ESSENTIAL DOMAINS
AND TWO CONJECTURES IN DIMENSION THEORY

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ABSTRACT. This note investigates two long-standing conjectures on the Krull
dimension of integer-valued polynomial rings and of polynomial rings in the
context of (locally) essential domains.

1. INTRODUCTION

Let R be an integral domain with quotient field K, and let $\mathrm{Int}(R) := \{f \in K[X] : f(R) \subseteq R\}$ be the ring of integer-valued polynomials over R. Considerable work, part of it summarized by Cahen and Chabert [7], has been concerned with various aspects of integer-valued polynomial rings. A central question concerning $\mathrm{Int}(R)$ is to describe its prime spectrum and, hence, to evaluate its Krull dimension. Several authors tackled this problem, and many satisfactory results were obtained for various classes of rings such as Dedekind domains [9, 10], Noetherian domains [10], valuation and pseudo-valuation domains [8], and pseudo-valuation domains of type n [33]. A well-known feature is that $\dim(R[X]) - 1 \leq \dim(\mathrm{Int}(R))$ for any integral domain R [5]. However, the problem of improving the upper bound $\dim(\mathrm{Int}(R)) \leq \dim_v(R[X]) = \dim_v(R) + 1$ [16], where $\dim_v(R)$ denotes the valuative dimension of R, is still elusively open in general. This is due, in part, to the fact that the fiber in $\mathrm{Int}(R)$ of a maximal ideal of R may have any dimension [5, Example 4.3] (this stands for the main difference between polynomial rings and integer-valued polynomial rings). It is noteworthy that all examples conceived in the literature for testing $\dim(\mathrm{Int}(R))$ satisfy the inequality $\dim(\mathrm{Int}(R)) \leq \dim(R[X])$. In [16, 17], we undertook an extensive study, using two different approaches, in order to grasp this phenomenon. We then obtained further evidence for the validity of the conjecture

$$(\mathcal{C}_1) \qquad \dim(\mathrm{Int}(R)) \leq \dim(R[X]) \text{ for any integral domain } R.$$
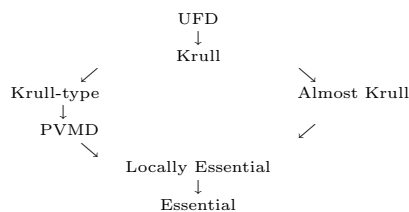
The current situation can be described as follows. $(\mathcal{C}_1)$ turned out to be true in three large (presumably different) classes of commutative rings, namely, (a) Krull-type domains, e.g., unique factorization domains (UFDs) or Krull domains [22, 17]; (b) pseudo-valuation domains of type n [17]; and (c) Jaffard domains [16, 5].

A finite-dimensional domain R is said to be Jaffard if $\dim(R[X_1, ..., X_n]) = n + \dim(R)$ for all $n \geq 1$; equivalently, if $\dim(R) = \dim_v(R)$ [1, 4, 14, 19, 27]. The class of Jaffard domains contains most of the well-known classes of finite-dimensional rings involved in dimension theory of commutative rings, such as Noetherian domains [29], Prüfer domains [19], universally catenarian domains [3], and stably strong S-domains [28, 30]. However, the question of establishing or denying a possible connection to the family of Krull-like domains is still unsolved. In this vein, Bouvier's conjecture (initially announced during a 1985 graduate course at the University of Lyon I) states that

$(\mathcal{C}_2)$ *finite-dimensional Krull domains, or more particularly UFDs, need not be Jaffard domains.*

Since the Krull property is stable under adjunction of indeterminates, the problem merely deflates to the existence of a Krull domain R with $1 + \dim(R) \lneqq \dim(R[X])$. It is notable that the rare non-Noetherian finite-dimensional UFDs or Krull domains existing in the literature do defeat $(\mathcal{C}_2)$, since all are Jaffard [2, 12, 13, 14, 18, 20]. So do the examples of non-Prüfer finite-dimensional Prüfer v-multiplication domains (PVMDs) [21, 26, 31, 34]; as a matter of fact, these mainly arise as polynomial rings over Prüfer domains or as pullbacks, and both settings either yield Jaffard domains or turn out to be inconclusive (in terms of allowing the construction of counterexamples) [1, 15]. In order to find the missing link, one has then to dig beyond the context of PVMDs.

