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The Dimension of Tensor Products of AF-Rings

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0. Introduction

All the rings and algebras considered in this paper will be commutative, with identity elements and ring-homomorphisms will be unital. If A is a ring, then $\dim A$ will denote the (Krull) dimension of A, that is the supremum of lengths of chains of prime ideals of A. An integral domain D is said to have valuative dimension n (in short, $\dim_v D = n$) if each valuation overring of D has dimension at most n and there exists a valuation overring of D of dimension n. If no such integer n exists, then D is said to have infinite valuative dimension (see [G]). It must be remembered that for any ring A, $\dim_v A = \sup\{\dim_v(A/P) \mid P \in \operatorname{Spec}(A)\}$. Furthermore, it must also be remembered by [ABDFK] that a finite-dimensional domain D is a Jaffard domain if $\dim D = \dim_v D$. As the class of Jaffard domains is not stable under localization, an integral domain D is defined to be

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a locally Jaffard domain if D_P is a Jaffard domain for each prime ideal P of D. Analogous definitions are given in [C] for a finite-dimensional ring.

R.Y. Sharp proved in [S] that if K_1 and K_2 are extension fields of a field k, then

$$\dim(K_1 \otimes_k K_2) = \min\{\text{t.d.}(K_1 : k),\ \text{t.d.}(K_2 : k)\}\ .$$

A.R. Wadsworth extended this result to AF-domains. We wish to recall, at this point, that a k-algebra A is an AF-ring (altitude formula) if

$$\text{ht}P + \text{t.d.}(A/P : k) = \text{t.d.}(A_P : k)$$

for each prime ideal P of A. He proved that if D_1 and D_2 are AF-domains, then

$$\dim(D_1 \otimes_k D_2) = \min\{\dim D_1 + \text{t.d.}(D_2 : k),\ \text{t.d.}(D_1 : k) + \dim D_2\}\ .$$

He also provided a formula for $\dim(D \otimes_k R)$ applicable to an AF-domain D, with no restriction on the ring R. He also proved that for any prime ideal P of an AF-ring A and for any $n \geq 1$, $\text{ht}\, P = \text{ht}\, P[X_1, \dots, X_n]$. This latter property characterizes the class of locally Jaffard rings, meaning that an AF-ring is a locally Jaffard ring.

In [Gi] the class of AF-domains is examined with respect to the class of k-algebras which are stably strong S-domains, and the behaviour of the class of AF-domains with respect to certain pull-back type constructions. An upper bound for the valuative dimension of the tensor product of two k-algebras is given, that is:
if A_1 and A_2 are k-algebras with $\text{t.d.}(A_1 : k) < \infty$ and $\text{t.d.}(A_2 : k) < \infty$, then

$$\dim_v(A_1 \otimes_k A_2) \leq \min\{\dim_v A_1 + \text{t.d.}(A_2 : k)\ ,\ \text{t.d.}(A_1 : k) + \dim_v A_2\}\ .$$

We wish to point out that this work is a continuation of Wadsworth's paper [W].

In this first section we extend some known results concerning the class of AF-domains [W] to the class of AF-rings and we show that the results do not extend trivially from domains to rings with zero-divisors. In particular, we provided a formula for the dimension of the tensor product $A \otimes B$, where A is an AF-ring and B is any ring. Once we have provided a technical formula for the dimension of tensor products of AF-rings, then we can prove that if A_1 and A_2 are AF-rings, then

$$\dim(A_1 \otimes_k A_2) = \min\{\dim A_1 + \text{t.d.}(A_2 : k)\ ,\ \text{t.d.}(A_1 : k) + \dim A_2\}$$

if and only if $m_1 \in \text{Max}(A_1)$ and $m_2 \in \text{Max}(A_2)$ exist such that either $\text{ht}\, m_1 = \dim A_1$, $\text{t.d.}(A_{2m_2} : k) = \text{t.d.}(A_2 : k)$ and $\text{t.d.}(A_2/m_2) \leq \text{t.d.}(A_1/m_1)$ or $\text{ht}\, m_2 = \dim A_2$, $\text{t.d.}(A_{1m_1} : k) = \text{t.d.}(A_1 : k)$ and $\text{t.d.}(A_1/m_1) \leq \text{t.d.}(A_2/m_2)$. Finally we consider the special case in which $A_1 = A_2$.

