

## On The Class Group of a Graded Domain

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### Abstract

This paper studies the class group of a graded integral domain  $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ . We prove that if the extension  $R_0 \subset R$  is inert, then  $Cl(R) = HCl(R)$  if and only if  $R$  is almost normal. As an application, we state a decomposition theorem for class groups of semigroup rings, namely,  $Cl(A[\Gamma]) \cong Cl(A) \oplus HCl(K[\Gamma])$  if and only if  $A[\Gamma]$  is integrally closed. This recovers the well-known results developed for the classic contexts of polynomial rings and Krull semigroup rings. Further, we obtain an interesting result on the natural homomorphism  $\phi : Cl(A) \rightarrow Cl(A[\Gamma])$ , that is,  $Cl(A[\Gamma]) = Cl(A)$  if and only if  $A$  and  $\Gamma$  are integrally closed and  $Cl(\Gamma) = 0$ . Our results are backed by original examples.

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### 0. Introduction

All rings considered in this paper are integral domains. Throughout,  $\Gamma$  will always denote a torsionless grading monoid. That is,  $\Gamma$  is a commutative cancellative monoid, written additively, and the quotient group generated by  $\Gamma$  is a torsion-free abelian group. By a graded domain  $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ , we mean an integral domain  $R$  graded by an arbitrary torsionless grading monoid  $\Gamma$ . Suitable background on torsionless grading monoids and  $\Gamma$ -graded rings is [21].

Let  $R$  be an integral domain. Following [6], we define the class group of  $R$ , denoted  $Cl(R)$ , to be the group of  $t$ -invertible fractional  $t$ -ideals of  $R$  under  $t$ -multiplication modulo its subgroup of principal fractional ideals. Divisibility properties of a domain  $R$  are often reflected in group-theoretic properties of  $Cl(R)$ . For  $R$  a Krull domain,  $Cl(R)$  is the usual divisor class group of  $R$ . In this case,  $Cl(R) = 0$  if and only if  $R$  is factorial. If  $R$  is a Prüfer domain, then  $Cl(R) = Pic(R)$  is the ideal class group of  $R$ . In this case,  $Cl(R) = 0$  if and only if  $R$  is a Bezout domain. We assume familiarity with class groups and related concepts, as in [6], [10], and [12].

On one hand, a well-known result is that if  $R$  is a  $Z_+$ -graded Krull domain, then  $Cl(R)$  is generated by the classes of homogeneous height-one prime ideals of  $R$  [10, Proposition 10.2], i.e.,  $Cl(R) = HCl(R)$ , where  $HCl(R)$  is the homogeneous class group of  $R$ . In [3, Theorem 4.2], D.F. Anderson showed that the same holds for any  $\Gamma$ -graded Krull domain, where  $\Gamma$  is an arbitrary torsionless grading monoid. So one may remove the “Krull assumption” and legitimately ask the following: *For an arbitrary graded domain  $R$ , how does the equality “ $Cl(R) = HCl(R)$ ” reflect in ring-theoretic properties of  $R$ ?*

On the other hand, many authors investigated the problem of characterizing ring-theoretic properties in terms of Picard groups. In [7], Bass and Murthy proved that for an integral domain  $A$ , if  $Pic(A[X, X^{-1}]) = Pic(A)$  then  $A$  is seminormal. However, Pedrini showed that the converse fails to be true in general [22, p. 96]. Many years later, Gilmer and Heitmann [14] solved completely the problem of characterizing “seminormality” in terms of Picard groups. They stated that  $Pic(A[X]) = Pic(A)$  if and only if  $A$  is seminormal. In the same line, in 1982, the Andersons [1] examined the property of almost normality for graded domains. They established that if  $R$  is an almost normal graded domain with  $R_0 \subset R$  inert, then  $Pic(R) = HPic(R)$ . So the problem remained somehow open. However, in 1987, Gabelli proved that for an integral domain  $A$ ,  $Cl(A[X]) = Cl(A)$  if and only if  $A$  is integrally closed [11, Theorem 3.6]. Recall for convenience that  $A[X]$ , graded in the natural way, is almost normal if and only if  $A[X]$  (and hence  $A$ ) is integrally closed. This motivates our second question: *For an arbitrary graded domain  $R$ , how does “almost normality” reflect in group-theoretic properties of  $Cl(R)$ ?*

This paper contributes to the study of class groups of graded integral domains. It particularly provides a satisfactory (and unique) answer to the previous two questions. As an application, we state a decomposition theorem for class groups of

semigroup rings. Indeed, Section 1 examines the interconnection between “almost normality” and the equality “ $Cl(R) = HCl(R)$ ” for a graded domain  $R$ . More precisely, we show, in Theorem 1.1, that if  $R_0 \subset R$  is inert, then  $Cl(R) = HCl(R)$  if and only if  $R$  is almost normal. Some interesting contexts for this result are  $Z_+$ -graded domains and polynomial rings. However, Example 1.11 illustrates its failure if one omits the condition “ $R_0 \subset R$  inert”. In the first part of Section 2, we focus on the specific case of semigroup rings, which provide an important class of graded domains. We establish the following decomposition theorem, Theorem 2.7: For an integral domain  $A$ , with quotient field  $K$ ,  $Cl(A[\Gamma]) \cong Cl(A) \oplus HCl(K[\Gamma])$  if and only if  $A[\Gamma]$  is integrally closed. This recovers most of the previous results stated for the classic contexts of polynomial rings [11] and Krull semigroup rings [3]. The second part of Section 2 is devoted to semigroups. Here we extend Chouinard’s results on Krull semigroups (cf. [8]) to arbitrary semigroups. As an application, we establish an interesting result, Theorem 2.12, on the natural homomorphism  $\phi : Cl(A) \rightarrow Cl(A[\Gamma])$ , that is,  $Cl(A[\Gamma]) = Cl(A)$  if and only if  $A$  and  $\Gamma$  are integrally closed and  $Cl(\Gamma) = 0$ .

## 1. The class group of a graded domain

The discussion which follows, concerning basic facts and notations connected with graded domains, will provide some background to the main result of this section and will be of use in its proof. In this section,  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  denotes a  $\Gamma$ -graded domain and  $S$  the multiplicatively closed subset of all nonzero homogeneous elements of  $R$ . Thus  $R_S$  is a  $\langle \Gamma \rangle$ -graded domain with

$$(R_S)_\alpha = \left\{ \frac{a}{b}; a \in R_\beta, 0 \neq b \in R_\gamma, \text{ and } \alpha = \beta - \gamma \right\}.$$

In particular,  $(R_S)_0$  is a field and each nonzero homogeneous element of  $R_S$  is a unit. It is well-known that the domain  $R_S$ , often called the homogeneous quotient field of  $R$ , is a completely integrally closed GCD-domain [2, Proposition 2.1]. If  $R$  is  $Z_-$  or  $Z_+$ -graded, then  $R_S \cong (R_S)_0[X, X^{-1}] \cong (R_S)_0[Z]$ . Semigroup rings  $A[\Gamma]$ , graded in the natural way with  $deg X^\alpha = \alpha$ , constitute perhaps the most important class of  $\Gamma$ -graded domains. The homogeneous quotient field of  $A[\Gamma]$  is the group ring  $K[G]$ , where  $K$  is the quotient field of  $A$  and  $G = \langle \Gamma \rangle$ , the quotient group of  $\Gamma$ .

