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The Dimension of Tensor Products of Commutative Algebras Over a Zero-Dimensional Ring

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0. Introduction

All the rings and algebras considered in this paper will be commutative, with identity elements and ring-homomorphisms will be unital. If A is a ring, then $\dim A$ will denote the (Krull) dimension of A, that is, the supremum of lengths of chains of prime ideals of A. An integral domain D is said to have valuative dimension n (in short, $\dim_v D = n$) if each valuation overring of D has dimension at most n and there exists a valuation overring of D of dimension n. If no such integer n exists, then D is said to have infinite valuative dimension (see [G]). For reader's convenience, recall that for any ring A, $\dim_v A = \sup\{\dim_v(A/P) \mid P \in \mathrm{Spec}(A)\}$, and that a finite-dimensional domain D is a Jaffard domain if $\dim D = \dim_v D$. As the class of Jaffard domains is not stable under localization, an integral domain D is defined to be a locally Jaffard domain if D_P is a Jaffard domain for each prime ideal P of D (see [ABDFK]). Analogous definitions are given in [C] for a finite-dimensional ring.

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In [S] Sharp proved that if K and L are two extension fields of a field k, then

$$\dim(K \otimes_k L) = \min(\text{t.d.}(K : k),\ \text{t.d.}(L : k)).$$

This result provided a natural starting point to explore dimensions of tensor products of somewhat general k-algebras and it was concretized by Wadsworth in [W], where the result of Sharp was extended to AF-domains (for "altitude formula"), that is, integral domains A such that

$$\text{ht}\, P + \text{t.d.}(A/P : k) = \text{t.d.}(A : k)$$

for all prime ideals P of A. He showed that if A_1 and A_2 are AF-domains, then

$$\dim(A_1 \otimes_k A_2) = \min(\dim A_1 + \text{t.d.}(A_2 : k),\ \dim A_2 + \text{t.d.}(A_1 : k)).$$

He also stated a formula for $\dim(A \otimes_k R)$ which holds for an AF-domain A, with no restriction on R. At this point, it is worthwhile to recall that an AF-domain is a (locally) Jaffard domain [Gi].

In [BGK1] we were concerned with AF-rings. A k-algebra A, where k is a field, is said to be an AF-ring provided $\text{ht}\, P + \text{t.d.}(A/P : k) = \text{t.d.}(A_P : k)$, for all prime ideals P of A (for non domains, $\text{t.d.}(A) = \sup\{\text{t.d.}(A/P : k) \mid P \in \text{Spec}\,(A)\}$). A tensor product of AF-domains is perhaps the most natural example of an AF-ring. We then developed quite general results for AF-rings, showing that the results do not extend trivially from integral domains to rings with zero-divisors.

The purpose of this note is to extend all the known results on the dimension of tensor products of k-algebras to the general case where k is any zero-dimensional ring, denoted by R. The most remarkable outcome is perhaps that $\dim(A_1 \otimes_R A_2)$, where A_1 and A_2 are two R-algebras, depends on a subtle relation which intertwines their two R-module structures. Such phenomenon does not hold in the field case since any k-algebra ($\neq \mathbf{0}$) contains k. Thus, our investigation relies on a mild new assumption that keeps under control most results involving the ideal structures of $A_1 \otimes_R A_2$.

In the first section we extend, in a natural way, the definition of the transcendence degree over a field as well as some basic Wadsworth's results to the zero-dimensional ring case. In the same regard, Section 2 and 3 establish the zero-dimensional ring case. In the same regard, Section 2 and 3 establish adequate analogues of all the results stated in [BGK1] for AF-rings. Some examples throw more light on the new phenomenon (cited above).

1. Background and preliminaries on tensor products of algebras over a zero-dimensional ring

Throughout this paper R denotes a zero-dimensional ring. We denote by (A, λ_A) an R-algebra A and its associated ring homomorphism $\lambda_A : R \longrightarrow A$; we denote by λ_A^* the associated spectral map $\text{Spec}(A) \longrightarrow \text{Spec}(R)$.

If P is a prime ideal of A, $\lambda_A^{-1}(P)$ is a maximal ideal of R; so we can consider the transcendence degree of the integral domain A/P over the field $R/\lambda_A^{-1}(P)$; we put:

$$\begin{aligned} \text{t}(A :_{\lambda_A} R) &= \sup\{\text{t.d.}\,(A/P : R/\lambda_A^{-1}(P)) \mid P \in \text{Spec}\,(A)\} \\ &= \sup\{\text{t.d.}\,(A/P : R/\lambda_A^{-1}(P)) \mid P \in \text{Min}\,(A)\} \end{aligned}$$

and we say that $\text{t}(A :_{\lambda_A} R)$ is the transcendence degree of the R-algebra A over R; we write $\text{t}(A : R)$ as an abbreviation for $\text{t}(A :_{\lambda_A} R)$, when there is no ambiguity. All along this note we consider only R-algebras (A, λ_A) such that $\text{t}(A :_{\lambda_A} R) < \infty$. This, of course, ensures that $\dim A < \infty$. If A is an integral domain, p_A denotes $\ker \lambda_A$.

First of all we note that the transcendence degree of an R-algebra A depends on its R-module structure, as it is shown by the next example:

Example 1. Let $R = k(X) \times k$ and $A = k(X)$, where k is a field. Let $\lambda_1 : R \longrightarrow A$ be the ring homomorphism defined by $\lambda_1(x, y) = x$, and $\lambda_2 : R \longrightarrow A$ be the ring homomorphism defined by $\lambda_2(x, y) = y$. We have $\text{t}(A :_{\lambda_1} R) = \text{t.d.}\,((k(X) : k(X)) = 0$ and $\text{t}(A :_{\lambda_2} R) = \text{t.d.}(k(X) : k) = 1$. Thus $\text{t}(A :_{\lambda_1} R) \neq \text{t}(A :_{\lambda_2} R)$.

We begin by giving a simple generalization of a well-known result [ZS] for algebras over a field.