Essential domains happen to offer such a suitable context for $(\mathcal{C}_2)$ as well as a common environment for both conjectures $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$, though these have developed in two dissimilar milieus. An integral domain R is said to be *essential* if R is an intersection of valuation rings that are localizations of R [23]. Since this notion does not carry up to localizations, R is said to be *locally essential* if R_p is essential for each $p \in \mathrm{Spec}(R)$. Notice that the locally essential domains correspond to the P-domains in the sense of Mott and Zafrullah [31]. PVMDs and almost Krull domains [19, p. 538] are perhaps the most important examples of locally essential domains. Recall that Heinzer constructed in [24] an example of an essential domain that is not locally essential. Also, it is worth noticing that Heinzer and Ohm's example [25] of an essential domain that is not a PVMD is, in fact, locally essential (cf. [31, Example 2.1]). Finally, recall that a Krull-type domain is a PVMD in which no nonzero element belongs to an infinite number of maximal t-ideals [21]. We have thus the implications shown in the diagram within the family of Krull-like domains:

$$
\begin{array}{ccc}
 & \mathrm{UFD} & \\
 & \downarrow & \\
 & \mathrm{Krull} & \\
\swarrow & & \searrow \\
\mathrm{Krull\text{-}type} & & \mathrm{Almost\ Krull} \\
\downarrow & & \\
\mathrm{PVMD} & & \swarrow \\
\searrow & & \\
 & \mathrm{Locally\ Essential} & \\
 & \downarrow & \\
 & \mathrm{Essential} &
\end{array}
$$

The purpose of this note is twofold. First, we state a result that widens the domain of validity of $(\mathcal{C}_1)$ to the class of locally essential domains. It is well known that $(\mathcal{C}_1)$ holds for Jaffard domains too [16, 5]. So one may enlarge the scope of study of $(\mathcal{C}_2)$, discussed above, and legitimately raise the following problem:

$(\mathcal{C}'_2)$ *Is every finite-dimensional (locally) essential domain Jaffard?*

Clearly, an affirmative answer to $(\mathcal{C}'_2)$ will definitely defeat $(\mathcal{C}_2)$, while a negative answer will partially resolve $(\mathcal{C}_2)$ for the class of (locally) essential domains. Our second aim is to show that the rare constructions of nontrivial (locally) essential domains (i.e., non-PVMD) existing in the literature yield Jaffard domains, making $(\mathcal{C}'_2)$ an open problem. Consequently, a settlement of $(\mathcal{C}_2)$ seems out of reach at present.

2. Result and example

In the first part of this section, we establish the following result.

Theorem 2.1. *For any locally essential domain R, $\dim(\mathrm{Int}(R)) = \dim(R[X])$.*

Proof. Assume that R is finite-dimensional and $R \neq K$, where K denotes the quotient field of R. Let $R = \bigcap_{p \in \Delta} R_p$ be a locally essential domain, where $\Delta \subseteq \mathrm{Spec}(R)$. Set

$$\Delta_1 := \{p \in \Delta : R_p \text{ is a DVR}\},$$
$$\Delta_2 := \{p \in \Delta : R_p \text{ is a valuation domain but not a DVR}\}.$$

