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n-Coherent Rings and Modules

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ABSTRACT : For each positive integer n, the notions of an n-coherent module and an n-coherent (commutative) ring are introduced, with the n=1 cases corresponding to the classical meanings of "coherence". Results are developed for various pullback contexts (the context of Greenberg and the classical D+M-constructions) in which coherence has been studied earlier.

1 INTRODUCTION

All rings considered below are commutative with unit, and all modules are unital. If n is a nonnegative integer, we say that an R-module M is n-presented if there is an exact sequence $F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$ of R-modules in which each F_i is finitely generated and free. (Our usage follows [4]; in [12], such M is said to "have a finite n-presentation".) In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Following [1], we let $\lambda(M) = \lambda_R(M) = \sup\{n/\ M$ is an n-presented R-module$\}$, so that $0 \leq \lambda(M) \leq \infty$; the properties of the function λ are recalled in Lemma2.2. Classically, the "n-presented concept allows both ideal-theoretic and module-theoretic approaches to coherent rings. Indeed (cf. [1], p.63, Exercise12), a ring R is coherent if and only if each finitely generated ideal of R is finitely presented; equivalently, if and only if each finitely presented R-module is 2-presented. Accordingly, as explained below, we use the λ-function to introduce both ideal and module theoretic approaches to "n-coherence" for any positive integer n. For background on coherence, we refer the reader to [8]. We also assume some familiarity with the studies of coherent rings in various pullback contexts ([7],[5],[3]); as well as with the (n,d)-properties introduced recently in [4].

Let n be a positive integer. We say that R is n-coherent (as a ring) if each $(n-1)$-presented ideal of R is n-presented; and that R is a strong n-coherent ring if each n-presented R-module is $(n+1)$-presented. (This terminology is not the same as that of [4], where Costa's "n-coherence" is our "strong n-coherence"; nor is our usage that of "r-coherence" mentioned in ([12], p.90))

Thus, the 1-coherent rings are just the coherent rings. Strong n-coherence arose naturally in Costa's study [4] of the (n,d)-properties. In general, any strong n-coherent ring is n-coherent (by, for instance, the version of Schanuel's Lemma in ([12], p.89). The converse holds if $n = 1$ (by the result ([1], p.63, Exercise12) cited earlier), but it is an open question for $n \geq 2$. Notice that each Bezout (for instance, valuation) domain R is n-coherent for each $n \geq 1$; indeed, each $(n-1)$-presented ideal of R is principal and hence infinitely-presented (in the obvious sense). Moreover, each Noetherian ring is n-coherent for any $n \geq 1$.

Section 2 begins, more generally, by defining n-coherent modules for each integer $n \geq 1$. As one might expect, the 1-coherent modules are just the "coherent modules" in the sense of [1]; and a ring R is an n-coherent ring if and only if R is an n-coherent R-module. Several results on transfer of n-coherence are developed in section 2, and these are used in section 3 to develop examples of n-coherent rings (and, more generally, to study associated properties) in the two pullback contexts cited above.

2 N-COHERENCE

If R is a ring and n is a positive integer, we say that an R-module M is an n-coherent module if M is n-presented and each $(n-1)$-presented submodule of M is n-presented. It follows from [1,p.62] that the 1-coherent modules are just the "coherent modules", in the sense of [1].

It will be helpful to isolate the following elementary result.

REMARK 2.1 Let R be a ring and let n be a positive integer. Then each $(n-1)$-presented submodule of an $n-$coherent $R-$module is itself an $n-$coherent $R-$module.

For reference purposes, we summarize some behavior of λ .