Lemma 1.1. *Let (A, λ_A) be an R-algebra and $P \in \text{Spec}(A)$. Then*

$$\text{ht}\, P + \text{t}(A/P : R) \leq \text{t}(A_P : R).$$

Proof. Clearly, $\text{t}(A/P : R) = \text{t.d.}(A/P : R/\lambda_A^{-1}(P))$. If $P' \in \text{Spec}(A)$ and $P' \subsetneq P$, then for the prime ideal P/P' of the $R/\lambda_A^{-1}(P)$-algebra A/P', by [ZS, p.10], we get

$$\begin{aligned} \text{ht}(P/P') + \text{t.d.}(A/P : R/\lambda_A^{-1}(P)) &\leq \text{t.d.}(A/P' : R/\lambda_A^{-1}(P)) \\ &= \text{t.d.}(A_P/P'A_P : R/\lambda_A^{-1}(P')). \end{aligned}$$

The result then follows. $\square$

The following elementary properties will be used frequently. These statements admit routine proofs.

Lemma 1.2. *Let (A, λ_A) be an R-algebra.*

(1) *If P is a prime ideal of A and $p = \lambda_A^{-1}(P)$, then*

$$\text{ht}\, P = \text{ht}(P/pA) \quad \text{and} \quad \text{t}(A_P : R) = \text{t}((A/pA)_{P/pA} : R).$$

(2) *If P is a prime ideal of A and $p = \lambda_A^{-1}(P)$, then for each $n \geq 1$*

$$\text{ht}\, P[X_1, \ldots, X_n] = \text{ht}(P/pA)[X_1, \ldots, X_n].$$

(3) *If A is a locally Jaffard ring, then A/pA is a locally Jaffard ring for each prime ideal p of R such that $pA \neq A$.*

Let (A_1, λ_1) and (A_2, λ_2) be R-algebras. For $i = 1, 2$, we denote by $\mu_i : A_i \to A_1 \otimes_R A_2$ the canonical A_i-algebra homomorphism. The R-algebra $A_1 \otimes_R A_2$, when not specifically indicated, has $\lambda_{A_1 \otimes_R A_2} = \mu_1 \circ \lambda_1 = \mu_2 \circ \lambda_2$ as its associated ring homomorphism. If $P_i \in \mathrm{Spec}(A_i)$, $i = 1, 2$, j_i denotes the inclusion of P_i into A_i, t_{P_i} denotes the transcendence degree of the local ring A_{iP_i} over R and $k(P_i)$ denotes the residue field of A_{iP_i}. At last, we set $\Gamma(A_1, A_2) = \{(P_1, P_2) \mid P_1 \in \mathrm{Spec}(A_1),\ P_2 \in \mathrm{Spec}(A_2) \text{ and } \lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2)\}$.

A tensor product of R-algebras may be zero. We are interested in R-algebras (A_1, λ_1) and (A_2, λ_2) such that $A_1 \otimes_R A_2 \neq \mathbf{0}$, and say that such algebras are *tensorially compatible*. The next result provides some elementary and useful characterizations of tensorially compatible R-algebras. For a more general result, we refer the reader to [GD, Corollary 3.2.7.1].

Proposition 1.3. *Let (A_1, λ_1) and (A_2, λ_2) be R-algebras. The following conditions are equivalent:*

(1) *(A_1, λ_1) and (A_2, λ_2) are tensorially compatible.*
(2) $\lambda_1^*(\mathrm{Spec}(A_1)) \cap \lambda_2^*(\mathrm{Spec}(A_2)) \neq \emptyset$.
(3) *There exists a prime ideal P_1 of A_1 such that $\lambda_1^{-1}(P_1)A_2 \neq A_2$.*
(4) *There exists a prime ideal P_2 of A_2 such that $\lambda_2^{-1}(P_2)A_1 \neq A_1$.*
(5) *There exists a prime ideal p of R such that $pA_1 \neq A_1$ and $pA_2 \neq A_2$.*
(6) $\ker \lambda_1 + \ker \lambda_2 \neq R$.

Proof. (1) $\Longrightarrow$ (2). If (1) holds, then there exists a prime ideal Q of $A_1 \otimes_R A_2$; therefore $\mu_1^{-1}(Q) \in \mathrm{Spec}(A_1)$ and $\mu_2^{-1}(Q) \in \mathrm{Spec}(A_2)$ are such that $\lambda_1^{-1}(\mu_1^{-1}(Q)) = \lambda_2^{-1}(\mu_2^{-1}(Q))$, and hence, $\lambda_1^*(\mathrm{Spec}(A_1)) \cap \lambda_2^*(\mathrm{Spec}(A_2)) \neq \emptyset$.

(2) $\Longrightarrow$ (3). Let P_1 be a prime ideal of A_1 and P_2 a prime ideal of A_2 such that $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2)$; then $\lambda_1^{-1}(P_1)A_2 \subseteq P_2$ and so $\lambda_1^{-1}(P_1)A_2 \neq A_2$.

The implications (3) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (6) are apparent.

Finally, assume (6). Since $\ker \lambda_1 + \ker \lambda_2 \neq R$, there exists a prime ideal p of R such that $\ker \lambda_1 + \ker \lambda_2 \subseteq p$; this ensures that there exist a prime idealP_1 of A_1 and a prime ideal P_2 of A_2 such that $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2) = p$. Then

$$\begin{aligned}(A_1 \otimes_R A_2)/\,(\mathrm{Im}(j_1 \otimes id_{A_2}) + \mathrm{Im}(id_{A_1} \otimes j_2)) &\cong (A_1/P_1) \otimes_R (A_2/P_2)\\ &\cong (A_1/P_1) \otimes_{R/p} (A_2/P_2) \neq \mathbf{0}\end{aligned}$$

and so $A_1 \otimes_R A_2 \neq \mathbf{0}$. □

By induction, we obtain the following:

Proposition 1.4. *Let $(A_1, \lambda_1), \dots, (A_n, \lambda_n)$ be R-algebras. Then the following conditions are equivalent:*

(1) $A_1 \otimes_R \cdots \otimes_R A_n \neq \mathbf{0}$.
(2) $\lambda_1^*(\mathrm{Spec}(A_1)) \cap \lambda_2^*(\mathrm{Spec}(A_2)) \cap \cdots \cap \lambda_n^*(\mathrm{Spec}(A_n)) \neq \emptyset$.
(3) *There exists a prime ideal p of R such that $pA_i \neq A_i$ for every $i = 1, 2, \dots, n$.*

The next result establishes an analogue to [W, Proposition 2.3].

