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ON THE KRULL AND VALUATIVE DIMENSION OF $D+XD_S[X]$ DOMAINS

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In this paper, we deal with the integral domain $D^{(S,r)} := D + (X_1, X_2, \ldots, X_r)D_S[X_1, X_2, \ldots, X_r]$, where D is an integral domain and S is a multiplicative set of D. The purpose is to pursue the study, initiated by Costa-Mott-Zafrullah in 1978, concerning the prime ideal structure of such domains. We characterize when $D^{(S,r)}$ is a strong S-domain, a stably strong S-domain, a catenarian domain and a universally catenarian domain. As a consequence, we obtain a new class of non-Noetherian universally catenarian domains. Moreover, we give an explicit formula for the Krull dimension of $D^{(S,r)}$ (depending on S and on the Krull dimensions of D and $D_S[X_1, X_2, \ldots, X_r]$) and we compute its valuative dimension.

0. Introduction

In [7] the integral domains $D+XD_S[X]$, where D is an integral domain, S is a multiplicative set of D and X is an indeterminate, were introduced and studied. Particular emphasis was placed on the transfer, from D to $T^{(S)} := D + XD_S[X]$, of the properties of being either Prüfer, Bézout, GCD, or coherent domains. The prime ideal structure of $T^{(S)}$ was also studied, and some useful bounds on the (Krull) dimension of $T^{(S)}$ were given. However, the problem of the determination of this dimension in the general situation, as a function of S and of the dimensions of D and $D[X]$, remained open.

In the present paper, we deal with a more general situation: we consider the domain

$$D^{(S,r)} := D + (X_1, X_2, \ldots, X_r)D_S[X_1, X_2, \ldots, X_r] = D + XD_S[X]$$

where D is an integral domain, S a multiplicative set of D and $X = \{X_1, X_2, \ldots, X_r\}$ is a finite set of indeterminates over D_S.

We notice that, as in the case of one indeterminate, the domain $D^{(S,r)}$ may be described in various ways: it is the direct limit of the direct system of domains $D[X_1/s, X_2/s, \dots, X_r/s]$, where $s \in S$ (and $s_1 \le s_2$ when $s_1 \mid s_2$); $D^{(S,r)}$ is the pullback of the canonical homomorphism $\varphi : D_S[X_1, X_2, \dots, X_r] \twoheadrightarrow D_S$, $X_i \mapsto 0$, $1 \le i \le r$, and of the embedding $\alpha : D \hookrightarrow D_S$:

$$(\square) \qquad \begin{tikzcd} D^{(S,r)} = \varphi^{-1}(\alpha(D)) \arrow[r, two heads, "\varphi'"] \arrow[d, hook, "\alpha'"] & D \arrow[d, hook, "\alpha"] \\ D_S[X_1, X_2, \dots, X_r] \arrow[r, two heads, "\varphi"] & D_S . \end{tikzcd}$$

Therefore, we can claim that many properties hold in $D^{(S,r)}$, because these properties are preserved by taking polynomial ring extensions and direct limits or by pullbacks of the special type ($\square$).

Similarly, as remarked in [7], it is possible to describe $D^{(S,r)}$ as the symmetric algebra of the D-module $D_S^{\oplus r}$ (using [2, Chapitre III, p. 73, Proposition 9]), but we will not use this last property in this paper.

The purpose of this work is to pursue the study, initiated by [7] when $r = 1$, of the prime ideal structure of the domain $D^{(S,r)}$. The main results of Section 2 (cf. Proposition 2.3 and Theorem 2.5) characterize when $D^{(S,r)}$ is a strong S-domain, a stably strong S-domain, a catenarian domain, or a universally catenarian domain. In particular, the domains of the type $D^{(S,r)}$ give rise to a new class of non-Noetherian universally catenarian domains (cf. [4]). Moreover, we give an explicit formula for the Krull dimension of $D^{(S,r)}$ (depending on S and on the Krull dimensions of D and $D_S[X_1, X_2, \dots, X_r]$) and we compute its Jaffard valuative dimension (cf. Theorem 3.2 and Proposition 3.4).

All rings considered below are (commutative integral) domains.

We recall that in [13] an integral domain R is called an S(eidenberg)-*domain* if for every height 1 prime ideal P of R, the height of $PR[Y]$, in the polynomial ring in one indeterminate $R[Y]$, is also 1. A *strong* S-*domain* is a domain R such that, for every prime ideal P of R, R/P is an S-domain. In [6], it is shown that there exists a strong S-domain for which $R[Y]$ is not a strong S-domain. In [15], a domain R is called a *stably strong* S-*domain* if $R[Y_1, Y_2, \dots, Y_n]$ is a strong S-domain for every finite family of indeterminates $\{Y_1, Y_2, \dots, Y_n\}$. A ring R is said to be *catenarian* in case for each pair $P \subset Q$ of prime ideals of R, all saturated chains of primes from P to Q have a common finite length. Note that each catenarian ring R must be locally finite-dimensional. In [3, Lemma 2.3], it is shown that if the polynomial ring $R[Y]$ is a catenarian domain, then R is a strong S-domain. We say that a (not necessarily Noetherian) ring is *universally catenarian* if the polynomial rings $R[Y_1, \dots, Y_n]$ are catenarian for each positive integer n.

Following Jaffard (cf. [14, Chapitre IV]), we define the *valuative dimension* of an integral domain R as

$$\dim_v(R) = \sup\{\dim(V) : V \text{ valuation overring of } R\}.$$

A *Jaffard domain* is a finite-dimensional integral domain R such that $\dim(R) = \dim_v(R)$ (see [1]).

