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Some Factorization Properties of $A + XB[X]$ Domains

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1 INTRODUCTION

Recall that an integral domain R satisfies the ascending chain condition for principal ideals (ACCP) if any ascending chain of principal ideals of R terminates. Some classical classes of domains satisfying ACCP are: Dedekind, Krull and Noetherian domains. These domains are atomic, i.e. any nonzero nonunit element can be written as a (finite) product of irreducible elements (cf. [C]). However, this is not always possible for any integral domain R. For example, if $R = \mathbb{Z}+X\mathbb{Q}[X]$, for any nonzero nonunit $p \in \mathbb{Z}$ and any nonzero $n \in \mathbb{Z}$, $\dfrac{X}{n} = p(\dfrac{X}{pn})$ is not irreducible in R, and hence X cannot be factored into a product of irreducible elements of R.

For an atomic domain R, a nonzero nonunit of R may have many factorizations into irreducibles of R and two factorizations may have different lengths (the length of a

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factorization is the number of its irreducible factors). Thus, following [Zs1] we say that an atomic domain R is a half-factorial domain (HFD), if two factorizations of each nonzero nonunit of R have the same length. An atomic domain R is called finite factorization domain (FFD) [AAZ 2], if every nonzero nonunit of R has only a finite number of non-associate irreducible factors.

Denoting with UFD a unique factorization domain, it is well known that the relations among the above concepts are the following:

$$\begin{array}{ccccc} UFD & \Rightarrow & FFD & & \\ \Downarrow & & \Downarrow & & \\ HFD & \Rightarrow & ACCP & \Rightarrow & Atomic \end{array}$$

and no one of the arrows may be conversed. A proof that HFD implies ACCP can be found as Theorem 2.2 in [Ch].

Let $A \subset B$ be an extension of integral domains and let X be an indeterminate over B. Consider $R = A + XB[X]$. This type of construction is useful in order to get examples of domains which satisfy or do not satisfy assigned factorization properties. For example, it has been shown that $R = \mathbb{R}+X\,\mathbb{C}[X]$ is a HFD such that the polynomial ring over R is not a HFD (cf. [AAZ 1, Example 5.4]).

This paper deals with the transfer of the previously recalled factorization properties among the domains A, B and $R = A + XB[X]$.

The same subject has been deeply studied in [AAZ 1] and [AAZ 2] and several of our results are slight generalizations of results contained in these papers. We point out however that with our results we get new examples of certain types of domains. For example, from Theorem 3.4 we obtain that $K + XK[X,Y]$, where K is a field, is a HFD.

1 ACCP CONDITION

Recall that an integral domain R satisfies the ascending chain condition for principal ideals (ACCP) if any ascending chain of principal ideals of R terminates. In this section, we shall determine when the $A+XB[X]$ construction yields domains which satisfy ACCP. Precisely, we establish a result that recovers [AAZ 1, Example 5.1].

PROPOSITION 1.1. *Let $A \subseteq B$ be any extension of domains and set $R = A + XB[X]$. If A is a field, then R satisfies ACCP.*

Proof. Let $f_1R \subseteq f_2R \subseteq \cdots$ be an ascending chain of principal ideals of R. Since for any i, $\deg f_{i+1} \leq \deg f_i$, there exists $n \geq 0$ such that for $i \geq n$, the f_i have the same degree. Thus, for any $i \geq n$, $f_i = a_i f_{i+1}$, where $a_i \in A$. Since A is a field, the chain terminates.

PROPOSITION 1.2. *Let $A \subseteq B$ be any extension of domains and set $R = A + XB[X]$. Consider the following conditions:*

(i) B satisfies ACCP and $U(B) \cap A = U(A)$;

(ii) R satisfies ACCP ;

(iii) A satisfies ACCP and $U(B) \cap A = U(A)$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. $(i) \Rightarrow (ii)$. Since B satisfies ACCP, $B[X]$ satisfies ACCP [Gr, p.321]. Moreover, $U(B[X]) \cap R = U(B) \cap A = U(A) = U(R)$ and so the ACCP property descends to R by [Gr, Proposition 2.1].

$(ii) \Rightarrow (iii)$. Since $U(A) = U(R)$, R satisfies ACCP implies that A satisfies ACCP. Suppose that there exists an element $a \in U(B) \cap A$ with $a \notin U(A)$. Then the ascending chain of principal ideals $R(\frac{X}{a^n})_{n \geq 0}$ shows that R does not satisfy ACCP, a contradiction.

