = D + M$. Then $(M : M) = T$. On the other hand, it is easy to see that $X^2/Y \in M^{-1} \setminus T$. □

For convenience, we record the following result as a corollary of Corollary 3. We hasten to add that the result is well known and follows easily from the fact that (in the situation considered) $M$ is the conductor of $T$ [F, Theorem 1.4].

**Corollary 5.** In the generic pullback diagram, assume that $M$ is a maximal ideal of $T$. Then $M$ is a divisorial ideal of $R$.

Proof: By Corollary 3, $M^{-1} = (M : M)$. Thus $M^{-1}$ is a ring, and by [HuP, Proposition 2.2], $M^{-1} = (M_v : M_v)$. Thus $T \subseteq (M_v : M_v)$, and $M_v$ is an ideal of $T$. Since $M$ is maximal in $T$, we have $M = M_v$. □ $M_v \subset R \subsetneq T$



We shall need the following result, which follows easily from [FG, Propositions 1.6 (a) and 1.8 (a)].

**Proposition 6.** In the generic pullback diagram, let $k$ denote the quotient field of $D$, let $J$ be a nonzero ideal of $D$, and let $I = \phi^{-1}(J)$. If $k \subseteq T/M$, then $I$ is an ideal of $R$ such that

(1) $I^{-1} = \phi^{-1}(J^{-1})$, and

(2) $(I : I) = \phi^{-1}(J : J)$.

**Theorem 7.** In the generic pullback diagram, assume that $M$ is a maximal ideal of $T$, and let $I$ be a nonzero ideal of $R$ with $I \subseteq M$. Consider the following statements.

(1) $(T : IT)$ is a ring.

(2) $I^{-1}$ is a ring.

(3) $I^{-1} = (R : IT)$ and $(R : IT)$ is a ring.

Then (1) implies (2), and (2) implies (3). Moreover, if $T$ is quasi-local with maximal ideal $M$ and $I_v \subsetneq M$, then the three statements are equivalent.

Proof: (1) $\Rightarrow$ (2): It suffices to show that $I^{-1} \supseteq (T : IT)$. Let $x \in (T : IT)$. Then $xI \subseteq xIT \subseteq T$. If $xI \nsubseteq M$, then, since $M$ is maximal in $T$, we can write $1 = tax + m$ for some $t \in T$, $a \in I$, and $m \in M$. Then $ax = ta^2x^2 + max$. Since $ax \in T$, $max \in M$, and since $(T : IT)$ is a ring, $tax^2 \in T$, and $ta^2x^2 \in IT \subseteq M$. It follows that $ax \in M$, a contradiction. Hence $xI \subseteq M \subseteq R$, and $I^{-1} \supseteq (T : IT)$, as desired.

(2) $\Rightarrow$ (3): Since $I^{-1}$ is a ring, we have $I^{-1} = (II^{-1} : II^{-1}) = (II^{-1})^{-1}$ by [HuP, Proposition 2.2] and [FHP3, Remark 2.3]. Since $I \subseteq M$, we have $T \subseteq M^{-1} \subseteq I^{-1}$, whence $I \subseteq IT \subseteq II^{-1}$, and $(II^{-1})^{-1} \subseteq (R : IT) \subseteq I^{-1}$. It follows that $(R : IT)$ is

a ring and that $I^{-1} = (R:IT)$.

(3) $\Rightarrow$ (1) (assuming that $T$ is quasi-local with maximal ideal $M$ and $I_v \subsetneq M$): It is clear that $(R:IT) \subseteq (T:IT)$. For the reverse inclusion, it suffices to show that $(T:IT)IT \subseteq M$, and for this it suffices to show that $IT$ is not principal in $T$. Suppose, on the contrary, that $IT = aT$ for some $a \in IT$. Since $(R:IT)$ is a ring, we have by [HuP, Proposition 2.2] that $(R:IT) = ((IT)_v:(IT)_v) = ((aT)_v:(aT)_v) = (T_v:T_v)$. Since $M = T^{-1}$ and $M^{-1}$ is a ring (Corollary 3), this yields $(R:IT) = (M^{-1}:M^{-1}) = M^{-1}$. Thus $M^{-1} = I^{-1}$, and $I_v = M_v = M$ (Corollary 5), a contradiction. Therefore, $IT$ is not principal. □