We wish to show first that $\dim(\mathrm{Int}(R)) \leq \dim(R[X])$. Let M be a maximal ideal of $\mathrm{Int}(R)$ such that $\dim(\mathrm{Int}(R)) = \mathrm{ht}(M)$ and let $\mathcal{M} := M \cap R$. Without loss of generality, we may assume that $\mathcal{M}$ is maximal in R with a finite residue field. We always have $R_{\mathcal{M}}[X] \subseteq (\mathrm{Int}(R))_{\mathcal{M}} \subseteq \mathrm{Int}(R_{\mathcal{M}})$ [6, Corollaires (4), p. 303]. If $\mathcal{M} \in \Delta_1$, then $R_{\mathcal{M}}[X]$ is a two-dimensional Jaffard domain ([32, Theorem 4] and [1, Proposition 1.2]). So the inclusion $R_{\mathcal{M}}[X] \subseteq (\mathrm{Int}(R))_{\mathcal{M}}$ yields $\dim(((\mathrm{Int}(R))_{\mathcal{M}}) \leq \dim_v(R_{\mathcal{M}}[X]) = \dim(R_{\mathcal{M}}[X])$. Thus, $\dim(((\mathrm{Int}(R))_{\mathcal{M}}) = \dim(R_{\mathcal{M}}[X]) = 2$. If $\mathcal{M} \in \Delta_2$, then $\mathrm{Int}(R_{\mathcal{M}}) = R_{\mathcal{M}}[X] = (\mathrm{Int}(R))_{\mathcal{M}}$ [6, Exemples (5), p. 302]. If $\mathcal{M} \notin \Delta$, then $R_{\mathcal{M}} = \bigcap_{p \in \Delta, p \subsetneq \mathcal{M}} R_p$, since R is a locally essential domain. So, by [6, Corollaires (3), p. 303], $\mathrm{Int}(R_{\mathcal{M}}) = \bigcap_{p \in \Delta, p \subsetneq \mathcal{M}} \mathrm{Int}(R_p) = \bigcap_{p \in \Delta, p \subsetneq \mathcal{M}} R_p[X] = R_{\mathcal{M}}[X] = (\mathrm{Int}(R))_{\mathcal{M}}$. In all cases, we have $\dim(\mathrm{Int}(R)) = \dim(((\mathrm{Int}(R))_{\mathcal{M}}) = \dim(R_{\mathcal{M}}[X]) \leq \dim(R[X])$, as desired.

We now establish the inverse inequality $\dim(R[X]) \leq \dim(\mathrm{Int}(R))$. Let M be a maximal ideal of $R[X]$ such that $\dim(R[X]) = \mathrm{ht}(M)$, and $\mathcal{M} := M \cap R$. Necessarily, $\mathcal{M}$ is maximal in R. Furthermore, we may assume that $\mathcal{M}$ has a finite residue field. If $\mathcal{M} \in \Delta_2$ or $\mathcal{M} \notin \Delta$, similar arguments as above lead to $R_{\mathcal{M}}[X] = (\mathrm{Int}(R))_{\mathcal{M}}$ and hence to the conclusion. Next, suppose $\mathcal{M} \in \Delta_1$. Then, $\mathrm{Int}(R_{\mathcal{M}})$ is a two-dimensional (Prüfer) domain [11]. Let $(0) \subsetneq P_1 \subsetneq P_2$ be a maximal chain of prime ideals in $\mathrm{Spec}(\mathrm{Int}(R_{\mathcal{M}}))$. Clearly, it contracts to $(0) \subsetneq \mathcal{M}R_{\mathcal{M}}$ in $\mathrm{Spec}(R_{\mathcal{M}})$. Furthermore, by [5, Corollaire 5.4], P_1 contracts to (0). Therefore, by [10, Proposition 1.3], $P_1 = fK[X] \cap \mathrm{Int}(R_{\mathcal{M}})$, for some irreducible polynomial $f \in K[X]$. This yields in $\mathrm{Spec}(R_{\mathcal{M}}[X])$ the maximal chain

$$(0) \subsetneq P_1 \cap R_{\mathcal{M}}[X] = fK[X] \cap R_{\mathcal{M}}[X] \subsetneq P_2 \cap R_{\mathcal{M}}[X],$$

which induces in $\mathrm{Spec}((\mathrm{Int}(R))_{\mathcal{M}})$ the following maximal chain:

$$(0) \subsetneqq P_1 \cap (\mathrm{Int}(R))_{\mathcal{M}} = fK[X] \cap (\mathrm{Int}(R))_{\mathcal{M}} \subsetneqq P_2 \cap (\mathrm{Int}(R))_{\mathcal{M}}.$$

Consequently, in all cases, we obtain $\dim(R[X]) = \dim(R_{\mathcal{M}}[X]) = \dim(((\mathrm{Int}(R))_{\mathcal{M}})$ $\leq \dim(\mathrm{Int}(R))$, to complete the proof of the theorem. $\square$

From [25, Proposition 1.8] and [19, Exercise 11, p. 539] we obtain the following.