COROLLARY 1.3. *Let $A \subseteq B$ be any extension of domains such that the quotient field of A, noted $qf(A)$, is contained in B. Then the following conditions are equivalent.*

(i) A is a field ;

(ii) $R = A + XB[X]$ satisfies ACCP.

Proof. By Proposition 1.1 we have $(i) \Rightarrow (ii)$. By Proposition 1.2 $(ii) \Rightarrow (iii)$, we have that $U(B) \cap A = U(A)$. Since $qf(A) \subseteq B$, A is necessarily a field.

REMARK 1.4. (a) Notice that, by Proposition 1.1 if A is a field, the ring $R = A + XB[X]$ satisfies ACCP. This is true even if B is very "bad", i.e. if B is very far from satisfying ACCP. So the converse of the implication $(i) \Rightarrow (ii)$ in Proposition1.2 does not hold in general.

(b) By implication $(ii) \Rightarrow (iii)$ in Proposition 1.2, we can construct easily examples of rings R that do not satisfy ACCP. Consider for example $A = \mathbb{Z}$ and $B = \mathbb{Q}$. Since $U(B) \cap A \neq U(A)$, the ring $R = \mathbb{Z}+X\mathbb{Q}[X]$ does not satisfy ACCP (an infinite ascending chain of principal ideals of R is for example $(\frac{X}{2^n})_{n \geq 0}$). Applying again Proposition 1.2 $(ii) \Rightarrow (iii)$ to $A = R_1 = \mathbb{Z} + X_1\mathbb{Q}[X_1]$ and $B = \mathbb{Q}[X_1]$, we get that the ring $R_2 = A + X_2B[X_2] = \mathbb{Z}+X_1\mathbb{Q}[X_1] + X_2\mathbb{Q}[X_1, X_2]$ does not satisfy ACCP either. After n applications of Proposition 1.2 $(ii) \Rightarrow (iii)$, we get that the ring $R_n = \mathbb{Z}+X_1\mathbb{Q}[X_1] + X_2\mathbb{Q}[X_1, X_2] + \cdots + X_n\mathbb{Q}[X_1, \cdots, X_n]$ does not satisfy ACCP.

(c) Notice that the converse of the implication $(i) \Rightarrow (ii)$ in Proposition 1.2 does not hold even if we suppose that A and B have the same quotient field. Consider for example $A = \mathbb{Q}+Y\mathbb{R}[Y]$, $B = \mathbb{Q}[\pi]+Y\mathbb{R}[Y]$ and $R = A+XB[X] = \mathbb{Q}+Y\mathbb{R}[Y]+X\mathbb{Q}[\pi]+XY\mathbb{R}[Y, X]$. We have that A and B have the same quotient field and B does not satisfy ACCP, since $\pi \in U(\mathbb{R}) \cap \mathbb{Q}[\pi]$, but π is not invertible in $\mathbb{Q}[\pi]$. Moreover, arguing as in the proof of Proposition 1.1, where $\deg f$ for $f \in R$ is taken to mean "total degree of f in X and Y", we can easily show that R satisfies ACCP.

(d) Notice finally that the converse of the implication $(ii) \Rightarrow (iii)$ in Proposition 1.2 does not hold. Consider $A = \mathbb{Z}$, $B = \mathbb{Z} + Y\mathbb{Q}[Y]$ and $R = A + XB[X] = \mathbb{Z}+X\mathbb{Z}+XY\mathbb{Q}[X, Y]$. We have that A satisfies ACCP and $U(B) \cap A = U(A)$, but R does not satisfy ACCP, as the infinite chain of principal ideals $(\frac{XY}{2^n})_{n \geq 0}$ shows.

COROLLARY 1.5. *Let $A \subseteq B$ be any extension of domains with B integral over A and set $R = A + XB[X]$. Consider the following conditions :*

(i) B satisfies ACCP ;

(ii) R satisfies ACCP ;

(iii) A satisfies ACCP.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. By Proposition 1.2, recall that if B is integral over A then $U(B) \cap A = U(A)$.