LEMMA 2.2([1, p.61, Exercise 6]) *Let R be a ring and let $0 \to P \to N \to M \to 0$ be an exact sequence of R modules. Then :*
a) $\lambda(N) \geq inf\{\lambda(P), \lambda(M)\}$.
b) $\lambda(M) \geq inf\{\lambda(N), \lambda(P)+1\}$.
c) $\lambda(P) \geq inf\{\lambda(N), \lambda(M)-1\}$.
d) *If $N = P \oplus M$ then $\lambda(N) = inf\{\lambda(M), \lambda(P)\}$.*

THEOREM 2.3 *Let R be a ring and let $0 \to P \xrightarrow{u} N \xrightarrow{v} M \to 0$ be an exact sequence of R-modules.*
1) *If $\lambda(P) \geq n-1$ and N is an n-coherent module, then M is an n-coherent module.*
2) *If $\lambda(M) \geq n$ and N is an n-coherent module, then P is an n-coherent module.*

Proof :1) P is $(n-1)$-presented and N is n-presented; therefore, M is n-presented by Lemma 2.2(b). Let M_1 be an $(n-1)$-presented submodule of M. Then the exact sequence : $0 \to P \xrightarrow{u} v^{-1}(M_1) \xrightarrow{v} M_1 \to 0$ shows that $\lambda(v^{-1}(M_1)) \geq inf\{\lambda(P), \lambda(M_1)\} \geq n-1$ (Lemma 2.2(a)); therefore, $\lambda(v^{-1}(M_1)) \geq n$ since $v^{-1}(M_1) \subseteq N$ and N is n-coherent. We conclude, by Lemma 2.2(b), that $\lambda(M_1) \geq inf\{\lambda(v^{-1}(M_1)), \lambda(P)+1\} \geq n$.

2) M and N are both n-presented; therefore, P is $(n-1)$-presented by Lemma 2.2(c). Every $(n-1)$-presented submodule of an n-coherent module is an n-coherent module by Remark 2.1; therefore, P is n-coherent.

THEOREM 2.4 *Let $m \geq n$ be positive integers and let $M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} M_2 \to \ldots \xrightarrow{u_m} M_m$ be an exact sequence of n-coherent R-modules. Then $Im(u_i)$, $Ker(u_i)$ and $Coker(u_i)$ are n-coherent R-modules for each $i = 1, 2, \ldots, m$.*

Proof : It suffices to prove the assertion for $m = n$. Let $M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} M_2 \to \ldots \xrightarrow{u_n} M_n$ be an exact sequence of n-coherent R-modules. We then have exact sequences :
$0 \to Ker(u_1) \to M_0 \to Im(u_1) \to 0$,
$0 \to Im(u_i) = Ker(u_{i+1}) \to M_i \to Im(u_{i+1}) \to 0$, for each $i = 1, \ldots, n-1$, and
$0 \to Im(u_n) \to M_n \to Coker(u_n) \to 0$.
$Im(u_1)$ is finitely generated since M_0 is finitely generated (for M_0 is n-coherent); therefore, $Im(u_2)$ is 1-presented; and by induction, we conclude that $Im(u_n)$ is

$(n-1)$-presented. Thus $Im(u_n)$ is an n-coherent module by Remark 2.1 since $Im(u_n)$ is a submodule of the n-coherent module M_n. Therefore, $Im(u_i)$ and $Ker(u_i)$ are n-coherent modules by applying Theorem 2.3 to the above exact sequences. Finally, Theorem 2.3 and the exactness of the sequence $0 \to Im(u_i) \to M_i \to Coker(u_i) \to 0$ show that $Coker(u_i)$ are n-coherent modules.

THEOREM 2.5 *Let $n \geq 1$, let the canonical ring homomorphism $R \to R/I$ satisfy $\lambda_R(R/I) \geq n$, and let M be an R-module such that $IM = 0$. Then M is n-coherent as an R/I-module if and only if M is n-coherent as an R-module.*

Before proving this theorem, we establish the following three Lemmas.

LEMMA 2.6 *Let $R \to S$ be a ring homomorphism such that $\lambda_R(S) \geq n$ and let M be an n-presented S-module. Then M is an n-presented R-module.*

Proof : By induction on n.

Case $n = 0$: If M is a finitely generated S-module and S a finitely generated R-module, it is clear that M is a finitely generated R-module.