We say that the graded domain  $R$  is almost normal [1] if for each homogeneous element  $x \in R_S$  of nonzero degree which is integral over  $R$  is actually in  $R$ . Any

integrally closed graded domain is almost normal; moreover, it is well-known that  $R$  is integrally closed if and only if  $R$  is almost normal and  $R_0$  is integrally closed in  $(R_S)_0$ . On the other hand, following Cohn [9], we say that an extension  $A \subset B$  of integral domains is inert if whenever  $xy \in A$  for some  $x, y \in B$ , then  $x = ru$  and  $y = su^{-1}$  for some  $r, s \in A$  and  $u$  a unit of  $B$ . The extension  $R_0 \subset R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  is inert if and only if for each  $\alpha \in \Gamma \cap (-\Gamma)$ ,  $R_\alpha$  contains a unit. This, particularly, happens if either  $R = R_0[\Gamma]$  is a semigroup ring, or  $R_0$  is a field, or  $R$  is  $\Gamma$ -graded with  $\Gamma \cap (-\Gamma) = 0$ .

Now, Let's review some terminology related to the  $v$ - and  $t$ -operations. Let  $A$  be any domain with quotient field  $K$ . By an ideal of  $A$  we mean an integral ideal of  $A$ . Let  $I$  and  $J$  be two nonzero fractional ideals of  $A$ . We define the fractional ideal  $(I : J) = \{x \in K \mid xJ \subset I\}$ . We denote  $(A : I)$  by  $I^{-1}$  and  $(I^{-1})^{-1}$  by  $I_v$ . We say that  $I$  is divisorial or a  $v$ -ideal of  $A$  if  $I_v = I$ . The ideal  $I$  is  $v$ -finite if  $I = J_v$  for some finitely generated fractional ideal  $J$  of  $A$ . For a nonzero fractional ideal  $I$  of  $A$ , we define  $I_t = \cup\{J_v \mid J \subset I \text{ finitely generated}\}$ . The ideal  $I$  is a  $t$ -ideal if  $I_t = I$ . Under the operation  $(I, J) \mapsto (IJ)_t$ , the set of  $t$ -ideals of  $A$  is a semigroup with unit  $A$ . An invertible element for this operation is called a  $t$ -invertible  $t$ -ideal of  $A$ . For more details about these notions, see [12, Sections 32 and 34].

A fractional ideal  $I$  of the graded domain  $R$  is homogeneous if there exists a nonzero homogeneous element  $s$  of  $R$  such that  $sI$  is a homogeneous (integral) ideal of  $R$ . Each homogeneous fractional ideal of  $R$  is contained in  $R_S$ . Moreover, if  $I$  and  $J$  are nonzero homogeneous fractional ideals of  $R$ , then  $(I : J)$  is also a homogeneous fractional ideal of  $R$ , and so are  $I^{-1}$  and  $I_v$  [2, Proposition 2.5]. Let  $T(R)$  (resp.,  $HT(R)$ ) denote the group of  $t$ -invertible fractional  $t$ -ideals (resp., homogeneous  $t$ -invertible fractional  $t$ -ideals) of  $R$ , and  $P(R)$  (resp.,  $HP(R)$ ), its subgroup of principal fractional ideals. We have  $Cl(R) = T(R)/P(R)$  and  $HCl(R) = HT(R)/HP(R)$ , a subgroup of  $Cl(R)$ . Now, let  $x \in R_S$  with  $x = x_{\alpha_1} + \dots + x_{\alpha_n}$ , where  $x_{\alpha_i} \in (R_S)_{\alpha_i}$  and  $\alpha_1 < \dots < \alpha_n$ . We define the content of  $x$ , denoted  $C(x)$ , to be  $C(x) = (x_{\alpha_1}, \dots, x_{\alpha_n})$ , the homogeneous fractional ideal of  $R$  generated by the homogeneous components of  $x$  in  $R_S$ . If  $I \subset R_S$  is a fractional ideal of  $R$ , then  $I$  is homogeneous if and only if  $C(x) \subset I$  for each  $x \in I$ . A well-known result due to Northcott [20] states that, for each  $x, y \in R$ ,  $C(x)^n C(xy) = C(x)^{n+1} C(y)$  for some integer  $n \geq 0$ .

We now announce our main result of this section. It sheds light on the inter-connection between “almost normality” and the equality “ $Cl(R) = HCl(R)$ ” for a graded domain  $R$ .

**Theorem 1.1.** *Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a  $\Gamma$ -graded domain such that  $R_0 \subset R$  is inert. Then  $Cl(R) = HCl(R)$  if and only if  $R$  is almost normal.*

The proof of this theorem requires some preliminaries.

Let  $I$  be a fractional ideal of  $R$  and assume that there exists  $s \in S$  such that  $sI \subset R$ . We define the content of  $I$ , denoted  $C(I)$ , to be the homogeneous fractional ideal of  $R$  generated by the homogeneous components in  $R_S$  of all elements of  $I$ . We have  $C(I) = \sum_{x \in I} C(x)$ , and  $I$  is homogeneous if and only if  $C(I) = I$ . The next two lemmas deal with technical properties of the content of a fractional ideal.

**Lemma 1.2.** *Let  $I$  be a fractional ideal of  $R$  with  $sI \subset R$  for some  $s \in S$ . Then*

- (1)  $C(HI) = HC(I)$ , for each homogeneous fractional ideal  $H$  of  $R$ .
- (2)  $C(I_v)_v = C(I)_v$ .
- (3)  $C(I_t)_t = C(I)_t$ .

**Proof.** (1) is straightforward.

(2) We have  $I \subset C(I)$ , hence  $I_v \subset C(I)_v$  and  $C(I_v)_v \subset C(I)_v$ . The reverse inclusion is trivial.