We recall that a *spectral space* $\mathscr{X} = \mathrm{Spec}(A)$ (i.e. the set of all the prime ideals of a ring A equipped with the Zariski topology) is an ordered set under the set-theoretical inclusion. Following EGA's terminology [9, 0.2.1.1], we say that a subset $\mathscr{Y}$ of a spectral space $\mathscr{X}$ is *stable for generalizations* (resp., *specializations*) if $y \in \mathscr{Y}$ and $y' \leq y$ (resp., $y \leq y''$) imply that $y' \in \mathscr{Y}$ (resp., $y'' \in \mathscr{Y}$).

1. Prime ideal structure

We start collecting some basic facts concerning the prime ideal structure of $D^{(S,r)} = D + (X_1, \dots, X_r)D_S[X_1, \dots, X_r] = D + XD_S[X]$. Most of these are consequences of the general properties of pullback diagrams studied in [8].

We denote by

$$\begin{aligned} u &:= {}^a\varphi : \mathscr{Z} := \mathrm{Spec}(D_S) \longrightarrow \mathscr{Y} := \mathrm{Spec}(D_S[X_1, \dots, X_r]), \\ v &:= {}^a\alpha : \mathscr{Z} \longrightarrow \mathscr{X} := \mathrm{Spec}(D), \\ i &:= {}^a\lambda : \mathscr{W} := \mathrm{Spec}(D^{(S,r)}) \longrightarrow \mathscr{P} := \mathrm{Spec}(D[X_1, \dots, X_r]) \end{aligned}$$

the continuous maps (of spectral spaces) canonically associated to the natural ring homomorphisms $\varphi : D_S[X_1, \dots, X_r] \to D_S$, $X_i \mapsto 0$ $1 \leq i \leq r$, $\alpha : D \hookrightarrow D_S$, and $\lambda : D[X_1, \dots, X_r] \hookrightarrow D^{(S,r)}$, respectively.

Theorem 1.1. *With the previous notation, the spectral space $\mathscr{W}$ is canonically homeomorphic to the topological amalgamated sum $\mathscr{X} \amalg_{\mathscr{Z}} \mathscr{Y}$. More precisely,*

(1) *$XD_S[X]$ is a prime ideal of $D^{(S,r)}$ and $D^{(S,r)}/XD_S[X]$ is canonically isomorphic to D. From a topological point of view, the continuous map $u' := {}^a\varphi' : \mathscr{X} \to \mathscr{W}$, associated to the surjective ring homomorphism $\varphi' : D^{(S,r)} \twoheadrightarrow D$, is a closed embedding, and establishes an order isomorphism $\mathscr{X} \xrightarrow{\sim} \mathscr{X}' := \{Q \in \mathscr{W} : Q \supset XD_S[X]\}$, $P \mapsto P + XD_S[X]$. In particular, $\mathscr{X}'$ is a subspace of $\mathscr{W}$ stable under specializations.*

(2) *$(D^{(S,r)})_S$ is canonically isomorphic to $D_S[X_1, \dots, X_r]$. From a topological point of view, the continuous map $v' := {}^a\alpha' : \mathscr{Y} \to \mathscr{W}$ associated to the natural ring homomorphism $\alpha' : D^{(S,r)} \to D_S[X_1, \dots, X_r]$, is injective and establishes an order isomorphism $\mathscr{Y} \xrightarrow{\sim} \mathscr{Y}' := \{Q \in \mathscr{W} : Q \cap S = \emptyset\}$, $P \mapsto P \cap D^{(S,r)}$, where $\mathscr{Y}'$ is a subspace of $\mathscr{W}$ stable under generalizations.*

(3) *$(D_S^{(S,r)}/XD_S[X])$ is canonically isomorphic to D_S. A topological interpretation of this fact is that $v' \circ u : \mathscr{Z} \to \mathscr{W}$ establishes an order isomorphism $\mathscr{Z} \xrightarrow{\sim} \mathscr{Z}' := \mathscr{X}' \cap \mathscr{Y}'$, $P \mapsto (P \cap D) + XD_S[X]$, where $\mathscr{Z}'$ is a closed subspace of $\mathscr{Y}'$ (but not, in general, of $\mathscr{W}$).*

(4) *The topological amalgamated sum $\mathscr{X} \amalg_{\mathscr{Z}} \mathscr{Y}$ is canonically homeomorphic (via the continuous map σ defined by $\sigma|_{\mathscr{X}} = u'$ and $\sigma|_{\mathscr{Y}} = v'$) to $\mathscr{W}$. In particular, these two topological spaces are order isomorphic.*

(5) *The canonical continuous map* $i : \mathcal{W} \to \mathcal{S}$ *is injective but, in general, it is not a topological embedding. As a matter of fact, it is not an order isomorphism with its image. But, if* $M \in \mathcal{X}' \subset \mathcal{W}$ *is a closed point of* $\mathcal{W}$, *then* $i(M)$ *is still a closed point of* $\mathcal{S}$. *Moreover,* $i(\mathcal{Y}')$ *is a subspace of* $\mathcal{S}$ *stable under generalizations.*

Proof. The proof of the statements (1), (2) and (3) is straightforward. For the first claim of (5), we shall give a counterexample (see the following Remark 1.4). The second claim follows from the fact that, if M is a maximal ideal of $D^{(S,r)}$ containing $XD_S[X]$, then $M \cap D[X]$ is a maximal ideal of $D[X]$ (containing $XD[X]$). The third claim follows by noticing that $D[X]$ and $D^{(S,r)}$ have the same localization at their multiplicative set S. For statement (4), it is easy to see that σ is a continuous bijection. Moreover, σ is also a closed map as a consequence of Corollary 1.3, which follows from:

Proposition 1.2. *Consider the following pullback of ring-homomorphisms*:

$$\begin{array}{ccc} R & \xrightarrow{\psi'} & B \\ {\scriptstyle\delta'}\downarrow & & \downarrow{\scriptstyle\delta} \\ A & \xrightarrow{\psi} & C, \end{array}$$

where ψ *is surjective,* $I = \mathrm{Ker}(\psi)$, *and* δ *is injective. Suppose that* R *is quasi-local with maximal ideal* M. *Then*

(a) $I \subset J(A)$ (= *Jacobson radical of* A);

(b) $\mathrm{Max}(A) = {}^a\psi(\mathrm{Max}(C))$;

(c) *For every* $P \in \mathrm{Spec}(R)$, *with* $P = \delta'^{-1}(P')$ *for some* $P' \in \mathrm{Spec}(A)$, *there exists* $Q \in \mathrm{Spec}(R)$ *with* $P \subset Q$ *and* $Q = (\psi \circ \delta')^{-1}(Q')$ *for some* $Q' \in \mathrm{Spec}(C)$.

Proof. For ease of notation, we identify R and B with their images in A and C. It is straightforward to see that I also coincides with $\mathrm{Ker}(\psi')$ and R/I is isomorphic to B. Therefore, B is also a quasi-local ring.

(a) Clearly $1 + I \subset 1 + M \subset U(R)$ (= units of R) since R is quasi-local. Thus $1 + I = 1 + IA \subset U(A)$, and the previous inclusion implies that $I \subset J(A)$.

(b) Obviously ${}^a\psi(\mathrm{Max}(C)) \subset \mathrm{Max}(A)$, because ${}^a\psi$ is a closed embedding. By (a) and by the isomorphism $A/I \cong C$, we deduce statement (b).

(c) is an easy consequence of (b). □

Corollary 1.3. *With the notation of Proposition* 1.2, *without supposing* R *quasi-local, if we take* $P_1, P_2 \in \mathrm{Spec}(R)$ *with* $P_1 \subset P_2$ *and* $P_1 = \delta'^{-1}(P_1')$ *for some* $P_1' \in \mathrm{Spec}(A)$ *and* $P_2 = \psi'^{-1}(P_2')$ *for some* $P_2' \in \mathrm{Spec}(B)$, *then there exists* $Q \in \mathrm{Spec}(R)$ *with* $P_1 \subset Q \subset P_2$ *and* $Q = (\psi \circ \delta')^{-1}(Q')$ *for some* $Q' \in \mathrm{Spec}(C)$.

Proof. After tensorizing by $\otimes_R R_{P_2}$, we are in the situation of Proposition 1.2 (cf. also [5, Lemma 2]). Using the statement (c) of the previous proposition, the conclusion follows from the properties of the correspondence between the prime ideals of R and those of R_{P_2}. $\square$

Remark 1.4. If we consider $D=\mathbb{Z}_{(2)}$, $S=\mathbb{Z}_{(2)}\setminus\{0\}$, and $r=1$, then it is easy to verify that $i:\operatorname{Spec}(\mathbb{Z}_{(2)}+X\mathbb{Q}[X])\to\operatorname{Spec}(\mathbb{Z}_{(2)}[X])$ is neither open nor closed (even though, in this particular case, the canonical map $\operatorname{Spec}(\mathbb{Q}[X])\to\operatorname{Spec}(\mathbb{Z}_{(2)}[X])$ is open, in fact universally open [9, 1.7.3.10], and not, simply, stable for generalizations). Moreover, the continuous injective map i is not an order isomorphism with its image, because, for instance, $P:=(2+X)\mathbb{Q}[X]\cap(\mathbb{Z}_{(2)}+X\mathbb{Q}[X])$ and $M:=2\mathbb{Z}_{(2)}+X\mathbb{Q}[X]$ are both maximal ideals of $\mathbb{Z}_{(2)}+X\mathbb{Q}[X]$, but $i(P)=(2+X)\mathbb{Z}_{(2)}[X]\subset i(M)=2\mathbb{Z}_{(2)}+X\mathbb{Z}_{(2)}[X]$. We also notice that $Q:=X\mathbb{Q}[X]$ and P are co-maximal in $\mathbb{Z}_{(2)}+X\mathbb{Q}[X]$, but $i(P)$ and $i(Q)$ are both contained in $i(M)$, as prime ideals of $\mathbb{Z}_{(2)}[X]$.

Another interesting property of the domains of the type $D^{(S,r)}$ is described in the following:

Proposition 1.5. *Let $Y_1, Y_2, \ldots, Y_n$ be a finite set of indeterminates over a given domain $D^{(S,r)}$. Then, the polynomial ring $D^{(S,r)}[Y_1, Y_2, \ldots, Y_n]$ is canonically isomorphic to $(D[Y_1, \ldots, Y_n])^{(S,r)}$.*

Proof. By flatness, the following diagram, obtained from the diagram ($\square$) by tensorizing with $\otimes_D D[Y_1, Y_2, \ldots, Y_n]$,

$$\begin{array}{ccc} D^{(S,r)}[Y_1, Y_2, \ldots, Y_n] & \twoheadrightarrow & D[Y_1, Y_2, \ldots, Y_n] \\ \downarrow & & \downarrow \\ D_S[X_1, \ldots, X_r; Y_1, \ldots, Y_n] & \twoheadrightarrow & D_S[Y_1, Y_2, \ldots, Y_n] \end{array}$$

is still a pullback diagram (cf. [5, Lemma 2]). The conclusion is now straightforward, after noticing that $D_S[Y_1, \ldots, Y_n]$ coincides with $D[Y_1, Y_2, \ldots, Y_n]_S$. $\square$

2. Transfer of some properties concerning prime chains

In this section, we will study the transfer of the properties of being an S-domain, a strong S-domain, or a catenarian domain to the integral domains of the type $D^{(S,r)}=D+(X_1, \ldots, X_r)D_S[X_1, \ldots, X_r]$ and to the polynomial rings with coefficients in a $D^{(S,r)}$.