In the second section we first prove that if A is an AF-ring and B is a locally Jaffard ring, then $A \otimes_k B$ is a locally Jaffard ring; then we give some formulas for computing the valuative dimension of the tensor product of an AF-ring and any ring. We conclude this section by giving an example of a tensor product of an AF-ring and a Jaffard ring which is not a Jaffard ring.

1. Tensor products of AF-rings

Throughout this paper k will indicate a field, $\text{t}(A)$ will denote the transcendence degree of a k-algebra A over k and for $P \in \text{Spec}\,(A)$ t_P will denote the transcendence degree of A_P over k. The tensor products, when not specifically indicated otherwise, will be taken as being relative to k.

In this section we will extend some of the properties of the dimension of the tensor product of AF-domains (see [W]) to the case of AF-rings.

Lemma 1.1. *Let $A_1, \dots, A_n$ be AF-rings and $T = A_1 \otimes \dots \otimes A_n$; for any $Q \in \text{Spec}\,(A_1 \otimes \dots \otimes A_n)$ let $P_i = Q \cap A_i$. Then*

$$\text{t}(T_Q) = \text{t}(A_{1P_1}) + \text{t}(A_{2P_2}) + \dots + \text{t}(A_{nP_n})\ .$$

Proof. Since there is nothing to prove for $n = 1$, we may assume that $n > 1$ and, by induction, that $R = A_2 \otimes \dots \otimes A_n$ satisfies the given property. Let $P = Q \cap R$; since T_Q is a localization of $A_{1P_1} \otimes \dots \otimes A_{nP_n}$, it results from [W, Corollary 2.4] that

$$\text{t}(T_Q) \leq \text{t}(A_{1P_1} \otimes \dots \otimes A_{nP_n}) = \text{t}(A_{1P_1}) + \dots + \text{t}(A_{nP_n})\ .$$

By the proof of [W, Proposition 3.1] we have

$$\text{t}(T_Q) = \text{ht}\, Q + \text{t}(T/Q) \geq \text{ht}P_1 + \text{ht}\, P + \text{t}(A_1/P_1) + \text{t}(R/P) =$$
$$= \text{t}(A_{1P_1}) + \text{t}(R_P) = \text{t}(A_{1P_1}) + \dots + \text{t}(A_{nP_n})\ .$$

We can now obtain the following known result for AF-domains.

Corollary 1.2. *Let $D_1, \dots, D_n$ be AF-domains, and $Q \in \text{Spec}\,(D_1 \otimes \dots \otimes D_n)$. Then* $\text{t}((D_1 \otimes \dots \otimes D_n)_Q) = \text{t}(D_1 \otimes \dots \otimes D_n) = \text{t}(D_1) + \dots + \text{t}(D_n)$.

The following simple statement will have important consequences.

Lemma 1.3. *Let A be an AF-ring. If $P \in \text{Spec}\,(A)$ and P_0 is a minimal prime ideal of A contained in P such that $\text{ht}\, P = \text{ht}\,(P/P_0)$, then $\text{t}_P = \text{t}_{P_0}$.*

Proof. $\text{t}_P = \text{ht}\, P + \text{t}(A/P) = \text{ht}\,(P/P_0) + \text{t}((A/P_0)/(P/P_0)) \leq \text{t}(A/P_0) \leq \text{t}(A_{P_0}) = \text{t}_{P_0}$.