REMARK 1.6. In [AAZ 1, Example 5.1], it is shown that if $\overline{\mathbb{Z}}$ is the ring of all algebraic integers, then $R = \mathbb{Z} + X\overline{\mathbb{Z}}[X]$ satisfies ACCP. Since $\overline{\mathbb{Z}}$ is not even atomic, $\overline{\mathbb{Z}}[X] = R'$ (the integral closure of R) does not satisfy ACCP. Hence, R is an example of an integral domain R which satisfies ACCP, but whose integral closure does not satisfy ACCP. This example shows also that the converse of the implication $(i) \Rightarrow (ii)$ in Corollary 1.5 does not hold.

PROPOSITION 1.7. *Let $A \subseteq B$ be any extension of integral domains with A a Krull domain and B integral over A. Then $R = A + XB[X]$ satisfies ACCP.*

Proof. Let $(f_nR)_{n\geq 1}$ be any ascending chain of principal ideals of R. Since the degrees of f_n are nonincreasing, the degrees eventually stabilize. It follows that the chain of principal ideals $(b_nB)_{n\geq 1}$, where b_n are the leading coefficients of f_n, $(n \geq 1)$, is an ascending chain of principal ideals $b_1B \subset b_2B \subset \cdots$ where each $\frac{b_n}{b_{n+1}} \in A$. Thus, for each i, $b_i \in qf(A)[b_1]$. Since B is integral over A, $qf(A)[b_1]$ is a finite algebraic extension field of $qf(A)$. Consequently, the integral closure $\overline{A}$ of A in $qf(A)[b_1]$ is a Krull domain (cf. [G, Theorem 43.13]) and hence satisfies ACCP. It follows that the ascending chain $(b_n\overline{A})_{n\geq 1}$ terminates, say $b_{n_0}\overline{A} = b_{n_0+1}\overline{A} = \cdots$. So $\frac{b_n}{b_{n+1}} \in U(\overline{A}) \cap A = U(A)$, for each $n \geq n_0$. Thus the chain $(f_nR)_{n\geq 1}$ terminates too.

Notice that Proposition 1.7 generalizes [AAZ 1, Example 5.1], since any Dedekind domain is a Krull domain.

2 ATOMIC DOMAINS

Recall that an integral domain R is atomic if each nonzero nonunit of R is a (finite) product of irreducible elements of R. Dedekind, Noetherian, Krull domains and, in general, domains satisfying the ACCP are atomic. However, an atomic domain need not satisfy the ACCP [Gr, Example 2.1].

PROPOSITION 2.1. *Let $A \subseteq B$ be any extension of integral domains such that $qf(A) \subset B$. Then the following conditions are equivalent:*

(i) A is a field ;

(ii) $R = A + XB[X]$ is atomic.

Proof. $(i) \Rightarrow (ii)$. By Proposition 1.1, R satisfies the ACCP. So R is atomic.

$(ii) \Rightarrow (i)$. $T = qf(A) + XB[X]$ is always atomic , since T satisfies the ACCP (Proposition 1.1). By [AAZ 2, Proposition 1.2 (a)], we have that $A = qf(A)$ is a field.

REMARK 2.2. We may wonder whether Proposition 1.2 holds when we replace ACCP with atomic. However, this is not the case. For, let $A \subset B$ be any extension of domains and set $R = A + XB[X]$. Consider the following conditions :

(i) $B[X]$ is atomic and $U(B) \cap A = U(A)$;

(ii) R is atomic ;

(iii) A is atomic and $U(B) \cap A = U(A)$.

Then: **a)** $(ii) \Rightarrow (iii)$ holds. In fact, assume that R is atomic and let $a \in A$ be a nonzero nonunit. Then any factorization of a into irreducibles of R is also a factorization into irreducibles of A (since $U(A) = U(R)$). It follows that A is atomic. Now, let $a \in (U(B) \cap A) \setminus U(A)$. Then $X = \frac{X}{a}a$ is not a product of irreducible elements of R, a contradiction.

Let A be a domain. If $A[X]$ is atomic then so is A. The converse is false (cf. [R, Example 5.1]). Consequently, the converse of $(ii) \Rightarrow (iii)$ is false.

b) [AAZ 2, Example 6.1] and Proposition 1.1 show that $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$, respectively, do not hold.