Assume the result is true for n. Let M be an $(n+1)$-presented S-module and let $\lambda_R(S) \geq n+1$. We must show that $\lambda_R(M) \geq n+1$. Let $F_{n+1} \xrightarrow{u_{n+1}} F_n \xrightarrow{u_n} \dots \to F_1 \xrightarrow{u_1} F_0 \xrightarrow{u_0} M \to 0$ be a finite $(n+1)$-presentation of M as an S-module. The exact sequence of S-modules $0 \to Ker(u_0) \to F_0 \to M \to 0$ shows that $\lambda_S(Ker(u_0)) \geq n$; so by induction we have $\lambda_R(Ker(u_0)) \geq n$ since $\lambda_R(S) \geq n+1 \geq n$. Moreover, $\lambda_R(F_0) \geq n+1$ since $\lambda_R(S) \geq n+1$ and F_0 is a finitely generated free S-module. Therefore $\lambda_R(M) \geq inf\{\lambda_R(F_0), \lambda_R(Ker(u_0))+1\} \geq n+1$ by Lemma 2.2(b) and this completes the proof of Lemma 2.6.

LEMMA 2.7 *Let $R \to S$ be a ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be an S-module. If M is n-presented as an R-module, then it is n-presented as an S-module.*

Proof : By induction on n.

Case $n = 0$: If M is a finitely generated R-module, then M is also a finitely generated S-module.

We conclude the proof by induction on n. Let M be an S-module such that $\lambda_R(M) \geq n+1$ and $\lambda_R(S) \geq n$. We must show that $\lambda_S(M) \geq n+1$. By induction, we have $\lambda_S(M) \geq n$. The exact sequence of S-modules $0 \to K \to F_0 \to M \to 0$ (in which F_0 is a finitely generated free S-module), considered as an exact sequence of R-modules, shows that $\lambda_R(K) \geq inf\{\lambda_R(F_0);\lambda_R(M)-1\} \geq n$ (Lemma 2.2(c)). Moreover, we have $\lambda_R(S) \geq n \geq n-1$; then by induction we have $\lambda_S(K) \geq n$; therefore, $\lambda_S(M) \geq n+1$ by Lemma 2.2(b) and this completes the proof of Lemma 2.7.

LEMMA 2.8 *Let $R \to S$ be a ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be an S-module. If M is n-coherent as an R-module, then it is n -coherent as an S-module.*

Proof : Let $R \to S$ be a ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be an S-module such that M is n-coherent as an R-module. Lemma 2.7 shows that $\lambda_S(M) \geq n$ since $\lambda_R(M) \geq n$ and $\lambda_R(S) \geq n-1$. Let N be a submodule of the S-module M such that $\lambda_S(N) \geq n-1$. Then by Lemma 2.6, we have $\lambda_R(N) \geq n-1$. Thus, $\lambda_R(N) \geq n$ since M is an n-coherent R-module; therefore, $\lambda_S(N) \geq n$ by Lemma 2.7 and this completes the proof of Lemma 2.8.

Proof of Theorem 2.5 : Let $R \to R/I$ be the canonical homomorphism such that $\lambda_R(R/I) \geq n$ and let M be an R-module such that $IM = 0$. If M is n-coherent as an R-module, then it is n-coherent as an R/I -module by Lemma 2.8 since $\lambda_R(R/I) \geq n \geq n-1$. Conversely, let M be an n-coherent R/I-module. By Lemma 2.6, we have $\lambda_R(M) \geq n$ because $\lambda_R(R/I) \geq n$. Let N be a submodule of the R -module M such that $\lambda_R(N) \geq n-1$. By Lemma 2.7, we have $\lambda_{R/I}(N) \geq n-1$ since $\lambda_R(R/I) \geq n$. Thus $\lambda_{R/I}(N) \geq n$ since M is an n-coherent R/I-module and N is a submodule of M as an R/I-module. Therefore, $\lambda_R(N) \geq n$ by Lemma 2.6 ($\lambda_R(R/I) \geq n$) and this completes the proof of Theorem 2.5.

REMARK 2.9 Let the canonical ring homomorphism $R \to$ R/I satisfy $\lambda_R(R/I) \geq n-1$, and let M be an R module such that $IM = 0$. If M is n-coherent as R-module, then it is n-coherent as an R/I-module by Lemma 2.8.