In order to study the problem of the transfer of the S-property to $D^{(S,r)}$, we need to know better the behaviour of this property in passing to polynomial rings. This problem was surprisingly disregarded in the literature and only briefly studied in [15, Theorems 3.1, 3.3 and Corollary 3.4], where in particular the authors showed that if R is a Prüfer domain, then $R[Y_1, Y_2, \dots, Y_n]$ is an **S**-domain. M. Zafrullah, in a private communication, proved the following general result that improves dramatically the previous statement of [15] and some results of a first draft of this paper:

Proposition 2.1. *Let R be an integral domain and $Y_1, Y_2, \dots, Y_n$ a finite family of indeterminates over R, where $n \geq 1$. Then $R[Y_1, Y_2, \dots, Y_n]$ is an* **S**-*domain.*

Proof. It is enough to show that the statement holds when $n=1$. Let $Y := Y_1$. It is easy to see that an integral domain A is an **S**-domain if and only if A_p is an **S**-domain for every height 1 prime ideal p of A. In order to prove the statement, it is enough to show that $R[Y]_P$ is an **S**-domain, for every height 1 prime ideal P of $R[Y]$. Two cases are possible for $p := P \cap R$. If $p \neq (0)$, then p is an height 1 prime ideal of R and $P = p[Y]$. Thus $R[Y]_P = R_p[Y]_{p[Y]}$ and $PR[Y]_P = pR_p[Y]_{p[Y]}$, hence $pR_p[Y]$ is a height 1 prime ideal of $R_p[Y]$. We recall that in [3, Corollary 6.3] it is shown that for one-dimensional domains, the notions of (strong) **S**-domain and stable strong **S**-domain are equivalent. By applying this result to R_p, we deduce that in $R_p[Y, Z]$ (where Z is another indeterminate) $pR_p[Y, Z]$ is still a height 1 prime ideal. Thus $p[Y, Z] = P[Z]$ is also a height 1 prime ideal. If $p = (0)$, then there exists a unique height 1 prime ideal Q of $K[Y]$, where K denotes the field of quotients of R, such that $Q \cap R[Y] = P$. Since $K[Y]$ is an **S**-domain, so is $K[Y]_Q$, this fact implies that also $R[Y]_P$ is an **S**-domain. The proof is complete. □

From the preceding proposition we deduce immediately the following:

Corollary 2.2. *We keep the notation introduced in Section* 0. *Then $D^{(S,r)}$ is an* **S**-*domain for every S and $r \geq 1$.*

Proof. By Proposition 2.1, we know that $D_S[X_1, X_2, \dots, X_r]$, with $r \geq 1$, is an **S**-domain. For every height 1 prime ideal P of $D^{(S,r)}$, we can consider two cases. If $P \cap S = \emptyset$, then $(D^{(S,r)})_P = ((D^{(S,r)})_S)_P = D_S[X_1, X_2, \dots, X_r]_P$ and hence it is an **S**-domain. If $P \cap S \neq \emptyset$, then necessarily $r=1$ and $P = XD_S[X]$, hence this second case is impossible, because $XD_S[X] \cap S = \emptyset$. □

In order to build-up a new class of examples of universally catenarian domains which is different from all the classes already known, we deepen the study of the domains $D^{(S,r)}$.

Proposition 2.3. *We keep the notation introduced in Section* 0. *Let $r \geq 1$. The following statements are equivalent*:

(i) *$D^{(S,r)}$ is a strong* **S**-*domain* (*resp., a catenarian domain*);

(ii) *D and $D_S[X_1, X_2, \dots, X_r]$ are both strong* **S**-*domains* (*resp., catenarian domains*).

Proof. It is clear that (i) $\Rightarrow$ (ii), because the notion of strong S-domain (resp. catenarian domain) is stable under localization and under the passage to quotient-domains.

(ii) $\Rightarrow$ (i). We start with the case of strong S-domains. Let P_1 and P_2 be two prime ideals of $D^{(S,r)}$ with $P_1 \subset P_2$ and $\mathrm{ht}(P_2/P_1)=1$. Three cases are theoretically possible.

Case 1. $P_1 \in \mathscr{X}'$ (with the notation of Theorem 1.1). Thus also $P_2 \in \mathscr{X}'$. In this case, $\mathrm{ht}(P_2[Y]/P_1[Y])=1$ because $\mathscr{X}' \cong \mathscr{X} = \mathrm{Spec}(D)$ and D is a strong S-domain.

Case 2. $P_2 \in \mathscr{Y}'$ (with the notation of Theorem 1.1). Thus also $P_1 \in \mathscr{Y}'$. Also in this case $\mathrm{ht}(P_2[Y]/P_1[Y])=1$ because $\mathscr{Y}' \cong \mathscr{Y} = \mathrm{Spec}(D_S[X_1,\dots,X_r])$ and $D_S[X_1,\dots,X_r]$ is a strong S-domain.

Case 3. $P_1 \in \mathscr{Y}'$ and $P_2 \in \mathscr{X}' \setminus \mathscr{Y}'$. This case is impossible when $\mathrm{ht}(P_2/P_1)=1$ by Corollary 1.3.

Finally, we notice that the implication (ii) $\Rightarrow$ (i) holds in the case of a catenarian domain. As a matter of fact, we can apply [5, Lemma 1], after remarking that the glueing condition (γ) is verified by Corollary 1.3. □

As an easy consequence of Proposition 2.3, we have

Corollary 2.4. *If $D[X_1,X_2,\dots,X_r]$ is a strong* S*-domain (resp., a catenarian domain), then $D^{(S,r)}$ is a strong* S*-domain (resp., a catenarian domain).* □

We will show (Example 2.7) that the converse of Corollary 2.4 does not hold in general, however it is possible to prove a 'universal' converse of the previous corollary.