3 HALF-FACTORIAL DOMAINS

Recall that R is a half-factorial domain (HFD) if each nonzero nonunit x of R is a product of irreducible elements of R and if $x = x_1 \cdots x_n = y_1 \cdots y_r$ are two factorizations into irreducible elements , then $n = r$. Any unique factorization domain (UFD) is a HFD and a HFD satisfies ACCP. In [AAZ 1, Theorem 5.3], it is shown that, if $K \subseteq L$ are fields, then $K + XL[X]$ is a HFD. We give here a slightly stronger result (cf. Theorem 3.4).

LEMMA 3.1. *Let $R = K + XB[X]$, where $K \subseteq B$, K is a field and $B[X]$ a UFD. If*

$$f(X) = X(b_1 + Xh_1(X)) \cdots (b_n + Xh_n(X))$$

where, for each $i = 1, \cdots, n$, $0 \neq b_i \in B \setminus U(B), h_i(X) \in B[X]$ and $b_i + Xh_i(X)$ is an irreducible element of $B[X]$, then f is an irreducible element of R.

Proof. Suppose that f is not irreducible in R. Then, since K is a field, $f = (1 + Xg(X))Xm(X)$ (where $g(X)$ and $m(X)$ are nonzero elements of $B[X]$). Thus, among the irreducible factors of $f(X)$ in $B[X]$, there is a factor of type $(1+Xg'(X))$ (where $g'(X) \in B[X]$), that is neither an associate of X nor of $b_i + Xh_i(X)$ for any i, a contradiction.

REMARK 3.2. Notice that if $R = A + XB[X]$, and if $f \in B[X] \setminus R$ is irreducible in $B[X]$, then it is not necessarily the case that Xf is irreducible in R. Consider for example $R = \mathbb{Z}+X\mathbb{Q}[X]$ and $f = X + \frac{1}{2} = \frac{1}{2}(2X+1)$.

LEMMA 3.3. *Let $R = A + XB[X]$, where $A \subseteq B$ is an extension of integral domains such that $U(B) \cap A = U(A)$ and let $f \in R$. If f is irreducible in $B[X]$, then f is irreducible in R.*

Proof. If $f = gh$ with $g, h \in R$, then $g, h \in B[X]$. So one of the factors (for example g) is irreducible in $B[X]$. But, since $U(B[X]) \cap R = U(R)$, we get that h is invertible in R and hence f is irreducible in R.

THEOREM 3.4. *Let $R = K + XB[X]$, where $K \subseteq B$, K is a field and B a UFD. Then R is a HFD.*

Proof. If $f \in R$, then $f(X) = X^r(b + Xh(X))$, where $r \geq 0$, $0 \neq b \in B$ and $h(X) \in B[X]$. If $r = 0$, then $b \in K$. Since K is a field, b is a unit of R and so $f(X)$ is an associate of an element of R of type $1 + Xh'(X)$, where $h'(X) \in B[X]$. In this case, the factorization of f as a product of irreducible elements in $B[X]$ is also a factorization of f as a product of irreducible elements in R. Indeed, any irreducible factor of f is of type $1 + Xh_i(X)$ and so is an irreducible element of R (cf. Lemma 3.3). Suppose $r \neq 0$ and $b \in U(B)$, then $f = (bX)X^{r-1}(1 + Xh'(X))$. Since bX and X are irreducible elements of R, decomposing the factor $(1+Xh'(X))$ in B[X], we get also, in this case, that f is a product of irreducible elements of R. Now, suppose $r \neq 0$ and $b \in B \setminus U(B)$. Consider the (unique) factorization of f into irreducible elements of $B[X]$:

$f(X) = X^r(b + Xh(X)) = uX^r(b_1 + Xh_1(X)) \cdots (b_n + Xh_n(X))$, where $u \in U(B)$, $b_1, \cdots, b_n \in B$ (at least one non invertible) and $h_1(X), \cdots, h_n(X) \in B[X]$. Since the factors $b_i + Xh_i(X)$ with $b_i \in U(B)$ are associates of elements of type $(1 + Xh_i'(X))$, we get

$$f(X) = vX^r(b_1 + Xh_1(X)) \cdots (b_k + Xh_k(X))(1 + Xh_1'(X)) \cdots (1 + Xh_s'(X)),$$

where $v \in U(B)$, $b_1, \cdots, b_k \in B \setminus U(B)$ and all the factors are irreducible elements in $B[X]$. By Lemma 3.1, $X(b_1 + Xh_1(X)) \cdots (b_k + Xh_k(X))$ is an irreducible element of R and hence f is a product of $r + s$ irreducible elements of R.