(3) We first show that  $I_t$  is homogeneous whenever  $I$  is. For, let  $x \in I_t$ . Then there exists a finitely generated fractional ideal  $F \subset I$  such that  $x \in F_v$ . Since  $I$  is homogeneous,  $C(F) \subset I$ . Hence by (2),  $C(x) \subset C(F_v) \subset C(F)_v \subset I_t$ . Therefore,  $I_t$  is homogeneous. Now, since  $I_t \subset C(I)_t$  and  $C(I)_t$  is homogeneous, then  $C(I_t) \subset C(I)_t$ . Hence  $C(I_t)_t \subset C(I)_t$ . The reverse inclusion is trivial.  $\diamond$

**Lemma 1.3.** *Let  $x_1, \dots, x_n \in R$  such that  $(C(x_1) + \dots + C(x_n))_v = R$ , and  $a \in S$ .*

*Then  $(a, x_1, \dots, x_n)_v = R$ .*

**Proof.** Let  $u \in qf(R)$  such that  $a, x_1, \dots, x_n \in uR$ . Then  $u = \frac{a}{r}$  for some  $r \in R$  and, for each  $i$ ,  $x_i = \frac{a}{r}r_i$  for some  $r_i \in R$ . By the Northcott’s result, for each  $i = 1, \dots, n$ , there exists a positive integer  $N_i$  such that  $C(x_i)^{N_i}C(rx_i) = C(x_i)^{N_i+1}C(r)$ . Let  $N = N_1 + \dots + N_n$ . Then  $C(x_i)^N C(rx_i) = C(x_i)^{N+1}C(r)$ , for each  $i = 1, \dots, n$ . On the other hand, for each  $i = 1, \dots, n$ ,  $C(rx_i) = C(ar_i) =$

$aC(r_i) \subset aR$ , hence  $C(x_i)^{N+1}C(r) \subset aR$ . Therefore  $(\sum_{i=1}^n C(x_i)^{N+1})C(r) \subset aR$ . Since  $(\sum_{i=1}^n C(x_i))_v = R$ , then  $(\sum_{i=1}^n C(x_i)^{N+1})_v = R$ . It follows that  $C(r)_v \subset aR$  and hence  $\frac{r}{a} \in R$ . Thus  $1 \in uR$  and hence  $(a, x_1, \dots, x_n)_v = R$ , as desired.  $\diamond$

Next we give a criterion for  $t$ -invertibility in a graded domain (see also [15, section 4]).

**Lemma 1.4.** *Let  $I$  be an ideal of  $R$  such that  $C(I)$  is  $t$ -invertible. Then  $I$  is  $t$ -invertible if and only if  $I_t R_S$  is principal.*

**Proof.** If  $I$  is  $t$ -invertible, by [5, Proposition 2.2],  $I_t R_S$  is a  $t$ -invertible  $t$ -ideal. Since  $R_S$  is a GCD-domain,  $I_t R_S$  is principal. Conversely, set  $J = C(I)^{-1}I_t$ . By Lemma 1.2,  $C(J)_t = C(C(I)^{-1}I_t) = (C(I)^{-1}C(I))_t = R$ . Further, since  $JR_S = I_t R_S$  is principal, it suffices to show that if  $C(I)_t = R$  and  $IR_S$  is principal, then  $I$  is  $t$ -invertible. Let  $x_1, \dots, x_n \in I$  such that  $(\sum_{i=1}^n C(x_i))_v = R$ . Since  $IR_S$  is principal, then  $IR_S = aR_S$  for some  $a \in I$ . Now, if  $x \in I$ , then  $x = \frac{ar}{s}$  for some  $r \in R$  and  $s \in S$ . Thus  $x(s, x_1, \dots, x_n)_v \subset (a, x_1, \dots, x_n)_v$ . By Lemma 1.3,  $(s, x_1, \dots, x_n)_v = R$ , so  $x \in (a, x_1, \dots, x_n)_v$  and hence  $I \subset (a, x_1, \dots, x_n)_v$ . On the other hand, there exists  $t \in S$  such that for each  $i$ ,  $x_i = \frac{ar_i}{t}$  for some  $r_i \in R$ . Hence  $tI \subset (ta, tx_1, \dots, tx_n)_v \subset aR$ , i.e.,  $\frac{t}{a} \in I^{-1}$ . Therefore,  $t = a\frac{t}{a} \in II^{-1}$  and, by Lemma 1.3,  $R = (t, x_1, \dots, x_n)_v \subset (II^{-1})_t$ . Hence,  $I$  is  $t$ -invertible.  $\diamond$

Notice that a useful case of Lemma 1.4 is when  $C(I) = R$ .

**Lemma 1.5.** *Let  $a \in R_S$  be a nonzero element and  $P_a = aR_S \cap R$ . Then  $P_a = uJ$  for some  $u \in R_S$  and some homogeneous ideal  $J$  of  $R$  if and only if  $P_a = aC(a)^{-1}$ .*

**Proof.** If  $P_a = uJ$ , then  $P_a R_S = aR_S = uR_S$ ; so there exist  $s, t \in S$  such that  $u = \frac{s}{t}a$ . Now, since  $\frac{s}{t}aJ \subset R$ , then  $\frac{s}{t}J \subset C(a)^{-1}$ , and hence  $P_a = \frac{s}{t}aJ \subset aC(a)^{-1}$ . The reverse inclusion is trivial. Conversely, if  $P_a = aC(a)^{-1}$ , let  $s \in C(a)$  be a nonzero homogeneous element. Then  $P_a = \frac{a}{s}(sC(a)^{-1})$ , and hence we may take  $J = sC(a)^{-1}$ .  $\diamond$

We next state our key lemma. Its main effect is to link, under the stated hypothesis,  $t$ -invertibility to almost normality.

**Lemma 1.6.** Assume that  $R_0 \subset R$  is inert. The following statements are equivalent.

- (i)  $R$  is almost normal;
- (ii) For each  $v$ -finite  $v$ -ideal  $I$  of  $R$ , there exist  $u \in R_S$  and a homogeneous  $v$ -finite  $v$ -ideal  $J$  of  $R$  such that  $I = uJ$ ;
- (iii) For each  $t$ -invertible  $t$ -ideal  $I$  of  $R$ , there exist  $u \in R_S$  and a homogeneous  $t$ -invertible  $t$ -ideal  $J$  of  $R$  such that  $I = uJ$ .