APPLICATION 2.10 *Let R be an n-coherent ring (i.e : R is n-coherent as an R-module) and let I be an $(n-1)$-presented ideal of R. Since R is an n-coherent R-module, it follows from Theorem 2.3(1) that R/I is an n-coherent R-module; therefore, by Theorem 2.5, R/I is an n-coherent ring. The case $n = 1$ recovers the known fact that if I is a finitely generated ideal of a coherent ring R, then R/I is a coherent ring.*

THEOREM 2.11 *Let $R \to S$ be a ring homomorphism making S a faithfully flat R-module and let M be an R-module. If $M \otimes S$ is an n-coherent S-module, then M is an n-coherent R-module.*

Proof : We have $\lambda_S(M\otimes S)\geq n$ since $M\otimes S$ is an n-coherent S-module; therefore, $\lambda_R(M)\geq n$ since S is a faithfully flat R-module. Let N be an $(n-1)$-presented submodule of M. Because S is a flat R-module, $\lambda_S(N\otimes S)\geq n-1$ and we may assume that $N\otimes S\subseteq M\otimes S$. Thus, $\lambda_S(N\otimes S)\geq n$ (since $M\otimes S$ is an n-coherent S-module); therefore, $\lambda_R(N)\geq n$ since S is a faithfully flat R -module.

Recall that a ring R is called n-coherent (as ring) if each $(n-1)$-presented ideal of R is n-presented. For example, each valuation domain and each Noetherian ring are n-coherent for each $n\geq 1$.

THEOREM 2.12 *Let $R\to S$ be a ring homomorphism making S a faithfully flat R-module. If S is an n-coherent ring then R is an n-coherent ring.*

Proof : Take $M=R$ in Theorem 2.11.

THEOREM 2.13 *Let $(R_i)_{i=1,2,\dots,m}$ be a family of rings. Then $\prod_{i=1}^{m} R_i$ is an n-coherent ring if and only if R_i is an n-coherent ring, for each $i=1,\dots,m$.*

To prove this Theorem, we need the following Lemma.

LEMMA 2.14 *Let R_1 and R_2 be two rings. Then R_i is an infinitely presented ideal of $R_1\times R_2$, for $i=1,2$.*

Proof : The rings R_1 and R_2, more acurately $R_1\times 0$ and $0\times R_2$, are two finitely generated ideals of $R_1\times R_2$ because $0\to R_1\to R_1\times R_2\to R_2\to 0$ and $0\to R_2\to R_1\times R_2\to R_1\to 0$ are exact sequences. We finish the proof of this Lemma by induction on the degrees of presentation of the R_i using the above two exact sequences.

Proof of Theorem 2.13 : Using induction on m, it suffices to prove the assertion for $m=2$. Let R_1 and R_2 be two rings such that $R_1\times R_2$ is an n-coherent ring. Since $R_1\cong(R_1\times R_2)/R_2$, $R_2\cong(R_1\times R_2)/R_1$, and the R_i are infinitely presented ideals of $R_1\times R_2$ (Lemma 2.14), then Application 2.10 shows that $R_i(i=1,2)$ are n-coherent rings. Conversely, let R_1 and R_2 be two n-coherent rings and let $I=I_1\times I_2$ be an $(n-1)$-presented ideal of $R_1\times R_2$, where I_i is an ideal of R_i; then for each $i=1,2$: $\lambda_{R_1\times R_2}(I_i)\geq inf\{\lambda_{R_1\times R_2}(I_1),\lambda_{R_1\times R_2}(I_2)\}=\lambda_{R_1\times R_2}(I)\geq n-1$ (Lemma 2.2(d)). By Lemma 2.7, we have $\lambda_{R_i}(I_i)\geq n-1$ ($\lambda_{R_1\times R_2}(R_i)=\infty$ (Lemma 2.14)). Thus, $\lambda_{R_i}(I_i)\geq n$ since R_i is an n-coherent ring and by Lemma 2.6, we have $\lambda_{R_1\times R_2}(I_i)\geq n$ because $\lambda_{R_1\times R_2}(R_i)=\infty$ (Lemma 2.14). Therefore : $\lambda_{R_1\times R_2}(I)=\lambda_{R_1\times R_2}(I_1\times I_2)=inf\{\lambda_{R_1\times R_2}(I_1),\lambda_{R_1\times R_2}(I_2)\}\geq n$ and this completes the proof of Theorem 2.13.