Theorem 2.5. *With the notation of Section* 0, *and $r \geq 1$, the following statements are equivalent*:

(i) *$D^{(S,r)}$ is a stably strong* S*-domain (resp., a universally catenarian domain)*;

(ii) *D is a stably strong* S*-domain (resp., a universally catenarian domain).*

Proof. (ii) $\Rightarrow$ (i). As a matter of fact, if for every $n \geq 1$, $D[Y_1,\dots,Y_n]$ is a strong S-domain (resp., a catenarian domain), then the conclusion follows from Corollary 2.4, after recalling that $(D[Y_1,\dots,Y_n])^{(S,r)} = D^{(S,r)}[Y_1,\dots,Y_n]$ (cf. Proposition 1.5).

(i) $\Rightarrow$ (ii). For every $n \geq 1$, we know that

$$D^{(S,r)}[Y_1,\dots,Y_n]/(X_1,\dots,X_r)D_S[X_1,\dots,X_r,Y_1,\dots,Y_n] \cong D[Y_1,\dots,Y_n]$$

thus the claim is a consequence of the fact that the notion of strong S-domain (resp., catenarian domain) is stable under passage to quotient-domains. □

The previous theorem leads to a further non-standard class of universally catenarian domains (besides those considered in [4]). In particular, it is possible now to exhibit a universally catenarian domain which is neither Noetherian nor a GD strong S-domain (thus not a Prüfer domain) with global dimension bigger than 2. As a matter of fact, when D is a universally catenarian domain and the multiplicative set S is non-trivial (i.e. $S \neq D \setminus \{0\}$ and $S \not\subset U(D)$) and $r \geq 1$, then $D^{(S,r)}$ is a universally catenarian domain of the announced kind, even if D is a universally catenarian domain of one of the 'classical' classes (i.e. CM, locally finite-dimensional Prüfer domain, or a domain of global dimension ≤ 2). For instance,

$$\mathbb{Z} + (X_1, X_2, \ldots, X_r)\mathbb{Z}_{(2)}[X_1, \ldots, X_r], \quad r \geq 1,$$

$$\mathbb{C}[U, V]_{(U,V)} + (X_1, X_2, \ldots, X_r)\mathbb{C}[U, V]_{(U)}[X_1, \ldots, X_r], \quad r \geq 1$$

are *new* examples of universally catenarian domains which are not Noetherian, not Prüfer, and have global dimension > 2.

Example 2.6. We give an example of a domain $D^{(S,r)}$ which is not a strong S-domain (still is an S-domain).

Let k be a field and X and Y two indeterminates over k and let

$$\begin{aligned} A_1 &:= k + Yk(X)[Y]_{(Y)}, & M_1 &:= Yk(X)[Y]_{(Y)}, \\ V_2 &:= k[Y]_{(Y)} + Xk(Y)[X]_{(X)}, & P &:= Xk(Y)[X]_{(X)}, \\ M_2 &:= Yk[Y]_{(Y)} + P. \end{aligned}$$

A_1 is a 1-dimensional pseudo-valuation domain, which is not an S-domain [10, Theorem 2.5], and V_2 is a 2-dimensional valuation domain. Set $D := A_1 \cap V_2$. It is not difficult to see that $\mathrm{Spec}(D) = \{(0), p = P \cap D, m_1 = M_1 \cap D, m_2 = M_2 \cap D\}$ and that

$$D_{m_1} = A_1, \qquad D_{m_2} = V_2,$$

with m_1 height 1 prime (maximal) ideal of D. Thus, D is not an S-domain. Thus $D + (X_1, X_2, \ldots, X_r)D_p[X_1, X_2, \ldots, X_r]$ is not a strong S-domain, but it is an S-domain (cf. Corollary 2.2 and Proposition 2.3).

Example 2.7. *There exists an integral domain D and a multiplicative set S of D such that D and $D^{(S,r)}$ are catenarian and strong* S*-domains, for every $r \geq 1$, but $D[X_1, \ldots, X_r]$ is not a strong* S*-domain for every $r \geq 1$ (hence, it is not a catenarian domain for $r \geq 2$).*

By [6, Example 3] (cf. also [1, Example 3.8]), we know that it is possible to give an example of a quasi-local 2-dimensional catenarian and strong S-domain D with a unique height 1 prime ideal P such that D_P is a (discrete) valuation domain, but $D[X_1, \ldots, X_r]$ is not a strong S-domain for $r \geq 1$ (hence, it is not catenarian for $r \geq 2$, cf. [3, Lemma 2.3]). In this case, since a finite-dimensional valuation domain is a universally catenarian domain [5] (in particular, a stably strong S-domain), then, by the previous Proposition 2.3, $D + (X_1, \ldots, X_r)D_P[X_1, \ldots, X_r]$, is catenarian and a strong S-domain for every $r \geq 1$.

3. Krull dimension and valuative dimension

In order to study the Krull dimension of $D^{(S,r)}$, we begin by giving some new definitions, related to the S-dimension introduced in [7], with the purpose of obtaining some useful bounds on the Krull dimension of $T^{(S)}:=D^{(S,1)}$.

Recalling the notation of Section 1, we identify for simplicity $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ with their canonical images (respectively, $\mathscr{X}'$, $\mathscr{Y}'$ and $\mathscr{Z}'$) in $\mathscr{W}$ (cf. Theorem 1.1).