Proposition 1.5. *Let (A_1, λ_1) and (A_2, λ_2) be R-algebras. Let $P_1 \in \mathrm{Spec}(A_1)$ and $P_2 \in \mathrm{Spec}(A_2)$ such that $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2) = p$. Let*

$$T = A_1 \otimes_R A_2 \quad \text{and} \quad \Omega = \{Q \in \mathrm{Spec}(A_1 \otimes_R A_2) \mid \mu_i^{-1}(Q) = P_i,\ i = 1, 2\}.$$

Then

(1) *Ω is lattice isomorphic to $\mathrm{Spec}(T')$, where $T' = k(P_1) \otimes_{R/p} k(P_2)$.*
(2) *A prime ideal Q of Ω is minimal in Ω if and only if $\mathrm{t}(T/Q : R) = \mathrm{t}(A_1/P_1 : R) + \mathrm{t}(A_2/P_2 : R)$.*
(3) *If $Q_0 \in \mathrm{Spec}(T)$ and $\mu_i^{-1}(Q_0) \supseteq P_i$, $i = 1, 2$, then there exists $Q \in \Omega$ such that $Q \subseteq Q_0$.*

Proof. (1) is an immediate consequence of [GD, Corollary 3.2.7.1(ii)]. The proof of (2) and (3) is quite similar to that of Wadsworth. □

Proposition 1.5 allows us to extend partially some results of [W] to R-algebras.

Corollary 1.6. *Let (A_1, λ_1) and (A_2, λ_2) be tensorially compatible R-algebras. Let $Q \in \mathrm{Spec}(A_1 \otimes_R A_2)$. Then*

$$\mathrm{ht}\, Q \geq \mathrm{ht}(\mu_1^{-1}(Q)) + \mathrm{ht}(\mu_2^{-1}(Q)).$$

We omit the proof because of its similarity to that of [W, Corollary 2.5].

Corollary 1.7. *Let (A_1, λ_1) and (A_2, λ_2) be tensorially compatible R-algebras. Then*

$$\mathrm{t}(A_1 \otimes_R A_2 : R) = \sup\{\mathrm{t}(A_1/P_1 : R) + \mathrm{t}(A_2/P_2 : R) \mid (P_1, P_2) \in \Gamma(A_1, A_2)\}.$$

Consequently,

$$\mathrm{t}(A_1 \otimes_R A_2 : R) \leq \mathrm{t}(A_1 : R) + \mathrm{t}(A_2 : R).$$

Proof. Let $(P_1, P_2) \in \Gamma(A_1, A_2)$. Then by Proposition 1.5 there exists a prime ideal Q of $A_1 \otimes_R A_2$ such that $\mathrm{t}((A_1 \otimes_R A_2)/Q : R) = \mathrm{t}(A_1/P_1 : R) + \mathrm{t}(A_2/P_2 : R)$. So
$\sup\{\mathrm{t}(A_1/P_1 : R) + \mathrm{t}(A_2/P_2 : R) \mid (P_1, P_2) \in \Gamma(A_1, A_2)\} \leq \mathrm{t}(A_1 \otimes_R A_2 : R)$.

Conversely, let Q be a prime ideal of $A_1 \otimes_R A_2$; the prime ideals $P_1 = \mu_1^{-1}(Q)$ and $P_2 = \mu_2^{-1}(Q)$ are such that $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2) = p$; let $T = A_1 \otimes_R A_2$; then, by using [W, Corollary 2.4], we obtain:

$$\begin{aligned}
\mathrm{t}(T/Q : R) &= \mathrm{t.d.}(T/Q : R/p) \\
&= \mathrm{t.d.}\left(\frac{T/(\mathrm{Im}(j_1 \otimes id_{A_2}) + \mathrm{Im}(id_{A_1} \otimes j_2))}{Q/(\mathrm{Im}(j_1 \otimes id_{A_2}) + \mathrm{Im}(id_{A_1} \otimes j_2))} : R/p\right) \\
&\leq \mathrm{t.d.}((A_1/P_1) \otimes_{R/p} (A_2/P_2)) : R/p) \\
&= \mathrm{t.d.}(A_1/P_1 : R/p) + \mathrm{t.d.}(A_2/P_2 : R/p) \\
&= \mathrm{t}(A_1/P_1 : R) + \mathrm{t}(A_2/P_2 : R),
\end{aligned}$$

as desired. $\square$

Remark 1. Let (A_1, λ_1) and (A_2, λ_2) be tensorially compatible R-algebras. Clearly, $\mathrm{t}(A_1 \otimes_R A_2 : R) = \mathrm{t}(A_1 : R) + \mathrm{t}(A_2 : R)$ if and only if there exist $P_1 \in \mathrm{Spec}(A_1)$, and $P_2 \in \mathrm{Spec}(A_2)$ such that $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2)$ and $\mathrm{t}(A_1 : R) = \mathrm{t}(A_1/P_1 : R)$, $\mathrm{t}(A_2 : R) = \mathrm{t}(A_2/P_2 : R)$. The second condition holds, for instance, if A_1 and A_2 are integral domains or if $\mathrm{Spec}(R)$ is reduced to only one prime ideal. In general, the equality fails as it is shown in the next example. Moreover, when R is a field, we have $\dim(A_1 \otimes_R A_2) \geq \dim A_1 + \dim A_2$ [W, Corollary 2.5]. This is not, in general, true in the zero-dimensional case. The next example deals with these matters.