3 N-COHERENCE IN PULLBACKS

Next we study n-coherent (and, to a lesser extent, strong n-coherent) rings for two pullback contexts where coherence has already been studied. First, we adopt the format and the assumptions of Greenberg [7], in considering :

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[d] \\ A/Q \arrow[r] & B/Q, \end{tikzcd}$$

where we assume that $A\to B$ is an injective flat ring homomorphism and Q is a flat ideal of A such that $QB=Q$.

THEOREM 3.1 *Under the above notation and hypotheses, let $n\geq 1$. If B is an n-coherent ring and A/Q is a strong $(n-1)$-coherent ring, then A is an n-coherent ring.*

Before proving this theorem, we establish the following Lemma.

LEMMA 3.2 *Let n be a nonnegative integer and M a submodule of a flat A–module. Then M is n-presented over A if and only if $B\otimes M$ and $(A/Q)\otimes M$ are n-presented over B and A/Q, respectively.*

Proof : For $n=0$, see[8, p.150, Theorem 5.1.1(3)].

Now, using induction on n, suppose the Lemma is true for n and let M be any $(n+1)$-presented A-module. We have the exact sequence $0\to K\to A^m\to M\to 0$, where $\lambda_A(K)\geq n$ (by Lemma 2.2(c)). By the hypothesis, B is a flat A-module. Moreover, $Tor_A^1(M,A/Q)=0$: since $M\otimes Q\to M$ is an injection because

$M \otimes Q \to F \otimes Q \to F$ are injections, where F is a flat A-module containing M. So tensoring with B and A/Q respectively, we get the following exact sequences :

(*) $\quad 0 \to B \otimes K \to B \otimes A^m (\cong B^m) \to B \otimes M \to 0$ and

$$0 \to A/Q \otimes K \to A/Q \otimes A^m (\cong (A/Q)^m) \to A/Q \otimes M \to 0$$

over B and A/Q-modules respectively. On the other hand, since $\lambda_A(K) \geq n$ and $K \subseteq A^m$, the induction hypothesis shows that $\lambda_B(B \otimes K) \geq n$ and $\lambda_{A/Q}(A/Q \otimes K) \geq n$. Therefore, the exact sequences (*) and Lemma 2.2(b) allow us to conclude that $\lambda_B(B \otimes M) \geq n+1$ and $\lambda_{A/Q}(A/Q \otimes M) \geq n+1$. Conversely, let M be any A-module such that $\lambda_B(B \otimes M) \geq n+1$ and $\lambda_{A/Q}(A/Q \otimes M) \geq n+1$. Consider the exact sequence $0 \to K \to A^m \to M \to 0$ of A-modules. The exact sequences (*) and Lemma 2.2(c) assert that $\lambda_B(B \otimes K) \geq n$ and $\lambda_{A/Q}(A/Q \otimes K) \geq n$. By the induction hypothesis, it follows that $\lambda_A(K) \geq n$ and the exactness of the sequence $0 \to K \to A^m \to M \to 0$ and Lemma 2.2(b) show that $\lambda_A(M) \geq n+1$.

Proof of Theorem 3.1 : Let J be any $(n-1)$-presented ideal of A. Since B is a flat A-module, $J \otimes B = JB$ is an $(n-1)$-presented ideal of B. Moreover, B is n-coherent and therefore $\lambda_B(J \otimes B) \geq n$. Since J is contained in the flat A-module A and $\lambda_A(J) \geq n-1$, we get $\lambda_{A/Q}(J/QJ) = \lambda_{A/Q}(J \otimes A/Q) \geq n-1$ (Lemma 3.2). From the fact that A/Q is strong $(n-1)$-coherent, we deduce that $\lambda_{A/Q}(J \otimes A/Q) \geq n$ and hence by Lemma 3.2 we have $\lambda_A(J) \geq n$.

Notice that for $n = 1$, Theorem 3.1 recovers [7, Theorem 2.4 (iii)]; and for $n = 2$ we obtain :

COROLLARY 3.3 *Under the notation and hypotheses of the beginning of this section, if A/Q is a coherent ring and B is a 2-coherent ring, then A is a 2-coherent ring.*

Proof : Recall that strong 1-coherence is equivalent to 1-coherence.