We define the *S-coheight* of a prime $P\in\mathscr{W}$ by

$$S\text{-coht}(P):=\sup\{t\geq 0:\ P=P_0\subset P_1\subset\cdots\subset P_t,\ \text{where } P_i\in\mathscr{X}\setminus\mathscr{Z} \text{ for } i\geq 1\},$$

and we set

$$S\text{-dim}(D):=\sup\{S\text{-coht}(P):\ P\in\mathscr{X}\}.$$

Obviously, $S\text{-coht}(P)\leq\text{coht}(P)$ for every $P\in\mathscr{X}$; moreover for $r=1$, the previously defined *S-dimension* coincides with that introduced in [7].

Finally, we define:

$$\mathscr{Z}\text{-dim}(D[X_1,\dots,X_r]):=\sup\{S\text{-coht}(P)+\text{ht}(P):\ P\in\mathscr{Z}\}$$

where $\text{ht}(P)$ is the height of P as a prime ideal of $D_S[X_1,\dots,X_r]$ or, equivalently, of $D[X_1,\dots,X_r]$.

Before producing a formula which gives the Krull dimension of $D^{(S,r)}$ as a function of the Krull dimension of $D_S[X_1,\dots,X_r]$ and of the $\mathscr{Z}$-dimension of $D[X_1,\dots,X_r]$, we give some bounds for $\dim(D^{(S,r)})$ analogous to those proved in [7] when $r=1$.

Proposition 3.1. *With the notation of Section* 0, *we have*:

$$\max\{\dim(D_S[X]),\ \dim(D)+r\}\leq\dim(D^{(S,r)})$$
$$\leq\min\{\dim(D[X]),\ \dim(D_s[X])+S\text{-dim}(D)\}.$$

Proof. It is clear that $\dim(D_S[X])\leq\dim(D^{(S,r)})\leq\dim(D[X])$ because of Theorem 1.1 and $D_S[X]=(D^{(S,r)})_S$. Moreover, in $D^{(S,r)}$ there always exists a chain of prime ideals of length $\geq\dim(D)+r$. As a matter of fact, we can choose a maximal ideal M of $D^{(S,r)}$ such that $M\supset XD_S[X]$ and $M/XD_S[X]$ corresponds to a maximal ideal of D which realizes the dimension of D. Then, M contains a chain of prime ideals of length $\text{ht}(M/XD_S[X])+\text{ht}(XD_S[X])\geq\dim(D)+r$. Finally, let Q be a prime ideal of $D^{(S,r)}$ corresponding to a closed point of $\mathscr{Z}$. By Corollary 1.3, to avoid the trivial cases we can consider a chain of prime ideals of $D^{(S,r)}$ passing through Q. This chain necessarily has length $\leq\dim(D_S[X])+S\text{-coht}(Q)\leq\dim(D_S[X])+S\text{-dim}(D)$. □

Theorem 3.2. *With the notation of Section* 0,

$$\dim(D^{(S,r)})=\max\{\dim(D_S[X_1,\dots,X_r]),\ \mathscr{Z}\text{-dim}(D[X_1,\dots,X_r])\}.$$

Proof. Let $M\in\text{Max}(D^{(S,r)})$. By Theorem 1.1, two cases are possible:

Case 1. $M\in\mathscr{Y}$ (with the notation of the beginning of this section). In this case,

$\mathrm{ht}(M) \leq \dim(D_S[X])$ and there exists a maximal ideal $\tilde{M} \in \mathrm{Max}(D^{(S,r)})$ with $\tilde{M} \in \mathscr{Y}$ such that $\mathrm{ht}(\tilde{M}) = \dim(D_S[X])$.

Case 2. $M \in \mathscr{X}$ (with the notation of the beginning of this section), that is, $M \supset XD_S[X]$. In such a case, we know that every chain of prime ideals of $D^{(S,r)}$ contained in M contains a prime ideal $Q \in \mathscr{Z}$ (Corollary 1.3). Therefore, the supremum of the length of the chains of prime ideals ending at a maximal ideal $M \in \mathscr{X}$ coincides with:

$$\sup\{S\text{-coht}(Q) + \mathrm{ht}(Q) \colon Q \in \mathscr{Z}\} = \mathscr{Z}\text{-}\dim(D[X]). \qquad \square$$

Before giving some important cases for which it is easy to compute $\mathscr{Z}$-$\dim(D[X_1, \dots, X_r])$, we draw some consequences from the previous theorem:

Corollary 3.3. *With the notation of Section* 0, *let D be a Jaffard domain. Then for every* $r \geq 1$

$$\dim(D^{(S,r)}) = \dim(D) + r.$$

In particular, $\mathscr{Z}\text{-}\dim(D[X_1, \dots, X_r]) = \dim(D[X_1, \dots, X_r]) = \dim(D) + r$.

Proof. We notice that when $\dim(D[X_1, \dots, X_r]) = \dim(D) + r$, then

$$\max\{\dim(D) + r,\ \dim(D_S[X_1, \dots, X_r])\} = \dim(D) + r.$$

Moreover,

$$\min\{\dim(D[X_1, \dots, X_r]),\ \dim(D_S[X_1, \dots, X_r]) + S\text{-}\dim(D)\} \\ = \dim(D[X_1, \dots, X_r]).$$

Otherwise, we would have

$$\dim(D) + r \leq \dim(D^{(S,r)}) \leq \dim(D_S[X_1, \dots, X_r]) + S\text{-}\dim(D) \\ \leq \dim(D[X_1, \dots, X_r]),$$

and thus $\dim(D_S[X_1, \dots, X_r]) + S\text{-}\dim(D) = \dim(D[X_1, \dots, X_r]) = \dim(D) + r$. Moreover, when D is Jaffard, $\dim(D[X_1, \dots, X_r]) = \dim_v(D) + r = \dim(D) + r$. Thus, by Proposition 3.1, $\dim(D^{(S,r)}) = \dim(D) + r$. The second statement follows easily, noticing that in general

$$\dim(D) + r \leq \mathscr{Z}\text{-}\dim(D[X_1, \dots, X_r]) \leq \dim(D[X_1, \dots, X_r]). \qquad \square$$

In order to study the transfer to $D^{(S,r)}$ of the Jaffard property, we need to compute the valuative dimension of $D^{(S,r)}$.