Example 2. There exist two tensorially compatible R-algebras (A_1, λ_1) and (A_2, λ_2) with

$$\mathrm{t}(A_1 \otimes_R A_2 : R) < \mathrm{t}(A_1 : R) + \mathrm{t}(A_2 : R)$$

and

$$\dim(A_1 \otimes_R A_2 : R) < \dim A_1 + \dim A_2.$$

Let $R = \mathbb{R} \times \mathbb{R}$, $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}[X]$. Let $\lambda_1 : R \longrightarrow A_1$ be the ring homomorphism defined by $\lambda_1(x, y) = x$ and let $\lambda_2 : R \longrightarrow A_2$ be the ring homomorphism defined by $\lambda_2(x, y) = (x, y)$. Then

$$\mathrm{t}(A_1 :_{\lambda_1} R) = \mathrm{t}(\mathbb{R} :_{\lambda_1} R) = \mathrm{t.d.}(\mathbb{R} : \mathbb{R}) = 0$$

and

$$\mathrm{t}(A_2 :_{\lambda_2} R) = \mathrm{t}(\mathbb{R} \times \mathbb{R}[X] :_{\lambda_2} R) = \sup\{\mathrm{t.d.}(\mathbb{R} : \mathbb{R})\,,\ \mathrm{t.d.}(\mathbb{R}[X] : \mathbb{R})\} = 1$$

and so

$$\mathrm{t}(A_1 :_{\lambda_1} R) + \mathrm{t}(A_2 :_{\lambda_2} R) = 1.$$

Moreover, by Corollary 1.7,
$\mathrm{t}(A_1 \otimes_R A_2 : R) = \sup\{\mathrm{t}(A_1/P_1 : R) + \mathrm{t}(A_2 : P_2 : R) \mid (P_1, P_2) \in \Gamma(A_1, A_2)\} = \mathrm{t}(A_1 :_{\lambda_1} R) + \mathrm{t}(A_2/((0) \times \mathbb{R}[X]) : R) = \mathrm{t.d.}(\mathbb{R} : \mathbb{R}) + \mathrm{t.d.}(\mathbb{R} : \mathbb{R}) = 0.$
Further

$$\dim(A_1 \otimes_R A_2 : R) \leq \mathrm{t}(A_1 \otimes_R A_2 : R) = 0 < \dim A_1 + \dim A_2 = 1. \quad \square$$

2. Tensor products of AF-rings

Definition 2.8. *An R-algebra (A, λ_A) is an AF-ring if for every $P \in \mathrm{Spec}(A)$*

$$\mathrm{ht}\, P + \mathrm{t}(A/P : R) = \mathrm{t}(A_P : R).$$

Remark 2. The AF-ring concept does not depend on the structure of algebra over R defined by the associated ring homomorphism.

Indeed, let A be a ring and let λ and $\overline{\lambda}$ be two ring homomorphisms defining two different structures of algebra over R on A. Let $P \in \mathrm{Spec}(A)$. Let $\pi : A \rightarrow A/P$ be the natural ring homomorphism. Let $p = \ker(\pi \circ \lambda) = \lambda^{-1}(P)$ and $q = \ker(\pi \circ \overline{\lambda}) = \overline{\lambda}^{-1}(P)$. We can view R/p and R/q as subfields of A/P. Let $k = R/p \cap R/q$. We have:

$$\mathrm{t}(A/P :_\lambda R) = \mathrm{t.d.}(A/P : R/p) = \mathrm{t.d.}(A/P : k) - \mathrm{t.d.}(R/p : k)$$

and

$$\mathrm{t}(A/P :_{\overline{\lambda}} R) = \mathrm{t.d.}(A/P : R/q) = \mathrm{t.d.}(A/P : k) - \mathrm{t.d.}(R/q : k).$$

On the other hand

$$\begin{aligned}
\mathrm{t}(A_P :_\lambda R) &= \sup\{\mathrm{t.d.}(A/Q : R/p) \mid Q \in \mathrm{Spec}(A) \text{ and } Q \subseteq P\} \\
&= \sup\{\mathrm{t}(A/Q : k) \mid Q \in \mathrm{Spec}(A) \text{ and } Q \subseteq P\} - \mathrm{t.d.}(R/p : k)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{t}(A_P :_{\overline{\lambda}} R) &= \sup\{\mathrm{t.d.}(A/Q : R/q) \mid Q \in \mathrm{Spec}(A) \text{ and } Q \subseteq P\} \\
&= \sup\{\mathrm{t}(A/Q : k) \mid Q \in \mathrm{Spec}(A) \text{ and } Q \subseteq P\} - \mathrm{t.d.}(R/q : k).
\end{aligned}$$

Therefore $\mathrm{t}(A/P :_\lambda R) - \mathrm{t.d.}(A_P :_\lambda k) = \mathrm{t}(A/P :_{\overline{\lambda}} R) - \mathrm{t}(A_P :_{\overline{\lambda}} R)$. Consequently, (A, λ) is an AF-ring if and only if $(A, \overline{\lambda})$ is an AF-ring.

Let $\mathcal{R}$ be the class of R-algebras that are AF-rings. Since $R[X_1, \ldots, X_n]$ satisfies the first chain condition for prime ideals [G, Corollary 31.17], any finitely generated R-algebra or any integral extension of such an algebra is an AF-ring. Moreover the class $\mathcal{R}$ is stable under localization and direct product.

The next result presents some properties of the class $\mathcal{R}$, and our proof of Proposition 2.3 uses the following lemma.

Lemma 2.9. *Let (A, λ_A) be an R-algebra. Then A is an AF-ring if and only if A/pA is an AF-ring over the field R/p, for each prime ideal p of R such that $pA \neq A$.*

Proof. Let P be a prime ideal of A and let $p = \lambda_A^{-1}(P)$. According to Lemma 1.2, $\operatorname{ht} P = \operatorname{ht}(P/pA)$ and $\mathrm{t}(A_P : R) = \mathrm{t.d.}((A/pA)_{P/pA} : R/p)$.

Assume that A is an AF-ring and let p be a prime ideal of R such that $pA \neq A$. Let P be a prime ideal of A containing pA; then

$$\begin{aligned} \operatorname{ht}(P/pA) + \mathrm{t.d.}((A/pA)/(P/pA) : R/p) &= \operatorname{ht} P + \mathrm{t}(A/P : R) \\ &= \mathrm{t}(A_P : R) \\ &= \mathrm{t.d.}((A/pA)_{P/pA} : R/p). \end{aligned}$$

Conversely, let P be a prime ideal of A and let $p = \lambda_A^{-1}(P)$. Then $pA \neq A$; so by hypothesis A/pA is an AF-ring over R/p, hence

$$\begin{aligned} \operatorname{ht} P + \mathrm{t}(A/P : R) &= \operatorname{ht}(P/pA) + \mathrm{t.d.}(A/P : R/p) = \\ &= \operatorname{ht}(P/pA) + \mathrm{t.d.}((A/pA)/(P/pA) : R/p) \\ &= \mathrm{t.d.}((A/pA)_{P/pA} : R/p) = \mathrm{t}(A_P : R). \quad \square \end{aligned}$$