Example 8. This example shows the necessity of the assumption $I^{-1} = (R:IT)$ in Theorem 7 (3). Let $k$ be a field, and let $X$ and $Y$ be indeterminates over $k$. Set $T = k(X)[[Y^2, \ Y^3]]$, $M = (Y^2, \ Y^3)T = Y^2k(X)[[Y]]$, $D = k[X^2, \ X^3]$, and $R = D + M$; and let $I$ be the ideal of $R$ given by $I = Y^2(k[X] + M)$. It is easy to see that $X^2/Y^2 \in I^{-1}$. On the other hand, $(X^2/Y^2)Y^3 = X^2Y \notin R$, so that $X^2/Y^2 \notin M^{-1}$. It follows that $I_v \subsetneq M$. Note that $IT = Y^2T$, so that $(T:IT)$ is not a ring. (Thus by Theorem 7, $I^{-1}$ is not a ring either.) Finally, $(R:IT) = (R:Y^2T) = Y^{-2}(R:T) = Y^{-2}M = Y^{-2}Y^2k(X)[[Y]] = k(X)[[Y]]$, and $(R:IT)$ is a ring. □

Theorem 9. In the generic pullback diagram, assume that $M$ is a maximal ideal of $T$ and that $M^{-1} = T$. Let $I$ be a nonzero ideal of $R$ such that $I_v \subsetneq M$ and $I^{-1}$ is a ring. Then $(T:IT)$ is not a ring $\Leftrightarrow$ $M$ is invertible in $T$ and $I_v = Q \cap M$ for some ideal $Q$ of $T$ with $Q \nsubseteq M$.

Proof: Since $I^{-1}$ is a ring, $I^{-1}I_v = I_v$ [HuP, Proposition 2.2]. Hence $I_vT = I_vM^{-1} \subseteq I_vI^{-1} = I_v$, and $I_v$ is an ideal of $T$. Next, we show that $(T:I_v) = (T:IT)$. Let $x \in (T:IT)$. Then $xI \subseteq T = M^{-1}$, and $xIM \subseteq R$. It follows that $xI_vM \subseteq R$, whence $xI_v \subseteq M^{-1} = T$. Thus $x \in (T:I_v)$, and we have $(T:IT) \subseteq (T:I_v)$. The reverse inclusion is trivial.



Now suppose that $(T:IT)$ is not a ring. Let $Q = I_v(T:I_v)$. Since $(T:IT)$ is not a ring (and $I^{-1}$ is a ring), we have $(T:IT) \supsetneq I^{-1}$. Hence $(T:IT)IT \nsubseteq M$. Thus $Q \nsubseteq M$. Since $Q + M = T$, $Q \cap M = QM$. Now $I_v = I_vI^{-1} \subseteq Q$, and $I_v \subseteq Q \cap M$. On the other hand, $I^{-1}QM = I^{-1}I_v(T:I_v)M = I_v(T:I_v)M = QM \subseteq R$, and $QM \subseteq I_v$. Hence $I_v = Q \cap M$. If $M$ is not invertible in $T$, then $M(T:M) = M \subseteq R$, and we have $(T:M) \subseteq M^{-1} = T$, whence $(T:M) = T$. Hence $(T:IT) = (T:I_v) = (T:QM) = ((T:M):Q) = (T:Q)$. However, by [FHP3, Remark 2.3], $(T:Q)$ is a ring, which yields a contradiction. Thus $M$ is invertible in $T$.

For the converse, note that $I_v = Q \cap M = QM$. Hence $I_v(T:I_v) = QM(T:QM) \supseteq QM(T:M) = Q$ (since $M$ is invertible in $T$). On the other hand, $I_v(R:I) = I_v \subsetneq Q$. Thus $(T:IT) = (T:I_v) \neq (R:I) = (R:IT)$. By the proof of (1) $\Rightarrow$ (2) in Theorem 7, $(T:IT)$ is not a ring. □

Example 10. This example shows that the situation described in Theorem 9 can actually occur. It follows that condition (3) does not imply condition (1) in Theorem 7. Let $k$ be a field, and let $X$ and $Y$ be indeterminates over $k$. Set $T = k[Y] + Xk(Y)[X]$, $M = (X+1)T$, and $Q = Xk(Y)[X]$. Then $M$ is a maximal ideal of $T$ by [CMZ, Theorem 4.21]. Let $I = Q \cap M = QM$. Since $1/(X+1) \in (T:I)$ but $1/(X+1)^2 \notin (T:I)$, $(T:IT) = (T:I)$ is not a ring. Note that $T/M \approx k(Y)$. Now let $R$ be the pullback of the following diagram.