Proposition 3.4. *With the notation of Section* 0,

$$\dim_v(D^{(S,r)}) = \dim_v(D) + r.$$

Proof. It is clear (using [14, Théorème 2, p. 60]) that

$$\dim(D) + r \leq \dim(D^{(S,r)}) \leq \dim_v(D^{(S,r)}) \leq \dim_v(D[X_1, \dots, X_r]) = \dim_v(D) + r.$$

Conversely, let V be a valuation overring of D realizing the valuative dimension of D and let K be the quotient field of D. We consider

$$R := V + (X_1, \dots, X_r)K[X_1, \dots, X_r].$$

It is easy to see that R is an overring of $D^{(S,r)}$ with

$$\dim_v(R) \geq \dim(R) \geq \dim(V) + r = \dim_v(D) + r.$$

The conclusion is now straightforward. □

Theorem 3.5. *With the notation of Section* 0,

(a) *The following statements are equivalent*:

(i) *D is a Jaffard domain*;

(ii) *$D^{(S,r)}$ is a Jaffard domain and* $\dim(D^{(S,r)}) = \dim(D) + r$, *for every* $r \geq 1$.

(b) *The following statements are equivalent*:

(j) *$D^{(S,r)}$ is a Jaffard domain*;

(jj) *$D[X_1, \dots, X_r]$ is a Jaffard domain and*

$$\dim(D^{(S,r)}) = \dim(D[X_1, \dots, X_r]) \ (= \mathcal{J}\text{-}\dim(D[X_1, \dots, X_r])).$$

Proof. (a) (i) ⇔ (ii). By Corollary 3.3 and Proposition 3.4.

(b) (j) ⇒ (jj). By Propositions 3.1 and 3.4, we know that

$$\dim(D[X_1, \dots, X_r]) \geq \dim(D^{(S,r)}) = \dim_v(D^{(S,r)}) = \dim_v(D) + r.$$

Moreover, it is well known that $\dim_v(D[X_1, \dots, X_r]) = \dim_v(D) + r$ ([14, Théorème 2, p. 60]). The conclusion follows from the fact that, in general, the valuative dimension is larger than the Krull dimension.

(jj) ⇒ (j) is a consequence of Proposition 3.4, since

$$\dim_v(D) + r = \dim_v(D[X_1, \dots, X_r]). \quad \square$$

We note that $D^{(S,r)}$ could be a Jaffard domain, even though D is not Jaffard, as the following example will show:

Example 3.6. Let $A_1 := k + Yk(X)[Y]_{(Y)}$ be the 1-dimensional pseudo-valuation domain considered in Example 2.6. We note that A_1 is not a Jaffard domain because $\dim_v(A_1) = 2$ [1, Proposition 2.5] and that the polynomial ring $A_1[Z]$ is a 3-dimensional Jaffard domain [1, 0.1(iv)]. Let $A_2 := k(Y)[X]_{(X)}$ and set $D := A_1 \cap A_2$. It is not difficult to see that D is a 1-dimensional quasi-semilocal domain with $\mathrm{Max}(D) = \{M := Yk(X)[Y]_{(Y)} \cap D,\ N := XA_2 \cap D\}$, $D_M = A_1$, and $D_N = A_2$. Hence $\dim_v(D) = \max\{\dim_v(A_1), \dim_v(A_2)\} = 2$. Set $S = D \setminus M$ and $r = 1$, and consider $D^{(S,1)} = D + ZA_1[Z]$. Since $D[Z]$ (like $A_1[Z]$) is a 3-dimensional Jaffard domain [1, Section 0], from Proposition 3.1 we deduce that $\dim(D^{(S,1)}) = 3$. From Proposition 3.4 we easily compute $\dim_v(D^{(S,1)})$; thus we can conclude that

$D^{(S,1)}$ is a 3-dimensional Jaffard domain, but D is not a Jaffard domain. Accordingly with Theorem 3.5, we have

$$\dim(D^{(S,1)}) = \dim(D[Z]) = 3 \gneq \dim(D) + 1.$$

Example 3.7. From Theorem 3.5(a), we deduce that

$$R_1 := \mathbb{Z}[Y_1, \dots, Y_n] + (X_1, \dots, X_r)\mathbb{Z}_{(2)}[X_1, \dots, X_r, Y_1, \dots, Y_n]$$

and

$$R_2 := \mathbb{C}[U, V]_{(U,V)}[Y_1, \dots, Y_n]$$
$$+ (X_1, \dots, X_r)\mathbb{C}[U, V]_{(U)}[X_1, \dots, X_r, Y_1, \dots, Y_n]$$

are both non-Noetherian, non-Prüfer Jaffard domains for every $r \geq 1$ and $n \geq 0$ with

$$\dim(R_1) = n + 1 + r, \qquad \dim(R_2) = n + 2 + r.$$

We end the paper with a result which allows one to compute the $\mathcal{Z}$-$\dim(D[X_1, \dots, X_r])$ in an important case.