Corollary 2.2. *Let R be a PVMD or an almost Krull domain. Then $\dim(\mathrm{Int}(R)) = \dim(R[X])$.*

In the second part of this section, we test the problem $(\mathcal{C}'_2)$, set up and discussed in the introduction, for the class of non-PVMD (locally) essential domains. These occur exclusively in the Heinzer-Ohm example [25] and Heinzer's example [24], both mentioned above. The first of them is a 2-dimensional Jaffard domain [31, Example 2.1]. Heinzer's example [24], too, is a 2-dimensional Jaffard domain by [14, Theorem 2.3]. Our next example shows that an enlargement of the scope of this construction still generates a large family of essential domains with nonessential localizations of arbitrary dimensions ≥ 2, but unfortunately they are Jaffard domains.

Example 2.3. For any integer $r \geq 2$, there exists an r-dimensional essential Jaffard domain D that is not locally essential.

Notice first that the case $r = 2$ corresponds to Heinzer's example mentioned above. In order to increase the dimension, we modify Heinzer's original setting by considering Kronecker function rings via the b-operation. For the sake of completeness, we give below the details of this construction.

Let R be an integral domain, K its quotient field, n a positive integer (or $n = \infty$), and $X, X_1, ..., X_n$ indeterminates over K. The b-operation on R is the a.b. star operation defined by $I^b := \bigcap \{IW : W \text{ is a valuation overring of } R\}$, for every fractional ideal I of R. Throughout, we shall use $\mathrm{Kr}_{K(X)}(R, b)$ to denote the Kronecker function ring of R defined in $K(X)$ with respect to the b-operation on R, and $R(X_1, ..., X_n)$ to denote the Nagata ring associated to the polynomial ring $R[X_1, ..., X_n]$, obtained by localizing the latter with respect to the multiplicative system consisting of polynomials whose coefficients generate R.

Let r be an integer ≥ 2. Let k_0 be a field, and let $\{X_n : n \geq 1\}, Y, \{Z_1, ..., Z_{r-1}\}$ be indeterminates over k_0. Let n be a positive integer. Set:

$$
\begin{aligned}
&k_n := k_0(X_1, ..., X_n), &\quad &k := \bigcup_{n \geq 1} k_n, \\
&F_n := k_n(Z_1, ..., Z_{r-1}), &\quad &F := \bigcup_{n \geq 1} F_n = k(Z_1, ..., Z_{r-1}), \\
&K_n := F_n(Y), &\quad &K := \bigcup_{n \geq 1} K_n = F(Y), \\
&M_n := YF_n[Y]_{(Y)}, &\quad &M := \bigcup_{n \geq 1} M_n = YF[Y]_{(Y)}, \\
&A_n := k_n + M_n, &\quad &A := \bigcup_{n \geq 1} A_n = k + M, \\
&V_n := F_n[Y]_{(Y)}, &\quad &V := \bigcup_{n \geq 1} V_n = F[Y]_{(Y)}.
\end{aligned}
$$

Note that, for each $n \geq 1$, V and V_n (resp., A and A_n) are one-dimensional discrete valuation domains (resp., pseudo-valuation domains) and $\dim_v(A) = \dim_v(A_n) = r$. For each $n \geq 1$, set $X'_n := \frac{1+YX_n}{Y}$. Clearly, $K_n = K_{n-1}(X_n) = K_{n-1}(X'_n)$. Next, we define inductively two sequences of integral domains $(B_n)_{n \geq 2}$

and $\big(D_n\big)_{n\geq 1}$ as follows:

$$D_1 := A_1,$$

$$B_2 := \mathrm{Kr}_{K_1(X_2')}(D_1, b), \qquad D_2 := B_2 \cap A_2,$$

$$B_n := \mathrm{Kr}_{K_{n-1}(X_n')}(D_{n-1}, b), \qquad D_n := B_n \cap A_n, \text{ for } n \geq 3.$$

For $n \geq 2$, let $\mathcal{M}_n := M_n \cap D_n$ ($\subset D_n = B_n \cap A_n \subseteq A_n$), where M_n is the maximal ideal of A_n.