Proposition 2.10. *The class $\mathcal{R}$ satisfies the following properties:*

(1) *Let $(A_1, \lambda_1), \ldots, (A_n, \lambda_n)$ be tensorially compatible R-algebras. If $A_1, \ldots, A_n$ are AF-rings, then $A_1 \otimes_R \cdots \otimes_R A_n$ is an AF-ring.*
(2) *Let A be an AF-ring. Then the polynomial ring $A[X]$ is an AF-ring and for each prime ideal P of A, $\operatorname{ht} P = \operatorname{ht} P[X]$.*
(3) *An AF-ring A is a locally Jaffard ring.*

Proof. (1) By induction, it suffices to consider the case $n = 2$. Let (A_1, λ_1) and (A_2, λ_2) be tensorially compatible AF-rings. Let p be a prime ideal of R such that $p(A_1 \otimes_R A_2) \neq A_1 \otimes_R A_2$. By Lemma 2.2 A_1/pA_1 and A_2/pA_2 are AF-rings over the field R/p ; hence by [W, Proposition 3.1] $(A_1/pA_1) \otimes_{R/p} (A_2/pA_2)$ is an AF-ring over R/p, so that $(A_1 \otimes_R A_2)/p(A_1 \otimes_R A_2) \cong (A_1/pA_1) \otimes_{R/p} (A_2/pA_2)$ is an AF-ring over R/p. The proof is complete via Lemma 2.2.

(2) Since $A[X] \cong A \otimes_R R[X]$, the result follows from (1). Let P be any prime ideal of A; so

$$\begin{aligned} \operatorname{ht} P &\leq \operatorname{ht} PA[X] \\ &= \mathrm{t}\left(A[X]_{PA[X]} : R\right) - \mathrm{t}(A[X]/PA[X] : R) \\ &\leq \mathrm{t}(A_P[X] : R) - \mathrm{t}((A/P)[X] : R) \\ &= \operatorname{ht} P. \end{aligned}$$

(3) Let A be an AF-ring. By (2) we obtain that for any prime ideal P of A and for each positive integer n, $\operatorname{ht} P = \operatorname{ht} P[X_1, \ldots, X_n]$. Hence, by [C, p.127], A is a locally Jaffard ring. $\square$

In the same regard, this section establishes adequate analogues of the main results stated in [BGK1] on the dimension of tensor products of AF-rings over a field. Let us consider for R-algebras the following functions (introduced in [W] for k-algebras) : let (A_1, λ_1) and (A_2, λ_2) be tensorially compatible R-algebras; let $P_1 \in \operatorname{Spec}(A_1)$ and $P_2 \in \operatorname{Spec}(A_2)$ such that $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2)$. Set

$$\delta(P_1, P_2) = \sup\{\operatorname{ht} Q \mid Q \in \operatorname{Spec}(A_1 \otimes_R A_2) \text{ and } \mu_i^{-1}(Q) = P_i,\ i = 1, 2\}.$$

One may easily check that

$$\dim(A_1 \otimes_R A_2) = \sup\{\delta(P_1, P_2) \mid (P_1, P_2) \in \Gamma(A_1, A_2)\},$$

and

$$\delta(P_1, P_2) = \delta(P_1/pA_1, P_2/pA_2),$$

where $p = \lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2)$.

Let (A, λ_A) be an R-algebra, $P \in \operatorname{Spec}(A)$ and d and s integers with $0 \leq d \leq s$. Set

$$\Delta(s, d, P) = \operatorname{ht} PA[X_1, \ldots, X_s] + \min(s,\ d + \mathrm{t}(A/P : R)),$$

$$D(s, d, A) = \sup\{\Delta(s, d, P) \mid P \in \operatorname{Spec}(A)\}.$$

Next we provide a formula for the dimension of the tensor product $A \otimes_R B$, where A is an AF-ring and B is any ring.

Theorem 2.11. *Let (A, λ_A) be an AF-ring and (B, λ_B) be any R-algebra such that $A \otimes_R B \neq \mathbf{0}$. Let $(P, I) \in \Gamma(A, B)$. Then*

$$\delta(P, I) = \Delta(\mathrm{t}_P, \operatorname{ht} P, I)$$

where $\mathrm{t}_P = \mathrm{t}(A_P : R)$, and consequently
$\dim(A \otimes_R B) = \sup\{D(\mathrm{t}_P, \operatorname{ht} P, B/pB) \mid P \in \operatorname{Spec}(A),\ p = \lambda_A^{-1}(P)$ *and* $pB \neq B\}$
$= \sup\{\operatorname{ht} I[X_1, \ldots, X_{\mathrm{t}_P}] + \min(\mathrm{t}_P,\ \operatorname{ht} P + \mathrm{t}(B/I : R)) \mid (P, I) \in \Gamma(A, B)\}$.

Proof. Let $P \in \operatorname{Spec}(A)$, $I \in \operatorname{Spec}(B)$ such that $\lambda_A^{-1}(P) = \lambda_B^{-1}(I) = p$. As noted previously, $\delta(P, I) = \delta(P/pA, I/pB)$; moreover, by Lemma 2.2, A/pA is an AF-ring over the field R/p; so we can apply Theorem 1.4 from [BGK1] to the (R/p)-algebras A/pA and B/pB, obtaining that

$$\delta(P/pA, I/pB) = \Delta(\mathrm{t}_P, \operatorname{ht}(P/pa), I/pB).$$

By Lemma 1.2, $\mathrm{ht}(P/pa) = \mathrm{ht}\,P$ and $t(A_P : R) = \mathrm{t}((A/pA)_{P/pA} : R))$. Further, for any $n \geq 1$, $\mathrm{ht}\,I[X_1, \dots, X_n] = \mathrm{ht}(I/pB)[X_1, \dots, X_n]$. Hence

$$\delta(P/pA, I/pB) = \Delta(\mathrm{t}_P, \mathrm{ht}\,P, I).$$

It follows that $\delta(P, I) = \Delta(\mathrm{t}_P, \mathrm{ht}\,P, I)$, as asserted. Consequently, using the definitions of δ, Δ and D and the stated condition on $\delta(P, I)$, yields $\dim(A \otimes_R B) = \sup\{\delta(P, I) \mid (P, I) \in \Gamma(A, B)\} = \sup\{\Delta(\mathrm{t}_P, \mathrm{ht}\,P, I) \mid (P, I) \in \Gamma(A, B)\} = \sup\{D(\mathrm{t}_P, \mathrm{ht}\,P, B/pB) \mid P \in \mathrm{Spec}(A),\ p = \lambda_A^{-1}(P)$ and $pB \neq B\} = \sup\{\mathrm{ht}\,I\,[X_1, \dots, X_{\mathrm{t}_P}] + \min(\mathrm{t}_P,\ \mathrm{ht}\,P + \mathrm{t}(B/I : R)) \mid (P, I) \in \Gamma(A, B)\}$ as we wished to show. □