**Proof.** The equivalence (i) $\Leftrightarrow$ (ii) follows from [1, Theorem 3.7(2) and Theorem 3.2].

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Let  $a \in R_S$  be homogeneous of nonzero degree and integral over  $R$ . Set  $P_a = (1 - a)R_S \cap R$  and let  $f(X) = X^n + r_{n-1}X^{n-1} + \cdots + r_0 \in R[X]$  such that  $f(a) = a^n + r_{n-1}a^{n-1} + \cdots + r_0 = 0$ . Regrouping terms of the same degree, we may assume that the  $r_i$ 's are homogeneous of pairwise distinct nonzero degrees. We have  $f(X) = (X - a)g(X)$ , where  $g(X) = X^{n-1} + b_{n-2}X^{n-2} + \cdots + b_0$  with the  $b_i$ 's are homogeneous elements of  $R_S$  of pairwise distinct nonzero degrees. On the other hand,  $f(1) = (1 - a)g(1) \in P_a$ , so  $1 + r_{n-1} + \cdots + r_0 \in P_a$ , moreover  $C(1 + r_{n-1} + \cdots + r_0) = R$ . It follows that  $C(P_a) = R$ . Since  $P_a R_S = (1 - a)R_S$  is principal and  $P_a$  is a  $t$ -ideal, then  $P_a$  is  $t$ -invertible (cf. Lemma 1.4). Therefore, there exist  $u \in R_S$  and  $J$  a homogeneous  $t$ -invertible  $t$ -ideal such that  $P_a = uJ$ . By Lemma 1.5,  $P_a = (1 - a)C(1 - a)^{-1}$ . We deduce that  $f(1) = (1 - a)(1 + b_{n-2} + \cdots + b_0) \in (1 - a)C(1 - a)^{-1}$ , i.e.,  $1 + b_{n-2} + \cdots + b_0 \in (1, a)^{-1}$ . It follows that  $a + ab_{n-2} + \cdots + ab_0 \in R$ , hence  $a \in R$  since the  $b_i$ 's are homogeneous of pairwise distinct nonzero degrees.  $\diamond$

**Proof of Theorem 1.1.** It follows from Lemma 1.6.  $\diamond$

**Remark 1.7.** (1) In [17], Matsuda constructed an example illustrating the fact that in [1, Theorem 3.7 (2)] and hence in Lemma 1.6 (i) $\Rightarrow$ (ii) the “ $R_0 \subset R$  inert” hypothesis cannot be deleted.

(2) In Lemma 1.6, we need this hypothesis only for the implication (i) $\Rightarrow$ (ii). However, [1, Theorem 3.2 and Theorem 3.7 (1)] shows that we can omit it if we assume that  $R$  contains a (homogeneous) unit of nonzero degree. In this case,  $R$  is almost normal if and only if  $R$  is integrally closed.

(3) By substituting the hypothesis “ $R$  contains a unit of nonzero degree” for “ $R_0 \subset R$  inert”, the statement of Theorem 1.1 remains true, that is,  $Cl(R) = HCl(R)$  if and only if  $R$  is integrally closed.

**Corollary 1.8.** *If  $R = R_0 \oplus R_1 \oplus \dots$  is  $Z_+$ -graded, then  $Cl(R) = HCl(R)$  if and only if  $R$  is almost normal.  $\diamond$*

Let  $A$  be an integral domain and  $X$  an indeterminate over  $A$ . Using [1, Proposition 5.8], one may show that  $HCl(A[X]) = Cl(A)$ . Thus we reobtain Gabelli's result:

**Corollary 1.9.** [11, Theorem 3.6]  *$Cl(A[X]) = HCl(A[X])(= Cl(A))$  if and only if  $A$  is integrally closed.  $\diamond$*

Let  $A \subset B$  be an extension of integral domains. Then  $R = A + XB[X]$  is a particular graded domain with the natural graduation. We reobtain [4, Corollary 1.2]:

**Corollary 1.10.**  *$Cl(A + XB[X]) = HCl(A + XB[X])$  if and only if  $B$  is integrally closed.*

**Proof.** This follows from Corollary 1.8 and the fact that  $A + XB[X]$  is almost normal if and only if  $B$  is integrally closed.  $\diamond$

Now, it seems natural to ask whether the equivalence " $Cl(R) = HCl(R) \Leftrightarrow R$  is almost normal" always holds for a graded domain  $R$ . By the proof of (iii) $\Rightarrow$ (i) of Lemma 1.6, the implication " $Cl(R) = HCl(R) \Rightarrow R$  is almost normal" is always true. However, the converse fails to be true in general as the following example shows.

**Example 1.11.** Let  $K$  be a field and let  $X, Y$ , and  $Z$  be three indeterminates over  $K$ . Set  $T = K[X, Y, Z]/(YZ + X - X^2)$ . Then  $T$  is an integral domain and  $T = K[x, y, z]$ , where  $yz = x(x - 1)$ . Let  $d$  be an integer and set  $T_d = K[x]y^d$  if  $d > 0$ ,  $T_0 = K[x]$ , and  $T_d = K[x]z^{-d}$  if  $d < 0$ . Then  $T = \bigoplus_{n \in \mathbb{Z}} T_d$  is a  $Z$ -graded integral domain (cf. [18, p. 13]). Now, let  $R = \bigoplus_{n \in \mathbb{Z}} R_d$  be the  $Z$ -graded subring of  $T$  defined as follows:  $R_d = T_d$  if  $d \neq 0$  and  $R_0 = K + x(x - 1)K[x]$ . Then  $R$  is almost normal and  $Cl(R) \neq HCl(R)$ .

Let  $S$  be the multiplicatively closed subset of nonzero homogeneous elements of  $R$ . Then  $R_S = K(x)[y, z]$  is a  $Z$ -graded integral domain with  $(R_S)_d = K(x)y^d$  if  $d > 0$ ,  $(R_S)_0 = K(x)$ , and  $(R_S)_d = K(x)z^{-d}$  if  $d < 0$ . Now, let  $\xi \in (R_S)_d$  be integral over  $R$  with  $d > 0$ . Then  $\xi^n + F_{n-1}\xi^{n-1} + \dots + F_0 = 0$  for some  $F_0, \dots, F_{n-1} \in R$ . Since  $\xi$  is homogeneous of degree  $d$ , we may assume that



$F_i \in R_{(n-i)d}$  for each  $i$ . It follows that  $\xi = \varphi(x)y^d$  for some  $\varphi(x) \in K(x)$  and, for each  $i$ ,  $F_i = f_i(x)y^{(n-i)d}$  for some  $f_i(x) \in K[x]$ . Hence  $\varphi^n + f_{n-1}\varphi^{n-1} + \dots + f_0 = 0$ . Since  $K[x]$  is integrally closed, then  $\varphi(x) \in K[x]$ , and hence  $\xi \in R_d$ . The case  $d < 0$  is similar. Therefore,  $R$  is almost normal.