By Proposition 2.1, R is atomic. To prove that R is a HFD, we have just to show that if $f \in R$ has the following factorization into irreducible elements of $B[X]$:

$$f(X) = uX^r(b_1 + Xf_1(X)) \cdots (b_k + Xf_k(X))(1 + Xg_1(X)) \cdots (1 + Xg_s(X)),$$

where $u \in U(B)$, $b_1, \cdots, b_k \in B - U(B)$ and $f_i(X), g_j(X) \in B[X]$, then any factorization of f into irreducible elements of R has $s + r$ factors.

Indeed, let $f = (a_1 + Xh_1(X)) \cdots (a_n + Xh_n(X))$ be another factorization of f into irreducible elements of R. Notice that if $a_i = 0$, then $a_i + Xh_i(X)$ is not divisible in $B[X]$ by any factor $1 + Xg_i(X)$. Furthermore, since $B[X]$ is a UFD, each factor $1 + Xg_i(X)$, where $1 \leq i \leq s$, is an associate of an element $a_j + Xh_j(X)$ with $a_j \neq 0$. Thus, $a_i \neq 0$ exactly s times. So the factorization of f into irreducible elements of R is:

$$f = a(1 + Xh_1(X)) \cdots (1 + Xh_s(X))(Xh_1'(X)) \cdots (Xh_{n-s}'(X)),$$

where $a \in K$ and $h_j'(X) \in B[X]$ for $j = 1, \cdots, n-s$. Furthermore, since the factors $Xh_j'(X)$ are irreducible elements of R, the polynomials $h_j'(X)$ are not divisible by X. Consequently, since X^r divides f and X^{r+1} does not divide f, we get that $n = s + r$ and the proof is complete.

EXAMPLE 3.5. Let K be a field and X and Y two indeterminates over K. Then $R = K + XK[X,Y]$ is a HFD.

COROLLARY 3.6. *Let $A \subseteq B$ be domains such that $qf(A) \subseteq B$ and B is a UFD. Then $R = A + XB[X]$ is a HFD if and only if A is a field.*

Proof. Suppose that R is a HFD. If A is not a field, then there is an irreducible element $p \in A$. So by hypothesis, p is a unit of B. It follows that, for any $n \geq 1$, $X = p^n \frac{X}{p^n}$ and hence R is not a HFD, a contradiction. The converse follows from Theorem 3.4.

REMARK 3.7. We don't know if Theorem 3.4 holds, if we replace the hypothesis "B UFD" with "$B[X]$ HFD". Indeed, it is not even easy to get examples of HFD's $B[X]$ that are not UFD's and such that B contains a field. Notice that, even if A and $B[X]$ are HFD, the domain $A + XB[X]$ may not be a HFD. For example, let $A = \mathbb{Z}$ and $B = \mathbb{Z}[\sqrt{-5}]$. Then A is a UFD, B is a Krull domain of class group $\mathbb{Z}/2\mathbb{Z}$ (cf. [Ry, Exemple p.78 and Proposition 2.15 p.74] and [No, Corollaire 1.12 p.29]) and hence $B[X]$ is a HFD [Zs2]. However, $R = \mathbb{Z}+X\mathbb{Z}[\sqrt{-5}][X]$ is not a HFD since $X^2(X^2+5) = X(X+\sqrt{-5})X(X-\sqrt{-5})$ are two factorizations of different lengths into irreducible elements of R.

We close the section with the following :

QUESTION : Let $A \subseteq B$ be an extension of integral domains such that $U(A) = U(B)$, each irreducible element of A is irreducible in B and B is a UFD. Is $R = A + XB[X]$ a HFD ?

4 FINITE FACTORIZATION DOMAINS

Recall that a domain R is a finite factorization domain (FFD) if R is atomic and any nonzero nonunit of R has only a finite number of nonassociate irreducible factors in R. Any UFD is a FFD. The converse is in general false (cf. [AAZ2]).