It is worthwhile to note that $\dim(A \otimes_R B)$ depends on the R-module structure of A and B. The next example illustrates this fact:

Example 3. Let (A, λ_A) and (B, λ_B) be R-algebras such that A is an AF-ring and $A \otimes_R B \neq \mathbf{0}$. Let p be a prime ideal of R and let $\pi : R \longrightarrow R/p$ be the canonical ring homomorphism. Let $\lambda_1 : R \times R \times R \longrightarrow R/p \times A$ and $\lambda_2 : R \times R \times R \longrightarrow R/p \times B$ be the ring homomorphisms defined respectively by $\lambda_1(x, y, z) = (\pi(x), \lambda_A(y))$ and $\lambda_2(x, y, z) = (\pi(x), \lambda_B(z))$. It is an easy matter to verify that $\Gamma(R/p \times A, R/p \times B) = \{((0) \times A,\ (0) \times B)\}$. Hence via Theorem 2.4, it is easy to check that the dimension of the tensor product of $((R/p \times A), \lambda_1)$ and $((R/p \times B), \lambda_2)$ is zero. On the other hand, let $\lambda_2' : R \times R \times R \longrightarrow R/p \times B$ be the ring homomorphism defined by $\lambda_2'(x, y, z) = (\pi(x), \lambda_B(y))$; now by Theorem 2.4 we obtain that the dimension of the tensor product of $((R/p \times A), \lambda_1)$ and $((R/p \times B), \lambda_2')$ is equal to $\dim(A \otimes_R B)$. Thus, it suffices to choose A and B such that $\dim(A \otimes_R B) > 0$ (for instance, when R is a field and A, B are non trivial R-algebras). Therefore the two values are different according to the $(R \times R \times R)$-module structure of $R/p \times A$ and $R/p \times B$.

With the further assumption that A is an AF-domain, we obtain the following :

Corollary 2.12. *Let (A, λ_A) be an AF-domain and let (B, λ_B) be any R-algebra such that $A \otimes_R B \neq \mathbf{0}$. Then*

$$\dim(A \otimes_R B) = D\,(\mathrm{t}(A : R), \dim A, B/p_A B)$$

where $p_A = \ker \lambda_A$. Furthermore, if B is an integral domain, then

$$\dim(A \otimes_R B) = D\,(\mathrm{t}(A : R), \dim A, B)\,.$$

Proof. Since A is an integral domain, for any prime ideal P of A, $\lambda_A^{-1}(P) = p_A$ and $\mathrm{t}(A_P : R) = \mathrm{t}(A : R)$; so Theorem 2.4 implies that $\dim(A \otimes_R B) = \sup\{D(\mathrm{t}(A : R), \mathrm{ht}\,P, B/p_A B) \mid P \in \mathrm{Spec}(A)\}$. Since $D(s, d, A)$ is a nondecreasing function of the second argument, then $\dim(A \otimes_R B) = D(\mathrm{t}(A : R), \dim A, B/p_A B)$, as asserted. □

Next, we state a technical result that allows us to determine a necessary and sufficient condition under which the dimension of the tensor product of AF-rings over a zero-dimensional ring satisfies the formula of Wadsworth's Theorem 3.8.

Proposition 2.13. *Let $(A_1, \lambda_1), \dots, (A_n, \lambda_n)$ be tensorially compatible AF-rings. Then*
$\dim(A_1 \otimes_R \cdots \otimes_R A_n) =$
$\sup\{\min(\mathrm{ht}\,M_1 + \mathrm{t}_{M_2} + \cdots + \mathrm{t}_{M_n},\ \mathrm{t}_{M_1} + \mathrm{ht}\,M_2 + \cdots + \mathrm{t}_{M_n},\ \cdots,\ \mathrm{t}_{M_1} + \dots + \mathrm{t}_{M_{n-1}} + \mathrm{ht}\,M_n) \mid M_i \in \mathrm{Max}(A_i)$ and $\lambda_1^{-1}(M_1) = \lambda_2^{-1}(M_2) = \cdots = \lambda_n^{-1}(M_n)\}$.

Proof. It is deduced from the fact that $\dim(A_1 \otimes_R \cdots \otimes_R A_n) =$
$\sup\{\dim((A_1/P_1) \otimes_R (A_2/P_2) \otimes_R \cdots \otimes_R (A_n/P_n)) \mid P_i \in \mathrm{Spec}(A_i)$ for $i = 1, \dots, n$ and $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2) = \cdots = \lambda_n^{-1}(P_n)\} =$
$\sup\{\dim((A_1/pA_1) \otimes_{R/p} (A_2/pA_2) \otimes_{R/p} \cdots \otimes_{R/p} (A_n/pA_n)) \mid p \in \mathrm{Spec}(R)$, and $pA_i \neq A_i$ for $i = 1, \dots, n\}$; now we conclude via [BGK1, Lemma 1.6 and Remark 1.7]. □

Theorem 2.14. *Let $(A_1, \lambda_1), \dots, (A_n, \lambda_n)$ be tensorially compatible AF-rings with $\mathrm{t}_i = \mathrm{t}(A_i : R)$ and $d_i = \dim A_i$. Then $\dim(A_1 \otimes_R \dots \otimes_R A_n) = \mathrm{t}_1 + \cdots + \mathrm{t}_n - \max\{\mathrm{t}_i - d_i \mid 1 \leq i \leq n\}$ if and only if there exist maximal ideals $M_1, \dots, M_n$ belonging respectively to $A_1, \dots, A_n$ such that $\lambda_1^{-1}(M_1) = \cdots = \lambda_n^{-1}(M_n)$, and there exists $r \in \{1, \dots, n\}$ such that $\mathrm{ht}\,M_r = d_r$ and for any $j \in \{1, \dots, n\} - \{r\}$, $\mathrm{t}_{M_j} = \mathrm{t}_j$ and $\mathrm{t}(A_j/M_j : R) \leq \mathrm{t}(A_r/M_r : R)$.*