We recall by [W, p. 394-395] the following functions:
let A_1 and A_2 be rings, $P_1 \in \text{Spec}\,(A_1)$ and $P_2 \in \text{Spec}\,(A_2)$, then

$$\delta(P_1, P_2) = \max\{\text{ht}\, Q \mid Q \in \text{Spec}\,(A_1 \otimes A_2) \quad \text{and } Q \cap A_1 = P_1\ , Q \cap A_2 = P_2\}\ ;$$

let A be a ring, $P \in \text{Spec}\,(A)$ and d and s integers with $0 \leq d \leq s$, then

$$\Delta(s, d, P) = \text{ht}\, PA[X_1, \dots, X_s] + \min(s, d + \text{t}(A/P))\ ,$$

$$D(s, d, A) = \max\{\Delta(s, d, P) \mid P \in \text{Spec}\,(A)\}\ .$$

Theorem 1.4. *Let A be an AF-ring and B any ring; let $p \in \operatorname{Spec}(A)$ and $q \in \operatorname{Spec}(B)$. Then*

$$\delta(p,q) = \Delta(\mathrm{t}_p, \mathrm{ht}\, p, q) \quad \text{and} \quad \dim(A \otimes B) = \max\{D(\mathrm{t}_P, \mathrm{ht}\, P, B) \mid P \in \operatorname{Spec}(A)\} .$$

Proof. Since $\delta(p,q) = \delta(pA_p, qA_q)$ and the class of AF-rings is stable under localizations, we may assume that p and q are maximal ideals in local rings. Let $\overline{B} = B/q$ and $\mathrm{t} = \mathrm{t}(\overline{B})$. By [W, Theorem 3.4]

$$\delta(p, 0\overline{B}) = \Delta(\mathrm{t}, 0, p) = \mathrm{ht}\, p[X_1, \ldots, X_t] + \min(\mathrm{t}, \mathrm{t}(A/p)) = \min(\mathrm{t}_p, \mathrm{ht}\, p + \mathrm{t}) .$$

Then

$$\Delta(\mathrm{t}_p, \mathrm{ht}\, p, q) = \mathrm{ht}\, q[X_1, \ldots, X_{\mathrm{t}_p}] + \delta(p, 0\overline{B}) .$$

Let $\overline{Q}_0 \subsetneq \overline{Q}_1 \subsetneq \cdots \subsetneq \overline{Q}_h$ be a chain of prime ideals of $A \otimes \overline{B}$ such that $h = \delta(p, 0\overline{B})$, $\overline{Q}_h \cap A = p$ and $\overline{Q}_h \cap \overline{B} = 0\overline{B}$. Then $\overline{Q}_0 \cap A = p_0$ is a minimal prime of A and $\overline{Q}_0 \cap \overline{B} = 0\overline{B}$. Let $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_h$ be the chain of inverse images in $A \otimes B$. Let $\tilde{A} = A/p_0$ and $\tilde{Q}_0 \subsetneq \tilde{Q}_1 \subsetneq \cdots \subsetneq \tilde{Q}_h$ be the chain of images in $\tilde{A} \otimes B$; so that $\tilde{Q}_0$ survives in the localization $\tilde{K} \otimes B$ of $\tilde{A} \otimes B$, where $\tilde{K}$ is the quotient field of $\tilde{A}$. Therefore according to [W, Remark 1.(a) p.398] $\mathrm{ht}\, \tilde{Q}_0 \geq \mathrm{ht}\, q[X_1, \ldots, X_{t_{p_0}}]$. Then we have

$$\mathrm{ht}\, Q_h \geq \mathrm{ht}\, Q_0 + \mathrm{ht}\,(Q_h/Q_0) \geq \mathrm{ht}\, q[X_1, \ldots, X_{\mathrm{t}_{p_0}}] + \delta(p, 0\overline{B}) .$$

Since $A \otimes \overline{B}$ is an AF-ring and $\mathrm{ht}\, \overline{Q}_h = \mathrm{ht}\, (\overline{Q}_h/\overline{Q}_0)$, by Lemma 1.3 we have $\mathrm{t}((A \otimes \overline{B})_{\overline{Q}_0}) = \mathrm{t}((A \otimes \overline{B})_{\overline{Q}})$. Since $A \otimes \overline{B}$ is a tensor product of an AF-ring and a field, by Lemma 1.1 we have

$$\mathrm{t}((A \otimes \overline{B})_{\overline{Q}_0}) = \mathrm{t}(A_{p_0}) + \mathrm{t} = \mathrm{t}((A \otimes \overline{B})_{\overline{Q}}) = \mathrm{t}(A_p) + \mathrm{t} .$$

So $\mathrm{t}_p = \mathrm{t}_{p_0}$. Therefore

$$\mathrm{ht}\, Q_h \geq \mathrm{ht}\, q[X_1, \ldots, X_{\mathrm{t}_p}] + \delta(p, 0\overline{B}) = \Delta(\mathrm{t}_p, \mathrm{ht}\, p, q) .$$

Therefore, it follows that $\delta(p,q) \geq \Delta(\mathrm{t}_p, \mathrm{ht}\, p, q)$.