LEMMA 4.1. *Let $A \subseteq B$ be an extension of integral domains such that $U(B) = U(A)$. Then, B is a FFD $\Rightarrow$ A is a FFD.*

Proof. Let a be a nonzero nonunit element of A. it is enough to show that a has only finitely many nonassociate divisors in A. Let d be such a divisor, i.e. d divides a in A. So d divides a in B. Since B is a FFD, a has only finitely many nonassociate divisors in B, suppose that they are $\{b_1, \cdots, b_n\}$. Thus $d = ub_i$ for some i, with $u \in U(B)$. Since $U(B) = U(A)$, we get $b_i = u^{-1}d \in A$. Thus the set of nonassociate divisors of a in A is a subset of the set of nonassociate divisors of a in B and hence it is also finite.

PROPOSITION 4.2. *Let $A \subseteq B$ be an extension of integral domains and set $R = A + XB[X]$. Consider The following conditions :*

(i) B is a FFD and $U(B) = U(A)$;

(ii) R is a FFD ;

(iii) A is a FFD and $U(B) \cap A = U(A)$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) .

Proof. $(i) \Rightarrow (ii)$ follows from Lemma 4.1 and from the fact that B is a FFD iff $B[X]$ is a FFD (cf. [AAZ 2, Proposition 5.3]).

$(ii) \Rightarrow (iii)$. It is easy to see that A is a FFD. Moreover, since FFD implies ACCP (cf. [AAZ2]) we get by Proposition 1.2 that $U(B) \cap A = U(A)$.

REMARK 4.3. (a) The converse of Lemma 4.1 is false. For example, the extension $\mathbb{Z} \subset \mathbb{Z} + X\mathbb{Q}[X]$ is such that $\mathbb{Z}$ is a FFD (since a UFD) and $U(\mathbb{Z}+X\mathbb{Q}[X]) = U(\mathbb{Z})$. However $\mathbb{Z}+X\mathbb{Q}[X]$ is not even atomic (cf. Proposition 2.3).

(b) The converse of $(ii) \Rightarrow (iii)$ in Proposition 4.2 is not true in general. For, let $R = \mathbb{R} + X\mathbb{C}[X]$, we have $U(\mathbb{C}) \cap \mathbb{R} = U(\mathbb{R})$, $\mathbb{R}$ is a FFD (since it is a field). However, R is not a FFD, since $\{(r+i)X, r \in \mathbb{R}\}$ is an infinite set of nonassociate irreducible divisors of X^2 in R [AAZ 2, Example 4.1]. Notice that the converse of $(ii) \Rightarrow (iii)$ fails to be true even when $A \subset B$ is integral.

(c) Let $K_1 \subset K_2$ be any extension of finite fields and set $R = K_1 + XK_2[X]$. Then R is both a FFD and a HFD [AAZ 2, p. 15]. However, $U(K_1) \subset U(K_2)$. Therefore the converse of $(i) \Rightarrow (ii)$ in Proposition 4.2 is false.

5 GCD DOMAINS

A domain R is said to be a GCD-domain if each pair of nonzero nonunits of R has a greatest common divisor. Such a domain is sometimes called HCF-domain (for high common factor) [C]. PID's, Bezout domains and, in general, UFD's are GCD-domains.

[CMZ, Theorem 1.1] states that for an integral domaind D and a multiplicatively closed subset S of D , the domain $D^{(S)} = D + XD_S[X]$ is a GCD-domain if and only if D is a GCD-domain and $GCD(X, d)$ exists in $D^{(S)}$ for each nonzero nonunit $d \in D$. Therefore, $D + Xqf(D)[X]$ is a GCD-domain if and only if D is a GCD-domain (See also [MS, Example 4.10]). However, this is not the case if we replace $qf(D)$ with any field extension of $qf(D)$. Precisely we have:

THEOREM 5.1. *Let $A \subseteq B$ be any extension of integral domains with $qf(A) \subseteq B$ and let $R = A + XB[X]$. Then R is a GCD-domain if and only if $B = qf(A)$ and A is a GCD-domain.*

Proof. If A is a GCD-domain with $B = qf(A)$, the result follows from [MS, Example 4.10] or [CMZ, Theorem 1.1].

Conversely, assume that R is a GCD-domain. Clearly, A is also a GCD-domain. Now, if A is a field, then R is atomic (Proposition 2.1). Since an atomic GCD-domain is a UFD [G, Proposition 16.4], necessarily $B = qf(A) = A$ (see Remark 5.3 (c) below). Suppose A is not a field and $qf(A) \subsetneq B$. Let $b \in B \setminus qf(A)$, it is clear that bX and X are nonassociate elements of R and that any nonzero nonunit of A divides both bX and X. Moreover, the only common divisors of bX and X are nonzero nonunits of A. It follows that $GCD(bX, X)$ does not exist in R, a contradiction.