Now we exhibit an invertible ideal of  $R$  which is not proportional to any invertible homogeneous ideal of  $R$ . Let  $a = x^2(x-1)$ ,  $b = x-y$ ,  $c = x(x-1) - xy$ ,  $d = 1 + (z/x)$ ,  $e = z + x - 1$ , and  $f = x - 1$ . Set  $I = (a, b, c)$  and  $J = (d, e, f)$ . Then  $I$  and  $J$  are two fractional ideals of  $R$  with  $IJ \subset R$  and  $1 = 16af - (4x(x-1) - 1)[(be - cd)^2 - (bd)^2 + 2bd] \in IJ$ . It follows that  $I$  is an invertible ideal of  $R$ . Now, assume that there exist  $q \in R_S$  and  $H$  a homogeneous (integral) ideal of  $R$  such that  $I = qH$ . Since  $a \in I$  and  $a$  is a homogeneous element of  $R$ , necessarily  $q$  is homogeneous in  $R_S$ . Therefore,  $I$  is a homogeneous fractional ideal of  $R$ . It follows that  $I = C(I) = (x, y)$  and  $J = C(J) = (1, z/x)$ . Thus,  $IJ = (x, x-1, y, z)$ , a contradiction since  $x \notin R$ .  $\diamond$

**Example 1.12.** Let  $T = C[X]$ , where  $C$  is the field of complex numbers. Then  $R = Z[\sqrt{-5}] + XC[X]$ , graded in the natural way, is an almost normal graded domain which is not integrally closed (and hence a non Krull domain) with  $R_0 \subset R$  inert. In this case,  $Cl(R) = HCl(R) = Cl(Z[\sqrt{-5}]) = Z/2Z$  (cf. [5, Theorem 3.12(2)]).  $\diamond$

## 2. Application: a decomposition theorem for semigroup rings

Let  $A$  denote an integral domain with quotient field  $K$  and  $\Gamma$  a nonzero torsionless grading monoid with quotient group  $G = \langle \Gamma \rangle$ . In the first part of this section, we focus on the specific case of semigroup rings. These constitute maybe the most important class of graded domains. We appeal to the main theorem of Section 1 to establish a decomposition theorem for the class group of a semigroup ring. Specifically, we show that  $Cl(A[\Gamma]) \cong Cl(A) \oplus HCl(K[\Gamma])$  if and only if  $A[\Gamma]$  is integrally closed. This recovers most of the well-known results stated for the classic contexts of polynomial rings [11] and Krull semigroup rings [3]. The problem breaks into two parts: first, we explore the question of when  $Cl(A[\Gamma])$  coincides with  $HCl(A[\Gamma])$ ; then state that  $HCl(A[\Gamma]) \cong Cl(A) \oplus HCl(K[\Gamma])$ .

**Lemma 2.1.** *Let  $A[\Gamma]$  be any semigroup ring. The following statements are equivalent.*

- (i)  $A[\Gamma]$  is almost normal ;
- (ii)  $A[\Gamma]$  is integrally closed ;
- (iii)  $A$  and  $\Gamma$  are integrally closed.

**Proof.** For (i) $\Rightarrow$ (ii), see the proof of [1, corollary 3.9]. The implication (ii) $\Rightarrow$ (i) is obvious. Finally for (ii) $\Leftrightarrow$ (iii), see [13, Corollary 12.11].  $\diamond$

It is known that the extension  $A \subset A[\Gamma]$  is always inert. Thus, as a consequence of Theorem 1.1 and Lemma 2.1, we have:

**Proposition 2.2.** *Let  $A[\Gamma]$  be a semigroup ring. Then  $Cl(A[\Gamma]) = HCl(A[\Gamma])$  if and only if  $A[\Gamma]$  is integrally closed.  $\diamond$*

In order to prove the second (and remaining) part of our main theorem, Theorem 2.7, we need some preliminary results. We begin our discussion by handling a number of technical points. Let  $\mathcal{F}(\Gamma)$  denote the set of all fractional ideals of  $\Gamma$ . Under ordinary addition of subsets of  $G$ , that is,  $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$ ,  $\mathcal{F}(\Gamma)$  is a commutative monoid with zero element  $\Gamma$ . If  $Y, Z \in \mathcal{F}(\Gamma)$ , then  $(Y : Z)$  is defined to be the fractional ideal  $(Y : Z) = \{g \in G \mid g + Z \subset Y\}$ . The fractional ideal  $Y^{-1} = (\Gamma : Y)$  (resp.,  $Y_v = (Y^{-1})^{-1}$ ) is called the inverse (resp., the  $v$ -closure) of  $Y$ . We say that  $Y$  is divisorial or a  $v$ -ideal if  $Y_v = Y$ . The ideal  $Y$  is  $v$ -finite if  $Y = (F + \Gamma)_v$  for some finite subset  $F$  of  $G$ . Note that finitely generated fractional ideals of  $\Gamma$  are of the form  $F + \Gamma$ , where  $F$  is a finite subset of  $G$ . For more details about the  $v$ -operation on semigroups, see [13, p. 215].

Now, let  $Y$  be a fractional ideal of  $\Gamma$ . We define  $Y_t = \cup\{(F + \Gamma)_v \mid F \subset Y \text{ a finite subset}\}$ . It is easily seen that if  $\alpha \in G$  and  $Y, Z \in \mathcal{F}(\Gamma)$ , then:

- (i)  $(\alpha + \Gamma)_t = \alpha + \Gamma$ ;  $(\alpha + Y)_t = \alpha + Y_t$ .
- (ii)  $Y \subset Y_t$ ; if  $Y \subset Z$ , then  $Y_t \subset Z_t$ .
- (iii)  $(Y_t)_t = Y_t$ .

Therefore,  $Y \mapsto Y_t$  defines a  $*$ -operation on  $\Gamma$  (cf. [19, section 10]), called the  $t$ -operation. It is done in analogy with the  $t$ -operation for domains. A fractional ideal  $Y$  of  $\Gamma$  is a  $t$ -ideal if  $Y_t = Y$ , or equivalently for each finite subset  $F \subset Y$ ,  $(F + \Gamma)_v \subset Y$ . Clearly, if  $Y$  is a fractional ideal of  $\Gamma$ ,  $Y_t \subset Y_v$  and hence a  $v$ -ideal is a  $t$ -ideal. Let  $t(\Gamma)$  denote the subset of  $\mathcal{F}(\Gamma)$  of all  $t$ -ideals of  $\Gamma$ . One may easily see that  $(Y + Z)_t = (Y_t + Z)_t = (Y_t + Z_t)_t$  for  $Y, Z$  fractional ideals of  $\Gamma$ , and hence  $t(\Gamma)$  forms a commutative monoid, with zero element  $\Gamma$ , under the operation  $(Y, Z) \mapsto (Y + Z)_t$ . An invertible element for this operation is called a  $t$ -invertible

(fractional)  $t$ -ideal of  $\Gamma$ . As in the case of domains, a  $t$ -invertible  $t$ -ideal is always divisorial and  $v$ -invertible. Conversely, a divisorial  $v$ -invertible ideal  $Y$  of  $\Gamma$  is  $t$ -invertible if and only if  $Y$  and  $Y^{-1}$  are  $v$ -finite.

In what follows, the graduation on the semigroup ring  $A[\Gamma]$  will always mean the natural graduation.