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & T/M. \end{array}$$

Here, we have identified $k$ with the isomorphic copy of $k$ contained in $T/M$. Of course, $I$ is an ideal of $R$. Since $M$ is a nonprincipal maximal ideal of $R$ and a principal ideal of $T$, $(R:M) = (M:M) = T$, and $(R:I) = (R:MQ) = ((R:M):Q) = (T:Q) = k(Y)[X]$. Thus $(R:I)$ is a ring. Also, $((R:(R:I)) = (R:k(Y)[X]) \subseteq (T:k(Y)[X]) = Q$. Thus $I_v \subseteq Q$. Hence $I_v \subseteq Q \cap M = I \subsetneq M$. □

For the remainder of the paper, we shall consider the case where $T$ is quasi-local with maximal ideal $M$. In this case, it is well known that each ideal of $R$ compares with $M$. It is clear from Proposition 6 (using the notation of that result) that if $I \supsetneq M$ the questions about whether $I^{-1}$ is a ring and whether $I^{-1} = (I:I)$ boil down to the corresponding questions about $J$ in the ring $D$. Of course, for $M$ itself, we have $M^{-1} = (M:M)$ by Corollary 3. Hence we shall concentrate on the case $I \subsetneq M$.

**Corollary 11.** In the classical $D+M$ construction, if $I$ is a nonzero ideal of $R$ with $I_v \subsetneq M$, then the following statements are equivalent.
(1) $I^{-1}$ is a ring.
(2) $I$ is a nonzero prime ideal of $R$.
(3) $I^{-1} = (I:I)$.
Proof: (1) $\Rightarrow$ (2): We have $IV \subseteq I_vV \subseteq MV = M \subseteq R$, so that $V \subseteq I^{-1}$. Since $I^{-1}$ is a ring, $I^{-1} = (I_v : I_v)$ [HuP, Proposition 2.2]. Hence $I_v$ is an ideal of $V$. By Theorem 7, $(V:IV)$ is a ring, whence $IV$ is a nonprincipal prime ideal of $V$ [HuP, Proposition 3.5]. Since $IV$ is not principal, $I = I_v$ by [BG, Theorem 4.3 (1)]. It follows that $I = IV$ is an ideal of $V$ and therefore a prime ideal of $R$.
(2) $\Rightarrow$ (3): Of course, $I$ is also a nonmaximal prime ideal of $V$. Hence $(V:IV) = (V:I) = (I:I)$ [HuP, Theorem 3.8]. Thus $(V:IV)$ is a ring, and by (the proof of) Theorem 7, $I^{-1} = (V:IV) = (I:I)$.
(3) $\Rightarrow$ (1): This is clear. □

**Example 12.** This example shows that it is not enough to assume that $T$ is quasi-local with maximal ideal $M$ in Corollary 11. Let $k$ be a field, and let $X$ be an indeterminate over $k$. Set $T = k[[X^3, X^5, X^7]]$ and $M = (X^3, X^5, X^7)T$. Let $D$ be any proper subring of $k$, set $R = D + M$, and let $I$ be the ideal of $R$ given by $I = (X^5, X^6)T$. It is straightforward to show that $I^{-1} = k[[X]]$. On the other hand, $XX^6 = X^7 \notin I$; hence $X \notin (I:I)$, and $I^{-1} \supsetneq (I:I)$. Finally, since $XX^3 = X^4 \notin T$, we have $X \notin M^{-1}$. Hence $I^{-1} \neq M^{-1}$, from which it follows that $I_v \subsetneq M$. □



**Theorem 13.** In the generic pullback diagram, assume that $T$ is quasi-local with maximal ideal $M$, and let $I$ be a nonzero ideal of $R$ with $I_v = M$. Then $I^{-1} = M^{-1} = (M:M) = (R:IT)$. If, in addition, $M$ is not principal in $(M:M)$, then $I^{-1} = (T:IT)$.
Proof: We have $I^{-1} = M^{-1} = (M:M) = (R:IT)$ by Theorem 7 and Corollary 3. Hence $(IT)_v = M_v = M$ by Corollary 5. Suppose that $M$ is not principal in $(M:M)$. We shall show that $IT$ is not principal in $T$. If $IT = aT$ is principal, then $(R:IT) = (R:aT) = a^{-1}M$, and we have $M = a(M:M)$, a contradiction. Hence $IT$ is not principal. It follows (as in the proof of (3) $\Rightarrow$ (1) in Theorem 7) that $I^{-1} = (R:IT) = (T:IT)$. (For the case of the classical $D+M$ construction, this last equality appears in [A2, Lemma 5].) □

**Example 14.** This example shows the necessity of the assumption in Theorem 13 that $M$ be nonprincipal in $(M:M)$. Let $k$ be a field, and let $X$ and $Y$ be indeterminates over $k$. Set $T = k(X)[[Y]]$ and $M = YT$. Let $D$ be any proper subring of $k$, let $R = D + M$; and let $I$ be the ideal of $R$ given by $I = Y(D[X] + M)$. By Proposition 18 below, $I_v = M$. However, since $IT = YT = M$ is a principal ideal of $T$, $(T:IT)$ is not a ring. □