Proposition 3.8. *With the notation of the beginning of this section, if $D_S[X_1, \dots, X_r]$ is a catenarian domain, then*

$$\mathcal{Z}\text{-}\dim(D[X_1, \dots, X_r]) = \dim(D) + r.$$

Proof. Let

$$M = P_t \supset P_{t-1} \supset \cdots \supset P_0 = Q = P'_h \supset P'_{h-1} \supset \cdots \supset P'_1 \supset (0)$$

be a prime chain of $D[X]$, realizing $\mathcal{Z}$-$\dim(D[X])$, where $Q \in \mathcal{Z}$, $P_i \in \mathcal{X} \setminus \mathcal{Z}$ for $i \geq 1$ and $P'_j \in \mathcal{Y}$ for $1 \leq j \leq h$. Since $P'_h = Q \supset XD_S[X]$ (because $Q \in \mathcal{Z}$), two cases are possible:

Case 1. $P'_h = Q = XD_S[X]$. In this case, $h = r$ since the height of Q in $D_S[X]$ (or, equivalently, in $D[X]$) is r. Moreover, S-coht$(Q) \leq \dim(D)$. Thus $\mathcal{Z}$-$\dim(D[X]) \leq \dim(D) + r$ and, since the opposite inequality always holds, then necessarily $\mathcal{Z}$-$\dim(D[X]) = \dim(D) + r$.

Case 2. $P'_h = Q \supsetneqq XD_S[X]$. We have the following diagram of inclusion of prime ideals:

$$\begin{tikzcd}[arrows=dash]
M \arrow[rr, "t"] \arrow[dd, "d"'] & & Q \arrow[dd, "h"] \\
& & \\
XD_S[X] \arrow[uurr, "l"'] \arrow[rr, "r"'] & & (0)
\end{tikzcd}$$

where d (resp., l) is the maximal length of the saturated chains between M and $XD_S[X]$ (resp., Q and $XD_S[X]$) inside $D^{(S,r)}$. Since $\mathscr{Y}$ is stable for generalizations and $D_S[X]$ is catenarian, $l+r=h$. Moreover, $d=\dim(D)$ and $\mathscr{X}$ is stable for specializations, thus $d \geq t+l$.

In conclusion, $d+r \geq t+l+r=t+h$; thus $d+r=t+h$ since the opposite inclusion always holds (cf. Proposition 3.1). □

From Corollary 3.3 and Proposition 3.8, we immediately deduce the following:

Corollary 3.9. *With the notation of Section* 0, *if D_S is a universally catenarian domain, then* $\dim(D^{(S,r)})=\dim(D)+r$, *for every* $r \geq 1$. □

The last example that we give is to show that it is possible to have

$$\begin{aligned}&\max\{\dim(D)+r, \dim(D_S[X_1,\dots,X_r])\}\\ &\quad\lneq \dim(D^{(S,r)})=\mathscr{J}\text{-}\dim(D[X_1,\dots,X_r])\\ &\quad\lneq \dim(D[X_1,\dots,X_r]).\end{aligned}$$

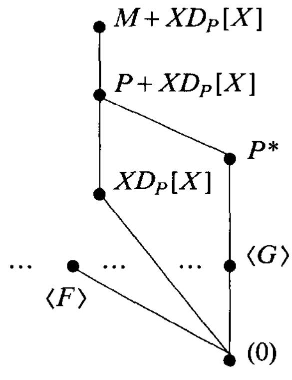
Example 3.10. Let k be a field and Z_1, Z_2, Z_3, Z_4 indeterminates. We consider $D:=k+Z_2k(Z_1)[Z_2]_{(Z_2)}+Z_4k(Z_1,Z_2,Z_3)[Z_4]_{(Z_4)}$. We know from [1] that $\dim(D)=2$, $\dim_v(D)=4$. Moreover, a direct verification shows that the polynomial ring $D[X]$ is a 5-dimensional Jaffard domain (see also below). Let $P:=Z_4k(Z_1,Z_2,Z_3)[Z_4]_{(Z_4)}$ be the height 1 prime ideal of D and let $S:=D\setminus P$. Clearly D_P is a 1-dimensional pseudo-valuation domain with $\dim_v(D_P)=2$ and thus $\dim(D_P[X])=3$ (cf. [1] and [10]). Let $D^{(S,1)}:=D+XD_P[X]$. Clearly

$$\max\{\dim(D)+1,\ \dim(D_P[X])\}=3$$

and

$$\min\{\dim D[X],\ S\text{-}\dim(D)+\dim(D_P[X])\}=5$$

because S-$\dim(D)=2$ [7, Definition 2.8]. More precisely, the prime spectrum of $D+XD_P[X]$, as partially ordered set, has the following form:



where M is the maximal ideal of D, $P^*:=PD_P[X]\cap D^{(S,1)}$, $F(X)$ is an irreducible polynomial with coefficients in $K:=k(Z_1,Z_2,Z_3,Z_4)$ (which is the quotient field of D), $\langle F\rangle:=FK[X]\cap D^{(S,1)}$ and $G(X)=Z_4X-Z_4Z_3\in K[X]$. In $D^{(S,1)}$ there are two kinds of prime ideals upper to (0): the height 1 maximal ideals and those contained in P^* (since $\mathrm{ht}(P^*)=2$). From Theorem 1.1 and Theorem 3.2, it follows that $\dim(D^{(S,1)})=\mathcal{J}\text{-}\dim(D[X])=4$.

Finally, we point out that the following question arises naturally from the theory developed in the present paper: Is $D^{(S,r)}$ a strong S-domain for every $r\geq 1$, when $D^{(S,1)}$ is? By our Proposition 2.3, this problem can be reduced to the following: Is $R[X,Y]$ a strong S-domain when $R[X]$ is? The question of the transfer of the strong S-property to polynomial rings is discussed in two recent papers by S. Kabbaj [11,12]. Although several partial affirmative results were obtained, the general question remains open.

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