The reverse inequality is deduced from the demonstration of the same inequality given in [W, Theorem 3.7] for an AF-domain. So $\delta(p,q) = \Delta(\mathrm{t}_p, \mathrm{ht}\, p, q)$.

The result upon $\dim(A \otimes B)$ derives directly from the definition of δ, Δ and D.

Corollary 1.5. *Let A_1 and A_2 be AF-rings; then*
(a) *If $p_1 \in \operatorname{Spec}(A_1)$ and $p_2 \in \operatorname{Spec}(A_2)$, then*

$$\delta(p_1, p_2) = \min(\mathrm{ht}\, p_1 + \mathrm{t}_{p_2}, \mathrm{t}_{p_1} + \mathrm{ht}\, p_2) .$$

(b) $\dim(A_1 \otimes A_2) = \max\{\min(\mathrm{ht}\, P_1 + \mathrm{t}_{P_2}, \mathrm{t}_{P_1} + \mathrm{ht}\, P_2) \mid P_1 \in \operatorname{Spec}(A_1), P_2 \in \operatorname{Spec}(A_2)\}$.

Proof. (a) According to Theorem 1.4 $\delta(p_1, p_2) = \Delta(\mathrm{t}_{p_1}, \mathrm{ht}\, p_1, p_2)$; furthermore

$$\begin{aligned}\Delta(\mathrm{t}_{p_1}, \mathrm{ht}\, p_1, p_2) &= \mathrm{ht}\, p_2[X_1, \ldots, X_{\mathrm{t}_{p_1}}] + \min(\mathrm{t}_{p_1}, \mathrm{ht}\, p_1 + \mathrm{t}(A_2/p_2)) \\ &= \mathrm{ht}\, p_2 + \min(\mathrm{t}_{p_1}, \mathrm{ht}\, p_1 + \mathrm{t}(A_2/p_2)) \\ &= \min(\mathrm{ht}\, p_1 + \mathrm{t}_{p_2}, \mathrm{t}_{p_1} + \mathrm{ht}\, p_2) .\end{aligned}$$

(b) Follows from the definition of $\delta(p_1, p_2)$.

Lemma 1.6. *Let $A_1, \ldots, A_n$ be AF-rings with $n \geq 2$. Then* $\dim(A_1 \otimes \cdots \otimes A_n) = \max\{\min(\mathrm{ht}\, P_1 + \mathrm{t}_{P_2} + \cdots + \mathrm{t}_{P_n}, \mathrm{t}_{P_1} + \mathrm{ht}\, P_2 + \mathrm{t}_{P_3} + \cdots + \mathrm{t}_{P_n}, \ldots, \mathrm{t}_{P_1} + \cdots + \mathrm{t}_{P_{n-1}} + \mathrm{ht}\, P_n) \mid P_i \in \operatorname{Spec}(A_i)$, *for* $i = 1, \ldots, n\}$.

Proof. We can define the following function for primes P_i of A_i with $i = 1, \ldots, n$:

$$\delta(P_1, \ldots, P_n) = \max\{\mathrm{ht}\, Q \mid Q \in \operatorname{Spec}(A_1 \otimes \cdots \otimes A_n) \text{ and } Q \cap A_i = P_i, \ i = 1, \ldots, n\} .$$