REMARK 3.4 a) In Lemma 3.2, the hypothesis "B is a flat A-module" is not necessary. We need only to assume that $wdim_A(B) \leq 1$: indeed, we need only the equality $Tor_A^1(B, M) = 0$, which is always true if $wdim_A(B) \leq 1$ [8, p.155, Theorem 5.1.2 (Proof)].

b) Notice that D. Costa [4] has given another definition for "n-coherence". Thus, a ring R is n-coherent (according to Costa) if any n-presented R-module is $(n+1)$-presented. This is what we call a strong n-coherent ring. So a ring that is n-coherent according to Costa is also n-coherent in our sense, with equivalence of the two definitions for $n = 1$ [8, p.45, Theorem 2.3.2].

QUESTION : *is strong n-coherence equivalent to n-coherence for $n \geq 2$?*

REMARK 3.5 Let $n \geq 1$ and let R be a ring. Then the answer to the above question is affirmative if and only if R n-coherent (in our sense) implies R^m is n-coherent as R-module, for each nonnegative integer m. Indeed, let R be a strong n-coherent ring and let $m \geq 0$. Our aim is to show that R^m is an n-coherent R-module. R^m is a k-presented R-module for each k, since it is free. Let M be an $(n-1)$-presented submodule of R^m; then the exact sequence $0 \to M \to R^m \to R^m/M \to 0$ shows that $\lambda_R(R^m/M) \geq n$ (Lemma 2.2(b)). Thus, $\lambda_R(R^m/M) \geq n+1$ since R is a strongly n-coherent ring by hypothesis; therefore, $\lambda_R(M) \geq n$. Conversely, let R be an n-coherent ring, we must show that R is a strong n-coherent ring. Let M be an n-presented R-module. There exists an exact sequence $0 \to P \to R^m \to M \to 0$; and $\lambda_R(P) \geq n-1$ (Lemma 2.2(c)). Thus $\lambda_R(P) \geq n$ since $P \subseteq R^m$ and R^m is an n-coherent R-module; therefore, $\lambda_R(M) \geq n+1$ (Lemma 2.2(b)) and so R is a strong n-coherent ring.

Next, motived by the work in [5] on coherence (the case $n = 1$), we consider n-coherent rings for the classical (pullback) $D + M$ -construction.

THEOREM 3.6 *Let $V = K + M$ be a valuation domain which is not a field, and let $R = D + M$, where D is a subring of the field K. Denote by $qf(D)$ the field of quotients of D.*

1) *If $qf(D) = K$, then R is n-coherent if and only if D is n-coherent.*

2) *If $qf(D) \neq K$, M is a flat R -module and $n \geq 2$, then :*

R n-coherent implies that D is n-coherent, and

D strong n-coherent implies that R is n-coherent.

The proof of this Theorem is based on Lemma 3.2 and the following Lemma :

LEMMA 3.7 [2, Theorem 2.1, (n)] *Let $V = K + M$ be a valuation domain and $R = D + M$ be a subring of V, where D is a subring of the field K. If I is an ideal of R contained in M, then either I is an ideal of V or IV is a principal ideal of V. Moreover, if I is not an ideal of V and if $IV = aV$, where $a \in I$, then $I = Wa + Ma$, for some D-submodule W of K such that $D \subseteq W \subset K$.*

Proof of Theorem 3.6 : **1)** Since $qf(D) = K$, by [5, Theorem 7] we deduce that M is a flat R-module. Moreover, for $S = D - \{0\}$, we have that $V = K + M = S^{-1}(D + M) = S^{-1}R$ is a flat R-module and then Lemma 3.2 may be applied to the pullback :

$$\begin{tikzcd} R = D+M \arrow[r, hook] \arrow[d] & V = K+M \arrow[d] \\ D = R/M \arrow[r] & K = V/M. \end{tikzcd}$$