**Lemma 2.3.** *Let  $A[\Gamma]$  be a semigroup ring and let  $I, J$  (resp.,  $Y, Z$ ) be two fractional ideals of  $A$  (resp.,  $\Gamma$ ). Then*

- (1)  $I[Y]$  is a homogeneous fractional ideal of  $A[\Gamma]$ .
  - (2)  $I[Y]$  is finitely generated if and only if  $I$  and  $Y$  are finitely generated.
- If, moreover,  $I$  and  $J$  are nonzero, then
- (3)  $(I[Y] : J[Z]) = (I : J)[(Y : Z)]$ .
  - (4)  $(I[Y])_v = I_v[Y_v]$ .
  - (5)  $(I[Y])_t = I_t[Y_t]$ .

**Proof.** (1) First, note that  $I[Y]$  is the subset of elements of  $K[G]$  of the form  $\sum a_i X^{\alpha_i}$ , where  $a_i \in I$  and  $\alpha_i \in Y$ . We have  $A[\Gamma]I[Y] \subset I[Y + \Gamma] = I[Y]$ . On the other hand, if  $0 \neq a \in A$  and  $\alpha \in \Gamma$  are such that  $aI \subset A$  and  $\alpha + Y \subset \Gamma$ , then  $aX^\alpha I[Y] \subset A[\Gamma]$ . Hence  $I[Y]$  is a fractional ideal of  $A[\Gamma]$ . The fact that  $I[Y]$  is homogeneous is obvious.

(2) If  $I$  and  $Y$  are finitely generated ideals, it is obvious that  $I[Y]$  is finitely generated. Conversely, suppose that  $I[Y]$  is finitely generated. Since  $I[Y]$  is homogeneous, there exist  $a_1, \dots, a_n \in I$  and  $\alpha_1, \dots, \alpha_n \in Y$  such that  $I[Y] = (a_1 X^{\alpha_1}, \dots, a_n X^{\alpha_n})$ . Now, one may easily check that  $I = (a_1, \dots, a_n)$  and  $Y = \{\alpha_1, \dots, \alpha_n\} + \Gamma$ .

(3) Clearly,  $(I : J)[(Y : Z)] \subset (I[Y] : J[Z])$ . For the reverse inclusion, let  $f \in (I[Y] : J[Z])$ . So  $fJ[Z] \subset I[Y]$  and then  $f \in (I : J)[G]$ . On the other hand,  $fJ[Z] \subset I[Y]$  implies that  $\alpha + Z \subset Y$  for each  $\alpha \in \text{Supp}(f)$ . Hence  $\text{Supp}(f) \subset (Y : Z)$ . It follows that  $f \in (I : J)[(Y : Z)]$ .

(4) Follows from (3).

(5) By (2) and (4), we have

$$\begin{aligned} (I[Y])_t &= \bigcup \{(F[T])_v \mid F \subset I \text{ and } T \subset Y, F \text{ and } T \text{ are ideals of finite type}\} \\ &= \bigcup \{F_v[T_v] \mid F \subset I \text{ and } T \subset Y, F \text{ and } T \text{ are of finite type}\} \\ &= I_t[Y_t]. \diamond \end{aligned}$$

**Corollary 2.4.** *Let  $A[\Gamma]$  be a semigroup ring,  $I$  a nonzero fractional ideal of  $A$ , and  $Y$  a fractional ideal of  $\Gamma$ . Then*

(1)  $I[Y]$  is a  $v$ -ideal (resp.,  $t$ -ideal) if and only if  $I$  and  $Y$  are  $v$ -ideals (resp.,  $t$ -ideals).

(2)  $I[Y]$  is  $v$ -finite if and only if  $I$  and  $Y$  are  $v$ -finite.

(3)  $I[Y]$  is  $v$ -invertible (resp.,  $t$ -invertible) if and only if  $I$  and  $Y$  are  $v$ -invertible (resp.,  $t$ -invertible).

**Proof.** (1) is a consequence of the statements (4) and (5) of Lemma 2.3.

(2) Follows from (2) and (4) of Lemma 2.3.

(3) Follows from the fact that  $(I[Y](I[Y])^{-1})_v = (II^{-1})_v[(Y + Y^{-1})_v]$  and  $(I[Y](I[Y])^{-1})_t = (II^{-1})_t[(Y + Y^{-1})_t]$ .  $\diamond$

Next we give a characterization of the homogeneous divisorial ideals of a semigroup ring.

**Proposition 2.5.** *Let  $A[\Gamma]$  be a semigroup ring. The following statements are equivalent.*

(i)  $I$  is a homogeneous fractional  $v$ -ideal of  $A[\Gamma]$  ;

(ii)  $I = J[Y]$  for some fractional  $v$ -ideals  $J$  and  $Y$  of  $A$  and  $\Gamma$ , respectively.

**Proof.** (i) $\Rightarrow$ (ii) Let  $I$  be a nonzero homogeneous fractional ideal of  $A[\Gamma]$ . Then there exist  $0 \neq c \in A$  and  $\gamma \in \Gamma$  such that  $cX^\gamma I \subset A[\Gamma]$ , so  $I \subset K[G]$ . Let  $Y$  be the set of degrees of all homogeneous elements of  $I$  and let  $J$  be the  $A$ -submodule of  $K$  generated by the coefficients of all elements of  $I$ . We have  $Y + \Gamma \subset Y$ ,  $\gamma + Y \subset \Gamma$  and  $cJ \subset A$ . Hence  $Y$  and  $J$  are fractional ideals. Next we show that  $I = J[Y]$ . The inclusion  $I \subset J[Y]$  is trivial. For the reverse inclusion, let  $f, 0 \neq g \in A[\Gamma]$  such that  $I \subset \frac{f}{g}A[\Gamma]$ . Let  $aX^\alpha \in I$  with  $a \neq 0$ . Then  $\frac{f}{g} = \frac{aX^\alpha}{h}$  for some  $0 \neq h \in A[\Gamma]$ . Now, let  $bX^\beta \in I$ . Then  $bX^\beta \in \frac{aX^\alpha}{h}A[\Gamma]$ . That is,  $bX^\beta h \in aX^\alpha A[\Gamma]$ , so  $bh \in aA[\Gamma]$ . Therefore,  $bX^\alpha \in \frac{f}{g}A[\Gamma]$  and hence  $bX^\alpha \in I$  (since  $I$  is divisorial). Hence  $J[Y] \subset I$  and  $I = J[Y]$ .