Claim 2.4. (1) B_n is an r-dimensional Bézout domain.
  (2) $B_n \cap K_{n-1} = D_{n-1} \subseteq A_{n-1} = A_n \cap K_{n-1}$.
  (3) $D_n \cap K_{n-1} = D_{n-1}$.
  (4) $D_n[X_n'] = B_n$ and $(D_n)_{\mathcal{M}_n} = D_n[\frac{1}{YX_n'}] = A_n$, with $\frac{1}{X_n'}$ and $YX_n' \in D_n$.
  (5) $\mathcal{M}_n$ is a height-one maximal ideal of D_n with $\mathcal{M}_n \cap K_{n-1} = \mathcal{M}_{n-1}$.
  (6) For each $q \in \mathrm{Spec}(D_n)$ with $q \neq \mathcal{M}_n$ there exists a unique prime ideal $Q \in \mathrm{Spec}(B_n)$ contracting to q in D_n with the property that $(D_n)_q = (B_n)_Q$.
  (7) $B_n = \bigcap\{(D_n)_q : q \in \mathrm{Spec}(D_n) \text{ and } q \neq \mathcal{M}_n\}$.
  (8) For each $q' \in \mathrm{Spec}(D_{n-1})$ with $q' \neq \mathcal{M}_{n-1}$ there exists a unique prime ideal $q \,(\neq \mathcal{M}_n) \in \mathrm{Spec}(D_n)$ contracting to q' in D_{n-1} such that $(D_n)_q = (D_{n-1})_{q'}(X_n')$.

Proof. Similar arguments as in [24] lead to (1)-(7).
  (8) By (7), $B_{n-1} \subseteq (D_{n-1})_{q'}$, and hence $(D_{n-1})_{q'}$ is a valuation domain in K_{n-1} of dimension $\leq r$ containing D_{n-1}. Since B_n is the Kronecker function ring of D_{n-1} defined in $K_{n-1}(X_n')$ by all valuation overrings of D_{n-1}, then $(D_{n-1})_{q'}$ has a unique extension in $K_{n-1}(X_n')$, that is a valuation overring of B_n, that is, $(D_{n-1})_{q'}(X_n')$. By (7), the center q of $(D_{n-1})_{q'}(X_n')$ in D_n is the unique prime ideal of D_n lying over q' with the property that $(D_n)_q = (D_{n-1})_{q'}(X_n')$. $\qquad\square$

Set $D := \bigcup_{n\geq 1} D_n$ and $\mathcal{M} := \bigcup_{n\geq 2} \mathcal{M}_n$. It is obvious that $D \subseteq A = \bigcup_{n\geq 1} A_n$.

Claim 2.5. $D_{\mathcal{M}} = A$, and $\mathcal{M}$ is a height-one maximal ideal in D.

Proof. This is an easy consequence of Claim 2.4(4), since $\mathcal{M}_n = \mathcal{M} \cap D_n$ for each n. $\qquad\square$

Let $q \in \mathrm{Spec}(D)$ with $q \neq \mathcal{M}$. Then, for some $m \geq 2$, we have in D_m, $q_m := q \cap D_m \neq \mathcal{M}_m = \mathcal{M} \cap D_m$. So, by Claim 2.4(6), $B_m \subseteq (D_m)_{q_m}$. Hence $(D_m)_{q_m}$ is a valuation overring of B_m of dimension $\leq r$, whence, by Claim 2.4(8), $D_q = \bigcup_{n\geq 1}(D_n)_{q_n} = (D_m)_{q_m}(X_{m+1}',...)$ is still a valuation domain of dimension $\leq r$.

Claim 2.6. $D = \bigcap\{D_q : q \in \mathrm{Spec}(D) \text{ and } q \neq \mathcal{M}\}$.

Proof. Similar to [24]. $\qquad\square$

From Claims 2.5 and 2.6 we obtain:

Conclusion 2.7. D is an essential domain with a nonessential localization, and $\dim(D) = \dim_v(D) = r$.

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