Proof. It is deduced from the fact that $\dim(A_1 \otimes_R \cdots \otimes_R A_n) =$
$\sup\{\dim((A_1/pA_1) \otimes_{R/p} (A_2/pA_2) \otimes_{R/p} \cdots \otimes_{R/p} (A_n/pA_n)) \mid p \in \mathrm{Spec}(R)$ and $pA_i \neq A_i$, for $i = 1, 2, \dots, n\}$ and [BGK1, Theorem 1.8]. □

Corollary 2.15. *Let $(A_1, \lambda_1), \dots, (A_n, \lambda_n)$ be tensorially compatible AF-rings with $\mathrm{t}_i = \mathrm{t}(A_i : R)$ and $d_i = \dim A_i$. If one of the following conditions is satisfied:*

1. *There exist maximal ideals $M_1, \dots, M_n$ belonging respectively to $A_1, \dots, A_n$ such that $\lambda_1^{-1}(M_1) = \cdots = \lambda_n^{-1}(M_n)$ and $\mathrm{ht}\,M_i = d_i$, $\mathrm{t}_{M_i} = t_i$ for $i = 1, 2, \dots, n$.*
2. *If $M_1, \dots, M_n$ are maximal ideals belonging respectively to $A_1, \dots, A_n$ such that $\lambda_1^{-1}(M_1) = \cdots = \lambda_n^{-1}(M_n)$, then $\mathrm{t}_{M_i} = t_i$ for $i = 1, \dots, n$.*

(3) *If $P_1, \ldots, P_n$ are minimal prime ideals belonging respectively to $A_1, \ldots, A_n$ such that $\lambda_1^{-1}(P_1) = \cdots = \lambda_n^{-1}(P_n)$, then $\mathrm{t}(A_i/P_i : R) = \mathrm{t}_i$, for $i = 1, \ldots, n$.*

(4) *$A_1, \ldots, A_n$ are equicodimensional.*

then

$$\dim(A_1 \otimes_R \cdots \otimes_R A_n) = \mathrm{t}_1 + \cdots + \mathrm{t}_n - \max\{t_i - d_i \mid 1 \leq i \leq n\}.$$

The proofs of (1), (2), (3) and (4) are similar to those of [BGK1, Corollaries 1.10, 1.11, 1.13, and 1.14], respectively.

Corollary 2.16. *Let $(A_1, \lambda_1), \ldots, (A_n, \lambda_n)$ be tensorially compatible AF-domains with $\mathrm{t}_i = \mathrm{t}(A_i : R)$ and $d_i = \dim(A_i)$. Then*

$$\dim(A_1 \otimes_R \cdots \otimes_R A_n) = \mathrm{t}_1 + \cdots + \mathrm{t}_n - \max\{\mathrm{t}_i - d_i \mid 1 \leq i \leq n\}.$$

Proof. Since $A_1 \otimes_R \cdots \otimes_R A_n \neq \mathbf{0}$, by Proposition 1.4 we have $p_{A_1} = p_{A_2} = \cdots = p_{A_n} = p$; then $A_1 \otimes_R \cdots \otimes_R A_n \cong A_1 \otimes_{R/p} \cdots \otimes_{R/p} A_n$. The result follows from [W, Theorem 3.8]. □

Now we consider the special case in which $(A_1, \lambda_1) = (A_2, \lambda_2)$.

Corollary 2.17. *Let (A, λ_A) be an AF-ring. Then $\dim(A \otimes_R A) = \dim A + \mathrm{t}(A : R)$ if and only if there exist maximal ideals M and N in A such that $\lambda_A^{-1}(M) = \lambda_A^{-1}(N)$, $\mathrm{ht}\, M = \dim A$, $\mathrm{t}(A_N : R) = \mathrm{t}(A : R)$ and $\mathrm{t}(A/N : R) \leq \mathrm{t}(A/M : R)$.*

3. The valuative dimension of tensor products and Jaffard rings

[BGK1, Theorem 2.1] establishes that if A is an AF-ring over a field k and B is a locally Jaffard ring, then $A \otimes_k B$ is a locally Jaffard ring. We next extend this result to AF-rings over a zero-dimensional ring.

Theorem 3.18. *Let (A, λ_A) be an AF-ring and (B, λ_B) a locally Jaffard ring such that $A \otimes_R B \neq \mathbf{0}$. Then $A \otimes_R B$ is a locally Jaffard ring.*

Proof. It is sufficient to prove that for each prime ideal Q of $A \otimes_R B$ and for each nonnegative integer n, $\mathrm{ht}\, Q[X_1, \ldots, X_n] = \mathrm{ht}\, Q$ (see [ABDFK] and [C]). Let $P = \mu_A^{-1}(Q)$, $I = \mu_B^{-1}(Q)$ and $p = \lambda_{A \otimes_R B}^{-1}(Q)$; according to Lemma 2.2, A/pA is an AF-ring over the field R/p; moreover, by Lemma 1.2 B/pB is a locally Jaffard ring; so we can apply Theorem 2.1 of [BGK1] to the (R/p)-algebras A/pA and B/pB obtaining that $(A/pA) \otimes_{R/p} (B/pB)$ is a locally Jaffard ring. Since $(A/pA) \otimes_{R/p} B/pB \cong (A \otimes_R B)/p(A \otimes_R B)$, then for each nonnegative integer n, it results that $\mathrm{ht}((Q/p(A \otimes_R B))[X_1, \ldots, X_n]) = \mathrm{ht}\,(Q/p(A \otimes_R B))$; so according to Lemma 1.2, $\mathrm{ht}\, Q = \mathrm{ht}\, Q[X_1, \ldots, X_n]$, as desired. □

Remark 3. Let (A, λ_A) be an AF-ring and (B, λ_B) any R-algebra such that $A \otimes_R B \neq \mathbf{0}$. Let $Q \in \mathrm{Spec}(A \otimes_R B)$, $P = \mu_A^{-1}(Q)$ and $I = \mu_B^{-1}(Q)$. We obtain from [BGK 1, Lemma 2.2] the following result:

$$\mathrm{ht}\, Q + \mathrm{t}\left((A \otimes_R B)/Q : R\right) = \mathrm{t}_P + \mathrm{ht}\, I[X_1, \ldots, X_{t_P}] + \mathrm{t}(B/I : R).$$

Let us recall that the valuative dimension of tensor products of algebras over a field does not seem to be effectively computable in general. However, [Gi, Proposition 3.1] states that provided A_1 and A_2 are two algebras over a field k, then

$$\dim_v(A_1 \otimes_k A_2) \leq \min\left(\dim_v A_1 + \mathrm{t.d.}(A_2 : k),\ \mathrm{t.d.}(A_1 : k) + \dim_v A_2\right).$$

The next result establishes the analogue of this result for the zero-dimensional case.