(ii) $\Rightarrow$ (i) Follows from Corollary 2.4.  $\diamond$

**Theorem 2.6.** *Let  $A[\Gamma]$  be a semigroup ring. We have the following splitting exact sequence of natural homomorphisms:*

$$0 \rightarrow Cl(A) \rightarrow HCl(A[\Gamma]) \rightarrow HCl(K[\Gamma]) \rightarrow 0$$

**Proof.** Since  $A[\Gamma]$  is a flat  $A$ -module, the natural homomorphism  $Cl(A) \rightarrow Cl(A[\Gamma])$ ,  $[J] \mapsto [J[\Gamma]]$  is well-defined (cf. [5, Proposition 2.2]), and it induces a

natural homomorphism  $\phi : Cl(A) \rightarrow HCl(A[\Gamma])$ . On the other hand, since  $K[\Gamma]$  is a quotient ring of  $A[\Gamma]$ , we have the natural homomorphism  $Cl(A[\Gamma]) \rightarrow Cl(K[\Gamma])$ ,  $[I] \mapsto [IK[\Gamma]]$ . It induces a natural homomorphism  $\psi : HCl(A[\Gamma]) \rightarrow HCl(K[\Gamma])$ . By Corollary 2.4 and Proposition 2.5, if  $I$  is a homogeneous  $t$ -invertible  $t$ -ideal of  $A[\Gamma]$ ,  $I = J[Y]$  for some  $t$ -invertible fractional  $t$ -ideals  $J$  and  $Y$  of  $A$  and  $\Gamma$ , respectively. Hence  $\psi([I]) = \psi([J[Y]]) = [K[Y]]$ . Thus we have the sequence:

$$Cl(A) \xrightarrow{\phi} HCl(A[\Gamma]) \xrightarrow{\psi} HCl(K[\Gamma])$$

We first show that  $\phi$  is injective. Let  $J$  be a  $t$ -invertible  $t$ -ideal of  $A$  such that  $J[\Gamma] = uA[\Gamma]$  for some homogeneous element  $u \in K[\Gamma]$ . Then  $u \in A$  and  $J = uA$ . Hence  $\phi$  is injective.

Next we show that  $Im\phi = Ker\psi$ . Clearly  $Im\phi \subset Ker\psi$ . To show that  $Ker\psi \subset Im\phi$ , let  $I$  be a homogeneous  $t$ -invertible  $t$ -ideal of  $A[\Gamma]$  such that  $IK[\Gamma] = fK[\Gamma]$  for some homogeneous element  $f \in K[\Gamma]$ . We may assume that  $f = X^\alpha$  for some  $\alpha \in \Gamma$ . Now, set  $I_1 = X^{-\alpha}I$ . Then  $I_1 \subset A[\Gamma]$  and  $I_1K[\Gamma] = K[\Gamma]$ . It follows that  $J = I_1 \cap A \neq 0$  and by [1, Proposition 5.7],  $I_1 = J[\Gamma]$ . Since  $I_1$  is a  $t$ -invertible  $t$ -ideal,  $J$  is a  $t$ -invertible  $t$ -ideal of  $A$  (cf. Corollary 2.4). Hence  $\phi([J]) = [I]$  and  $Ker\psi \subset Im\phi$ .

Now, let  $I$  be a homogeneous  $t$ -invertible  $t$ -ideal of  $K[\Gamma]$ , by Proposition 2.5,  $I = K[Y]$  for some ideal  $Y$  of  $\Gamma$ . From Corollary 2.4,  $Y$  is a  $t$ -invertible  $t$ -ideal of  $\Gamma$  and  $A[Y]$  is a  $t$ -invertible  $t$ -ideal of  $A[\Gamma]$ . Now, consider the map

$$\psi' : HCl(K[\Gamma]) \rightarrow HCl(A[\Gamma]), [I] = [K[Y]] \mapsto [A[Y]].$$

It is clear that  $\psi'$  is a well-defined homomorphism and  $\psi \circ \psi' = i$ , the identity map.

◇

Finally, we are able to announce our decomposition theorem.

**Theorem 2.7.** *Let  $A[\Gamma]$  be a semigroup ring. Then  $Cl(A[\Gamma]) \cong Cl(A) \oplus HCl(K[\Gamma])$  if and only if  $A[\Gamma]$  is integrally closed.*

**Proof.** It follows from Proposition 2.2 (i.e., Theorem 1.1) and Theorem 2.6. ◇

The following corollary is a straightforward consequence of the above results.

**Corollary 2.8.** *Let  $A[\Gamma]$  be an integrally closed semigroup ring. Then  $Cl(A[\Gamma]) \cong Cl(A) \oplus Cl(K[\Gamma])$ .* ◇

The second part of this section is devoted to semigroups. Here our ambient semigroup  $\Gamma$  is actually a nonzero torsionless grading monoid. This is indispensable to ensure that the associated semigroup ring  $A[\Gamma]$  will be an integral domain. Our purpose is to extend Chouinard's results on Krull semigroups (cf. [8]) to arbitrary semigroups. As an application, we establish an interesting result, Theorem 2.12, on the natural homomorphism  $\phi : Cl(A) \rightarrow Cl(A[\Gamma])$  along with a few consequences.

In [8, Lemma 1, p.1463], the author proved that  $Cl(K[\Gamma]) \cong Cl(\Gamma)$  for any field  $K$  and any Krull semigroup  $\Gamma$  with  $\Gamma \cap (-\Gamma) = 0$ . This agreed with the fact that  $Cl(K[\Gamma])$  is not controlled by  $K$  provided  $K[\Gamma]$  is a Krull domain [3, Proposition 7.3(2)].

In analogy with the case of integral domains, we define the  $(t)$ -class group of the semigroup  $\Gamma$ , denoted  $Cl(\Gamma)$ , to be the group of  $t$ -invertible fractional  $t$ -ideals of  $\Gamma$  under  $t$ -multiplication modulo its subgroup of principal fractional ideals. Also, in a Krull semigroup, the  $t$ -operation and the  $v$ -operation coincide. Thus, if  $\Gamma$  is a Krull semigroup,  $Cl(\Gamma)$  is just the divisor class group of  $\Gamma$  defined in [8], see also [13, Section 16].

**Theorem 2.9.** *Let  $K$  be a field. Then  $HCl(K[\Gamma]) \cong Cl(\Gamma)$ .*

**Proof.** Consider the map  $\varphi : Cl(\Gamma) \rightarrow HCl(K[\Gamma])$ ,  $[Y] \mapsto [K[Y]]$ . If  $Y$  is a  $t$ -invertible  $t$ -ideal of  $\Gamma$ , then by Corollary 2.4,  $K[Y]$  is a homogeneous  $t$ -invertible  $t$ -ideal of  $K[\Gamma]$ . Hence  $\varphi$  is a well-defined homomorphism. Now, let  $Y$  be a  $t$ -invertible  $t$ -ideal of  $\Gamma$  such that  $K[Y] = fK[\Gamma]$  for some homogeneous element  $f \in K[G]$ . We may suppose that  $f = X^\alpha$  for some  $\alpha \in Y$ . That is,  $K[Y] = X^\alpha K[\Gamma]$ . Therefore  $K[Y] = K[\alpha + \Gamma]$ , and hence  $Y = \alpha + \Gamma$  is principal. It follows that  $\varphi$  is injective. To show that  $\varphi$  is also surjective, let  $I$  be a homogeneous  $t$ -invertible  $t$ -ideal of  $K[\Gamma]$ . By Proposition 2.5 and Corollary 2.4, there exists  $Y$ , a  $t$ -invertible  $t$ -ideal of  $\Gamma$ , such that  $I = K[Y]$ . Thus  $\varphi([Y]) = [I]$ .  $\diamond$

As a consequence of Theorem 2.9, we have the following corollaries which recover [8, Lemma 1].