Now, assume that R is n-coherent and let J_0 be any nonzero $(n-1)$-presented ideal of D. Set $J = J_0+M$; J is an ideal of R. Since V is a flat R-module, we have : $V \otimes_R J = VJ = (J_0+M)(K+M) = (J_0K)+(J_0M+KM+M^2) = K+M = V$ which is an $(n-1)$-presented V-module. On the other hand, $J \otimes R/M = (J_0+M) \otimes R/M = (J_0+M)/(J_0+M)M = (J_0+M)/M \cong J_0$, and J_0 is an $(n-1)$-presented $R/M(=D)$-module. Hence by Lemma 3.2, $\lambda_R(J) \geq n-1$. But R is n-coherent, so $\lambda_R(J) \geq n$. Thus by Lemma 3.2, $\lambda_D(J_0) = \lambda_{R/M}(J \otimes R/M) \geq n$ and so D is n-coherent. Conversely, assume that D is n-coherent. As valuation domains are n-coherent, we may assume without loss of generality that D is not a field. Now, let J be any $(n-1)$-presented ideal of R. Two cases are possible :

Case 1 : J $= J_0+M$ with J_0 a nonzero ideal of D : Since $\lambda_R(J) \geq n-1$, Lemma 3.2 shows that $\lambda_D(J_0) = \lambda_{R/M}(J \otimes R/M) \geq n-1$. It follows that $\lambda_D(J_0) = \lambda_{R/M}(J \otimes R/M) \geq n$ (since D is n-coherent). On the other hand, because V is a flat R-module, we have $J \otimes V = JV = (J_0+M)(K+M) = V$ which is an n-presented V-module. By Lemma 3.2 we obtain $\lambda_R(J) \geq n$.

Case 2 : $J \subseteq M$. In this case we need to show that J is n-presented. It suffices, by Lemma 3.2, to prove that $\lambda_V(J \otimes V) \geq n$ and $\lambda_{R/M}(J \otimes R/M) \geq n$. Since $\lambda_R(J) \geq n-1$, Lemma 3.2 shows that $\lambda_V(J \otimes V) \geq n-1$. As V is a flat R-module, $J \otimes V = JV$ is an ideal of V, which is, in particular, finitely generated, and without loss of generality, we may take $J \neq 0$. Therefore, since V is a valuation domain, there exists $0 \neq a \in J$ such that $J \otimes V = JV = aV \cong V$ (as V-modules). Thus $\lambda_V(J \otimes V) = \infty \geq n$.

For the remaining inequality, Lemma 3.7 asserts that J is either an ideal of V or of the form $J = Wa+Ma$ with $a \in J$ and W a D-submodule of K such that $D \subseteq W \subset K$.

If J is an ideal of V, it is a finitely generated R-module and so it is a cyclic V-module (V is a valuation domain). We may assume $J \neq 0$ and so $J \cong V$ (as V-modules). Hence, $J/JM = J \otimes R/M \cong V \otimes R/M \cong V/M$ (as V/M-modules); that is $J/JM \cong K$ as K-modules, and so as D-modules. Therefore, $K \cong J/JM$ is a finitely generated D-module and since K is thus integral over D, D is a field, a contradiction.

If J is not an ideal of V then $J = Wa+Ma$ for some $a \in J$ and $D \subseteq W \subset K$. We have $J \otimes R/M = J/JM$ is an $(n-1)$-presented $R/M(=D)$-module and

$JM = (Wa+Ma)M = MWa+M^2a = Ma+M^2a = Ma$. Since $J \neq 0$, we may assume $a \neq 0$ and then $J \otimes R/M = J/JM = (Wa+Ma)/Wa \cong Wa \cong W$ as D-modules. It follows that $\lambda_{R/M}(J \otimes R/M) = \lambda_D(W) \geq n-1$. Because W is also a finitely generated $D-$module with $D \subseteq W \subset K$=qf(D), there exists an ideal I of D and a nonzero $d \in D$ such that $W = (1/d)I \cong I$ (as D-modules). Hence $\lambda_D(I) = \lambda_D(W) \geq n-1$ and so $\lambda_D(I) \geq n$ (since D is n-coherent). Therefore, $\lambda_{R/M}(J \otimes R/M) = \lambda_D(W) = \lambda_D(I) \geq n$. Thus we proved that $\lambda_V(J \otimes V) \geq n$ and $\lambda_{R/M}(J \otimes R/M) \geq n$. Hence Lemma 3.2 shows that $\lambda_R(J) \geq n$ and thus R is n-coherent.