As an immediate consequence of Theorem 5.1 we have:

COROLLARY 5.2. *Let $K \subseteq L$ be any field extension. The domain $K + XL[X]$ is a GCD-domain if and only if $K = L$.*

REMARK 5.3. (a) From Theorem 5.1, it follows that $\mathbb{Z} + X\mathbb{Q}[X]$ is a GCD-domain. However, this is not the case for $\mathbb{Z} + (X, Y)\mathbb{Q}[X, Y]$, since each nonzero nonunit of $\mathbb{Z}$ divides both X and Y (see also [MS, Example 4.10]).

(b) For a GCD-domain A, the polynomial extension $A[Y_1, \cdots, Y_n]$ is also a GCD-domain for any set of indeterminates $\{Y_1, \cdots, Y_n\}$ [MS, section 4, p.393]. Thus, one can ask whether the domain $R = A + XA[Y_1, \cdots, Y_n][X]$ is a GCD-domain. The answer is negative. For, let $P = X^2(X+Y_1)(1+Y_1)$ and $Q = X^2(X+Y_1)Y_1$. So, $X(X+Y_1)$ and X are the only common nonassociate irreducible factors of P and Q. However, $GCD(X, X(X+Y_1)) = 1$ (in R) and their product $X^2(X+Y_1)$ does not divide either P or Q.

(c) Let $A \subseteq B$ be any extension of integral domains. Then $R = A + XB[X]$ is a UFD if and only if $A = B$ and B is a UFD. This follows from the fact that a UFD is completely integrally closed and [AAZ 1, Theorem 2.7 (2)].

(d) If $R = A + XB[X]$ is a GCD-domain , then it is integrally closed [C]. It follows that B is integrally closed and A is integrally closed in B [AAZ1, Theorem 2.7 (1)]. Thus, even when $A \subset B$ is an extension of GCD-domains, R may not be a GCD-domain. For example, $R_1 = \mathbb{Z} + X\mathbb{Q}[i\sqrt{2}][X]$ and $R_2 = \mathbb{Z} + X\overline{\mathbb{Z}}[X]$ are not GCD-domains, since $\mathbb{Z}$ is integrally closed neither in $\mathbb{Q}[i\sqrt{2}]$ nor in $\overline{\mathbb{Z}}$.

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On Spectral Binary Relation

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1 INTRODUCTION

1.1 Spectral sets

I. Kaplansky [5] states the following problem " Under what conditions a partially ordered set is isomorphic to the prime spectrum of a ring ordered by inclusion ?". W.J. Lewis and J. Ohm called a partially ordered set *spectral* if it is order isomorphic to Spec(R) for some ring R. Although there are partial results about this subject [2], [3], [6] and [7], the problem of characterizing the spectral sets remains open. However the corresponding topological problem was completely answered by M. Hochster in his remarkable paper [4]. Let $(X, \leq)$ be a partially ordered set, a topology τ on X is said to be compatible with the ordering $\leq$ if the closure of $\{x\}$ is $\mathrm{cl}\{ x \} = [\, x, \rightarrow [\; = \{ y \in X \,/\, x \leq y \}$, which is equivalent to the following two conditions:

i) $[\, x, \rightarrow [$ is a closed subset of (X, τ), for all $x \in X$.

ii) every closed subset F of (X, τ) is closed under specialization (i.e: $[\, x, \rightarrow [\; \subseteq F$, for all $x \in F$).

Obviously $(X, \leq)$ is spectral if and only if there exists an order compatible spectral topology.

1. 2 Binary relation.

Let X be a set equipped with a binary relation R. Let $tc(R)$ be the transitive closure of R defined by: x $tc(R)$ y if and only if $x = y$ or there exist finitely many elements $x_1, \ldots, x_n$ of X such that $x_1 = x, x_n = y$ and $x_i R x_{i+1}$ for all $i < n$.

The relation $tc(R)$ is a quasi-order. Consider the equivalence relation $T(R)$ induced by the quasi-order $tc(R)$ defined by : $x\ T(R)\ y$ if and only if $x\ tc(R)\ y$ and $y\ tc(R)\ x$.

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