**Proposition 15.** In the classical $D+M$ construction, assume that $I$ is a nonzero ideal of $R$ with $I \subsetneq M$ and $I_v = M$. Then $I^{-1} = M^{-1} = (M:M) = V$, and $IV = M = cV$ for some $c \in I$; moreover, $I^{-1} \neq (I:I)$.
Proof: By Theorem 13, we have $I^{-1} = M^{-1} = (M:M)$, and $M^{-1} = V$ by [HuP, Corollary 3.4]. If $IV$ is not principal, then by [BG, Theorem 4.3 (1)], we have $I = I_v = M$, contrary to hypothesis. Thus $IV = cV$, and we may take $c \in I$. Now $IV$ is an ideal of $R$, and since $I \subseteq IV \subseteq M$, we must have $(IV)_v = M$.

Hence $M = (cV)_v = (cM^{-1})_v = cM^{-1} = cV$. Finally, we show that $I^{-1} \neq (I:I)$. Since $(I:I) \subseteq I^{-1} = V$, this amounts to showing that $I$ is not an ideal of $V$. However, this follows from the fact that any ideal $J$ of $V$ is automatically divisorial as an ideal of $R$. That this is true follows from [BG, Theorem 4.3 (1)] if $J$ is not principal in $V$; and if $J = aV$ is principal, then $J = aM^{-1}$, which is clearly divisorial. □

From Corollary 11 and Proposition 15, we have

**Corollary 16.** In the classical $D+M$ construction, assume that $M$ is nonprincipal in $V$. If $I$ is a nonzero ideal of $R$ with $I \subseteq M$, then the following statements are equivalent.

(1) $I^{-1}$ is a ring.

(2) $I$ is prime.

(3) $I^{-1} = (I:I)$. □

**Example 17.** This example shows that it is not enough to assume that $T$ is quasi-local with maximal ideal $M$ in Proposition 15. Let $k$ be a field, and let $X$ and $Y$ be indeterminates over $k$. Set $T = k[[X, Y]]$; then $T$ is a two-dimensional regular local ring with maximal ideal $M = (X, Y)T$. Let $D$ be any proper subring of $k$, set $R = D + M$, and let $I$ be the ideal of $R$ given by $I = M^2$. Then $I \subsetneq M$. Note that $I^{-1} \subseteq (T:I) = T \subseteq (I:I) \subseteq I^{-1}$; hence $I^{-1} = (I:I) = T$. Similarly, we have $M^{-1} = T$. Thus $I_v = M_v = M$. □

We wish to examine in greater detail the situation described in Proposition 15. Recall that in [HuP] Huckaba and Papick asked whether $I^{-1}$ a ring implies $I^{-1} = (I:I)$ and that D.F. Anderson gave a classical $D+M$ counterexample [A1, Example 3.2]. Among other things, our next result shows that Anderson's construction was essentially the only one possible.



**Proposition 18.** Let $I$ be an ideal in the classical $D+M$ construction, and assume that $M = cV$ is principal in $V$. Then $I$ is as in Proposition 15 ($I \subsetneq M$ and $I_v = M$) $\Leftrightarrow$ $I = cW + cM$, where $W$ is a $D$-submodule of $k$ such that $D \subseteq W \subsetneq k$ and such that $W$ is not a fractional ideal of $D$.

Proof: ($\Rightarrow$) From [BG, Theorem 4.3 (2)], such an ideal $I$ must have the form $I = cW + cM$ for some $D$-submodule $W$ of $k$ with $D \subseteq W \subsetneq k$; moreover, if $W$ is a fractional ideal of $D$, then $I_v = cW_v + cM$. However, since $W_v$ must be a proper subset of $k$, this gives $I_v \subsetneq cV = M$, a contradiction. Hence for such an $I$, we must have that $W$ is a $D$-module which is not a fractional ideal.

($\Leftarrow$) By [BG, Theorem 4.3 (2)], $I \subsetneq I_v = cV = M$, as desired. □

It is perhaps not surprising that one can have $I^{-1}$ be a ring with $I^{-1} \neq (I:I)$, since $I^{-1}$ is divisorial, whereas $(I:I)$ need not be divisorial. As the following example shows, however, it is possible for $I^{-1}$ to be a ring with $I^{-1} \neq (I:I)_v$.