**Corollary 2.10.** *Let  $K$  be a field. If  $\Gamma$  is integrally closed, then  $Cl(K[\Gamma]) \cong Cl(\Gamma)$ .*

**Proof.** It follows from Lemma 2.1, Proposition 2.2, and Theorem 2.9.  $\diamond$

**Corollary 2.11.** *Let  $A[\Gamma]$  be an integrally closed semigroup ring. Then  $Cl(A[\Gamma]) \cong Cl(A) \oplus Cl(\Gamma)$ .*

**Proof.** It follows from Theorem 2.7 and Theorem 2.9.  $\diamond$

We close this section by a brief study of the canonical homomorphism  $\phi : Cl(A) \rightarrow Cl(A[\Gamma])$ ,  $[J] \mapsto [J[\Gamma]]$ . Let  $R$  be a graded domain. It is well-known that, when defined, the homomorphism  $\phi : Cl(R_0) \rightarrow Cl(R)$ ,  $[I] \mapsto [(IR)_t]$  is not an isomorphism in general. For example, see [3, section 6]. Nevertheless, in the case of a semigroup ring we give a complete characterization for  $\phi$  to be an isomorphism.

**Theorem 2.12.** *Let  $A$  be an integral domain. Then  $Cl(A[\Gamma]) = Cl(A)$  if and only if  $A$  and  $\Gamma$  are integrally closed and  $Cl(\Gamma) = 0$ .*

**Proof.** Assume that  $Cl(A[\Gamma]) = Cl(A)$ . Since  $\phi$  maps into  $HCl(A[\Gamma])$ , then  $Cl(A[\Gamma]) = HCl(A[\Gamma])$ . Therefore,  $A[\Gamma]$  is integrally closed (cf. Proposition 2.2) and  $Cl(\Gamma) = 0$  (cf. Corollary 2.11). The converse follows from Lemma 2.1 and Corollary 2.11.  $\diamond$

We conclude with some corollaries and examples illustrating (the scope of) Theorem 2.12.

Theorem 2.12 generalizes [11, Theorem 3.6] (see also Corollary 1.9). To see this, let  $\Gamma = \oplus_{\alpha} Z_+ e_{\alpha}$ . Clearly,  $G = \oplus Z e_{\alpha}$ . Since  $\Gamma$  is factorial (cf. [13, Theorem 6.8]), then  $Cl(\Gamma) = 0$ . Hence

**Corollary 2.13.**  *$Cl(A[\{X_{\alpha}\}]) = Cl(A)$  if and only if  $A$  is integrally closed.  $\diamond$*

As an other consequence of Theorem 2.12, we have the following result on group rings which recovers [16, Proposition 5.3].

**Corollary 2.14.** *Let  $A$  be an integral domain and  $G$  a nonzero torsion-free abelian group. Then  $Cl(A[G]) = Cl(A)$  if and only if  $A$  is integrally closed.  $\diamond$*

**Corollary 2.15.** *Let  $A$  be an integral domain. Then  $Cl(A[\{X_{\alpha}, X_{\alpha}^{-1}\}]) = Cl(A)$  if and only if  $A$  is integrally closed.  $\diamond$*

**Example 2.16.** Let  $\Gamma = Z \times Z_+$  and let  $A$  be an integrally closed domain. By [13, Theorem 6.8],  $\Gamma$  is factorial. Hence  $\Gamma$  is integrally closed and  $Cl(\Gamma) = 0$ . Thus  $Cl(A[\Gamma]) = Cl(A[X, X^{-1}, Y]) = Cl(A)$ .  $\diamond$

**Example 2.17.** Let  $\Gamma = \cup_{n \geq 0} \frac{1}{p^n} Z_+$ , where  $p$  is a positive prime integer. Since  $\Gamma$  does not satisfy the a.c.c on subsemigroups, then  $\Gamma$  is not a Krull semigroup. Now let  $A = Z[\sqrt{-5}]$  and  $K$  its quotient field. By [13, Theorem 13.5],  $K[\Gamma]$  is a Bezout domain. Hence  $\Gamma$  is integrally closed and  $Cl(\Gamma) = Cl(K[\Gamma]) = 0$ . Thus  $Cl(Z[\sqrt{-5}][\Gamma]) = Cl(Z[\sqrt{-5}]) = Z/2Z$ .  $\diamond$

Theorem 2.12 shows that the class group of  $A[\Gamma]$  measures the failure of integral closure for the ring  $A$  and for the semigroup  $\Gamma$ . The following example illustrates this fact. It also illustrates the failure of Corollary 2.10 for non-integrally closed semigroups.

**Example 2.18.** Let  $\Gamma = \{0, 2, 3, 4, \dots\}$  and  $K$  be any field. Then  $K[\Gamma] = K[X^2, X^3]$  and  $\Gamma$  is not integrally closed. Let  $Y_0 = \{2, 3, 4, \dots\}$ . Then  $Y_0$  is an integral ideal of  $\Gamma$ , and if  $Y$  is a nonprincipal integral ideal of  $\Gamma$ ,  $Y = n + Y_0$ , for some integer  $n \geq 0$ . On the other hand, one can easily check that  $Y_0$  is a divisorial ideal of  $\Gamma$ . Hence all ideals of  $\Gamma$  are divisorial. Thus  $Cl(\Gamma) = Pic(\Gamma)$ . Now let  $Y$  be an invertible ideal of  $\Gamma$ . Then  $Y + (\Gamma : Y) = \Gamma$ . Let  $n \in Y$  and  $m \in (\Gamma : Y)$  such that  $n + m = 0$ . We have  $Y = n + m + Y \subset n + \Gamma$ , so  $Y = n + \Gamma$  is a principal ideal. Hence  $Cl(\Gamma) = Pic(\Gamma) = 0$ . Thus  $Cl(\Gamma) = HCl(K[\Gamma]) = 0$ , while  $Cl(K[\Gamma]) = K$  (cf. [5, Example 4.7(2)]).  $\diamond$

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