Proposition 3.19. *Let (A_1, λ_1) and (A_2, λ_2) be tensorially compatible R-algebras. Then*

$$\dim_v(A_1 \otimes_R A_2) \leq \min\left(\dim_v A_1 + \mathrm{t}(A_2 : R),\ \mathrm{t}(A_1 : R) + \dim_v A_2\right).$$

Proof. Let Q be any prime ideal of $A_1 \otimes_R A_2$; let $P_1 = \mu_1^{-1}(Q)$, $P_2 = \mu_2^{-1}(Q)$ and $p = \lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2)$. Let $T = A_1 \otimes_R A_2$. Then

$$\dim_v(T/Q) \leq \dim_v\left(T/\left(\mathrm{Im}(j_1 \otimes id_{A_2}) + \mathrm{Im}(id_{A_1} \otimes j_2)\right)\right).$$

Moreover, using the canonical isomorphism

$$T/(\mathrm{Im}(j_1 \otimes id_{A_2}) + \mathrm{Im}(id_{A_1} \otimes j_2)) \cong (A_1/P_1) \otimes_{R/p} (A_2/P_2)$$

and [Gi, Proposition 3.1], yields

$$\begin{aligned} \dim_v(T/Q) &\leq \dim_v\left((A_1/P_1) \otimes_{R/p} (A_2/P_2)\right) \\ &\leq \min\left(\dim_v A_1/P_1 + \mathrm{t}(A_2/P_2 : R),\ \dim_v A_2/P_2 + \mathrm{t}(A_1/P_1 : R)\right) \\ &\leq \min\left(\dim_v A_1 + \mathrm{t}(A_2 : R),\ \dim_v A_2 + \mathrm{t}(A_1 : R)\right). \ \square \end{aligned}$$

The next result handles the case where one of two R-algebras is an AF-ring.

Proposition 3.20. *Let (A, λ_A) and (B, λ_B) be tensorially compatible R-algebras and A an AF-ring. Then, for any $r \geq \dim_v B - 1$,*
$\dim_v(A \otimes_R B) =$
$\sup\{D(\mathrm{t}_P + r, \mathrm{ht}\, P + r, B/pB) \mid P \in \mathrm{Spec}(A),\ p = \lambda_A^{-1}(P) \text{ and } pB \neq B\} - r =$
$\sup\{htI\,[X_1, \ldots, X_r] + \min(t_P,\ \mathrm{ht}\, P + \mathrm{t}(B/I : R)) \mid (P, I) \in \Gamma(A, B)\}.$

Proof. Let $r \geq \dim_v B - 1$. Then, by [C, Proposition 1, ii)],$B[X_1,...,X_r]$ is a locally Jaffard ring. So, according to Theorem 3.1, $A \otimes_R B[X_1,...,X_r]$ is a locally Jaffard ring and hence a Jaffard ring. Therefore, by Corollary 2.5, $\dim_v(A \otimes_R B[X_1,...,X_r]) = \dim(A \otimes_R B[X_1,...,X_r]) =$ $\sup\{D(t_P, htP, (B/pB)[X_1,\ldots,X_r]) \mid P \in \operatorname{Spec}(A)$ with $\lambda_A^{-1}(P) = p$ and $pB \neq B\}$. Hence, according to [BGK1, Lemma 2.3], $\dim_v(A \otimes_R B) =$ $\sup\{D(t_P + r,\ htP + r,\ B/pB) \mid P \in \operatorname{Spec}(A)$ with $\lambda_A^{-1}(P) = p$ and $pB \neq B\} - r =$ $\sup\{htI[X_1,...,\ X_r] + \min(t_P,\ htP + t(B/I:R)) \mid (P,I) \in \Gamma(A,B)\}$. □

We conclude this section with two results on AF-domains.

Corollary 3.21. *Let (A, λ_A) be an AF-domain and B any R-algebra such that $A \otimes_R B \neq \mathbf{0}$. Then for any $r \geq \dim_v B - 1$* $\dim_v(A \otimes_R B) = D(\mathrm{t}+r,\ d+r,\ B/p_A B) - r = \sup\{\mathrm{ht}\, Q[X_1,\ldots X_r] + \min(\mathrm{t},\ d + \mathrm{t}(B/I:R)) \mid I \in \operatorname{Spec}(B)$ and $\lambda_B^{-1}(I) = p_A\}$, *where* $\mathrm{t} = \mathrm{t}(A:R)$ *and* $d = \dim A$.

Corollary 3.22. *Let (A, λ_A) and (B, λ_B) be R-algebras such that A is an AF-domain and $A \otimes_R B \neq \mathbf{0}$. If $\dim_v B \leq \mathrm{t}(A:R) + 1$, then $A \otimes_R B$ is a Jaffard ring.*

Remark 4. We thank the Referee for the following observation. Let A_{red} be the reduced ring associated to a ring A. Then $\mathrm{t}(A:R) = \mathrm{t}(A_{red}:R_{red})$ for any R-algebra (A, λ_A); moreover, if (A_1, λ_1) and (A_2, λ_2) are R-algebras, then $(A_1 \otimes_R A_2)_{red} = ((A_1)_{red} \otimes_{R_{red}} (A_2)_{red})_{red}$ [GD, Corollary 4.5.12]. One may therefore assume that R is absolutely flat and (A_1, λ_1), (A_2, λ_2) are reduced R-algebras.

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