**Example 19.** Let $F$ be a field, and let $k = F(t)$, where $t$ is an indeterminate over $F$. Let $V = k + M$ be a valuation domain with maximal ideal $M = cV$, and let $R = F + M$. Finally, let $W = F + Ft$, and set $I = cW + cM$. Then, since $W$ is an $F$-submodule of $k$ which is not a fractional ideal of $F$, Proposition 18 implies that $I \subsetneq M$, and $I^{-1} = M^{-1} = V$. However, $(I:I) = (W:_k W) + M = R$, whence $(I:I)_v = R \subsetneq I^{-1}$. □

In [DF] Dobbs and Fedder call a domain $R$ with quotient field $K$ conducive if each overring $T \neq K$ of $R$ satisfies $(R:T) \neq 0$. This provides a convenient framework for our last result.

**Proposition 20.** In the classical $D+M$ construction, $R$ does not admit ideals of the type described in Proposition 15 if and only if

(1) $M$ is a not a principal ideal of $V$; or

(2) $M$ is a principal ideal of $V$, and $D$ is of one of the following types:

(a) $D$ is a conducive domain with quotient field $k$.

(b) $D$ is a field and $k$ is an algebraic extension of $D$ of degree 2.

Proof: ($\Leftarrow$) If $M$ is not principal in $V$, then ideals of the type described in Proposition 15 cannot exist. Suppose that M is principal in V. If $D$ is as described in (a), then by [BG, Theorem 4.5] or [DF], each $D$-submodule $W$ of $k$ such that $D \subseteq W \underset{\neq}{\subset} k$ is a fractional ideal of $D$; hence by Proposition 18, there can be no ideals of the type described in Proposition 15. If $D$ is as described in (b), then any such $W$ is a $D$-vector space between $D$ and $k$, whence $W = D$ or $W = k$. If $W = D$, we have $I = cW + cM = cR$, so that $I_v = cR \underset{\neq}{\subset} M$; and if $W = k$, we have $I = cW + cM = cV = M$. Either way, there are no ideals of the type described in Proposition 15.

($\Rightarrow$) We assume that $M = cV$ is principal in $V$. Suppose that $D$ is not as described in (a) or (b). If $D$ is a field, then we may choose $u \in k \setminus D$ with $W = D + Du \underset{\neq}{\subset} k$. It is clear that $W$ is not a fractional ideal of $D$. Set $I = cW + cM$. It is easy to see that $IV = cV = M$, and by [BG, Theorem 4.3 (2)], $I \underset{\neq}{\subset} I_v = cV$. If $D$ is not a field but there exists a field $F$ properly between $D$ and $k$, then we may take $I = cF + cM$; again $I$ is of the type described in Proposition 15. Finally, if $D$ is a nonconducive domain with quotient field $k$, then by [BG, Theorem 4.5] or [DF] there is a $D$-module $W$ with $D \subseteq W \underset{\neq}{\subset} k$ such that $W$ is not a fractional ideal of $D$. As before, set $I = cW + cM$. Then $I$ is of the type described in Proposition 15. □

**Remark 21.** We note that it is not enough in Proposition 20 to assume that $V$ is quasi-local with maximal ideal $M$; this follows from Examples 14 and 17.



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# 21 Cancellation and Prime Spectra

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## 1. BACKGROUND AND DEFINITIONS

Two commutative rings with identity A and B are said to be stably equivalent if there is an integer n such that the polynomial rings $A[x_1,\ldots,x_n]$ and $B[y_1,\ldots,y_n]$ are isomorphic as rings. A natural question was asked by Coleman and Enochs (1971):

Can we cancel the indeterminates?

or

Does A and B stably equivalent imply that A and B are isomorphic?

Much of the fundamental work in this area was done in the 1970s and 1980s. In a series of articles beginning in 1972, Eakin and Heinzer (1972), Abhyankar, Eakin and Heinzer (1972), Brewer and Rutter (1972), and later others, examined this cancellation problem and obtained significant steps toward a positive solution. Hochster (1972) and Asanuma (1982) independently gave examples of stably equivalent rings which are not isomorphic. We outline Hochster's example to show how non-cancellation in one venue is transformed to non-cancellation in another. Hochster's example is based on generating non-isomorphic R-modules M and N and a free module F so that $M\oplus F \cong N\oplus F$. Stably equivalent rings are generated by the symmetric algebras:

$$S(M)[x_1,\ldots,x_n] = S(M\oplus F) \cong S(N\oplus F) = S(N)[y_1,\ldots,y_n].$$

Hochster realized that, if the modules M and N are not isomorphic, then the corresponding symmetric algebras S(M) and S(N) are not isomorphic. Thus the inability to "cancel" the free modules in the module problem led to the inability to cancel the indeterminates in the ring problem.