We prove by induction that

$$\begin{aligned}\delta(P_1, \ldots, P_n) = \min(&\mathrm{ht}\, P_1 + \mathrm{t}_{P_2} + \cdots + \mathrm{t}_{P_n}, \\ &\mathrm{t}_{P_1} + \mathrm{ht}\, P_2 + \mathrm{t}_{P_3} + \cdots + \mathrm{t}_{P_n}, \ldots, \mathrm{t}_{P_1} + \cdots + \mathrm{t}_{P_{n-1}} + \mathrm{ht}\, P_n) .\end{aligned}$$

For $n = 2$, this is Corollary 1.5. Let $n > 2$ and assume that $\delta(P_2, \ldots, P_n)$ satisfies the given formula. Of course,

$$\delta(P_1, \ldots, P_n) = \max\{\delta(P_1, Q') \mid Q' \in \operatorname{Spec}(A_2 \otimes \cdots \otimes A_n) \text{ and } Q' \cap A_j = P_j, j = 2, \ldots, n\};$$

moreover $\delta(P_1, Q') = \min(\mathrm{ht}\, P_1 + \mathrm{t}_{Q'}, \mathrm{ht}\, Q' + \mathrm{t}_{P_1})$ and $\mathrm{t}_{Q'} = \mathrm{t}_{P_2} + \cdots + \mathrm{t}_{P_n}$ according to Lemma 1.1. So

$$\begin{aligned}&\delta(P_1, \ldots, P_n) = \\ &= \max\{\min(\mathrm{ht}\, P_1 + \mathrm{t}_{P_2} + \cdots + \mathrm{t}_{P_n}, \mathrm{ht}\, Q' + \mathrm{t}_{P_1}) \mid \\ &\quad Q' \in \operatorname{Spec}(A_2 \otimes \cdots \otimes A_n) \text{ and } Q' \cap A_j = P_j\ ,\ j = 2, \ldots, n\} = \\ &= \min(\mathrm{ht}\, P_1 + \mathrm{t}_{P_2} + \cdots + \mathrm{t}_{P_n}, \delta(P_2, \ldots, P_n) + \mathrm{t}_{P_1}) = \\ &= \min(\mathrm{ht}\, P_1 + \mathrm{t}_{P_2} + \cdots + \mathrm{t}_{P_n}, \ \mathrm{t}_{P_1} + \mathrm{ht}\, P_2 + \mathrm{t}_{P_3} + \cdots + \mathrm{t}_{P_n}, \ldots, \mathrm{t}_{P_1} + \cdots + \mathrm{t}_{P_{n-1}} + \mathrm{ht}\, P_n).\end{aligned}$$

Then

$$\begin{aligned}&\dim(A_1 \otimes \cdots \otimes A_n) = \\ &= \max\{\delta(P_1, \ldots, P_n) \mid P_i \in \operatorname{Spec}(A_i), \text{ for } i = 1, \ldots, n\} = \\ &= \max\{\min(\mathrm{ht}\, P_1 + \mathrm{t}_{P_2} + \cdots + \mathrm{t}_{P_n}, \mathrm{t}_{P_1} + \mathrm{ht}\, P_2 + \mathrm{t}_{P_3} + \cdots + \mathrm{t}_{P_n}, \ldots, \\ &\quad \mathrm{t}_{P_1} + \cdots + \mathrm{t}_{P_{n-1}} + \mathrm{ht}\, P_n) \mid P_i \in \operatorname{Spec}(A_i), \text{ for } i = 1, \ldots, n\}.\end{aligned}$$

Remark 1.7. (a) Since $D(s, d, A)$ is a nondecreasing function of the first two arguments, then in Theorem 1.4 it suffices to consider the maximal ideals of A for $\dim(A \otimes B)$ only and in Corollary 1.5 it suffices to consider the maximal ideals of A_1 and A_2 for $\dim(A_1 \otimes A_2)$ only.
(b) With the notation as in Lemma 1.6, it is very easy to prove :
 (i) $\dim(A_1 \otimes \cdots \otimes A_n) = \max\{\min(\mathrm{ht}\, M_1 + \mathrm{t}_{M_2} + \cdots + \mathrm{t}_{M_n}, \ \mathrm{t}_{M_1} + \mathrm{ht}\, M_2 + \mathrm{t}_{M_3} + \cdots + \mathrm{t}_{M_n}, \ldots, \mathrm{t}_{M_1} + \cdots + \mathrm{t}_{M_{n-1}} + \mathrm{ht}\, M_n) \mid M_i \in \operatorname{Max}(A_i), \text{ for } i = 1, \ldots, n\}$.
 (ii) $\dim(A_1 \otimes \cdots \otimes A_n) \leq \mathrm{t}_1 + \mathrm{t}_2 + \cdots + \mathrm{t}_n - \max\{\mathrm{t}_i - \mathrm{d}_i, 1 \leq i \leq n\}$, where $\mathrm{d}_i = \dim A_i$.