2) Set $k = qf(D)$ and $V_0 = k+M$; V_0 is a strong 2-coherent ring [4, p.12, Corollary 5.2]. As $V_0 = S^{-1}R$ is a flat R-module (where $S = D - \{0\}$), we may apply Lemma 3.2 to the pullback :

$$\begin{tikzcd} R = D+M \arrow[r, hook] \arrow[d] & V_0 \arrow[d] \\ D = R/M \arrow[r] & k = V_0/M. \end{tikzcd}$$

Now, assume that R is n-coherent. Then, if we replace V with V_0 in part 1), the above argument allows us to conclude that D is n-coherent.

Now, let D be a strong n-coherent ring. We will show that R is n-coherent. Let J be any $(n-1)$-presented ideal of R. Two cases are possible :

Case 1 : $J = J_0+M$ with J_0 a nonzero ideal of D. If we replace V with V_0 in part 1), the same argument shows that J is n-presented.

Case 2 :$J \subseteq M$: From Lemma 3.2, $\lambda_{V_0}(J \otimes V_0) \geq n-1$ (since $\lambda_R(J) \geq n-1$). Since V_0 is a flat R-module, we have that $J \otimes V_0 = JV_0$ is an ideal of V_0 which is finitely presented (since $n \geq 2$). As V_0 is strong 2-coherent, $J \otimes V_0 = JV_0$ is an infinitely presented V-module, that is, $\lambda_{V_0}(J \otimes V_0) = \infty \geq n$.

By Lemma 3.7, J is either an ideal of V or of the form $J = Wa+Ma$ where $a \in J$ and W is a D-submodule of K such that $D \subseteq W \subset K$. If J is an ideal of V, then after replacing V with V_0 in part 1), the same arguments hold : because V_0 is strong 2-coherent [4, Corollary 5.2], V_0 is also strong n-coherent and therefore n-coherent (for $n \geq 2$). If $J = Wa+Ma$ (with $a \in J$ and $D \subseteq W \subset K$), by replacing V with V_0 in part 1), the above reasoning applies and we get $\lambda_{R/M}(J \otimes R/M) = \lambda_D(W) \geq n-1$. Since W is a finitely generated D-module with $D \subseteq W \subset K$, then $k \otimes W = kW$ is a k-vector space of finite dimension and therefore there exists an integer m such that $W \subseteq kW \cong k^m$. Therefore, there exists $0 \neq d \in D$ so that $(1/d)W \subseteq D^m$. It follows that $\lambda_D(D^m/(1/d)W) \geq n$. So $\lambda_D(D^m/(1/d)W) = \infty \geq n+1$ (since D is strong n-coherent) and we have

$\lambda_D((1/d)W) \geq n$. We proved that $\lambda_{V_0}(J \otimes V_0) \geq n$ and $\lambda_{R/M}(J \otimes R/M) \geq n$, and so Lemma 3.2 allows us to complete the proof.

REMARK 3.8 **a)** It follows by [5, Theorem 3] that if $qf(D) = K$, then $R = D + M$ is coherent if and only if D is coherent. This assertion is generalized to "n-coherence" in Theorem 3.6(1).
b) In regard to Theorem 3.6(2), note via [5, Theorem 7] that if $qf(D) \neq K$, then M is a flat R-module if and only if $M = M^2$. Also, by [5, p.51], if D is a field, then the 1-coherence of R implies that M is not a flat R-module.
c) For $n = 2$, Application 2.10 shows that if R is a 2-coherent ring and I is a 1-presented ideal of R, then R/I is a 2-coherent ring. For $R = D + M \subseteq V = K + M$ in which $V = K + M$ is a domain, but not necessarily valuation (cf.[3]), we have a special result in which $I (= M)$ need only be assumed finitely generated over R. It addresses a context not covered by Theorem 3.6.

REMARK 3.9 Let $T = K + M$ be any domain with D a subring of K. If $R = D + M$ is a 2 -coherent ring and M a finitely generated R-module, then $D = R/M$ is a 2-coherent ring. Indeed, since M is finitely generated, by [3, Lemma 1], D is a field and thus is a 2-coherent ring.

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