In the following result, we determine a necessary and sufficient condition under which the dimension of the tensor product of the AF-rings $A_1, \ldots, A_n$ satisfies the formula of Wadsworth's Theorem 3.8, that is

$$\dim(A_1 \otimes \cdots \otimes A_n) = \mathrm{t}_1 + \mathrm{t}_2 + \cdots + \mathrm{t}_n - \max\{\mathrm{t}_i - \mathrm{d}_i, 1 \leq i \leq n\} .$$

Theorem 1.8. *Let $A_1, \dots, A_n$ be AF-rings, with $t_i = t(A_i)$ and $d_i = \dim A_i$. Then $\dim(A_1 \otimes \cdots \otimes A_n) = t_1 + t_2 + \cdots + t_n - \max\{t_i - d_i, 1 \le i \le n\}$ if and only if for any $i = 1, \dots, n$ there is $m_i \in \mathrm{Max}\,(A_i)$ and there is $r \in \{1, 2, \dots, n\}$ such that $\mathrm{ht}\, m_r = d_r$ and for any $j \in \{1, 2, \dots, n\} - \{r\}$, $t_{m_j} = t_j$ and $t(A_j/m_j) \le t(A_r/m_r)$.*

Proof. ($\Longrightarrow$) We may assume that

$$\dim(A_1 \otimes \cdots \otimes A_n) = d_1 + t_2 + \cdots + t_n ;$$

on the basis of Remark 1.7 (b) for $i = 1, 2, \dots, n$ let $m_i \in \mathrm{Max}\,(A_i)$ such that $\dim(A_1 \otimes \cdots \otimes A_n) = \min(\mathrm{ht}\, m_1 + t_{m_2} + \cdots + t_{m_n}, t_{m_1} + \mathrm{ht}\, m_2 + t_{m_3} + \cdots + t_{m_n}, \dots, t_{m_1} + \cdots + t_{m_{n-1}} + \mathrm{ht}\, m_n)$. Then $d_1 + t_2 + \cdots + t_n \le \mathrm{ht}\, m_1 + t_{m_2} + \cdots + t_{m_n}$. So

$$0 \le d_1 - \mathrm{ht}\, m_1 \le (t_{m_2} - t_2) + \cdots + (t_{m_n} - t_n) .$$

Then $\mathrm{ht}\, m_1 = d_1$ and $t_{m_j} = t_j$ for any $j = 2, \dots, n$. Furthermore for any $j = 2, \dots, n$, being $d_1 + t_{m_2} + \cdots + t_{m_n} \le t_{m_1} + \cdots + t_{m_{j-1}} + \mathrm{ht}\, m_j + t_{m_{j+1}} + \cdots + t_{m_n}$, it follows that

$$\mathrm{ht}\, m_1 + t_2 + \cdots + t_n \le t_{m_1} + \cdots + t_{j-1} + \mathrm{ht}\, m_j + t_j + \cdots + t_n ;$$

so

$$t(A_j/m_j) = t_j - \mathrm{ht}\, m_j \le t_{m_1} - \mathrm{ht}\, m_1 = t(A_1/m_1) .$$

($\Longleftarrow$) We may assume that for any $i = 1, \dots, n$ an $m_i \in \mathrm{Max}\,(A_i)$ exists so that $\mathrm{ht}\, m_1 = d_1$ and for any $j = 2, \dots, n$ $t_{m_j} = t_j$ and $t(A_j/m_j) \le t(A_1/m_1)$. Therefore, for any $j = 2, \dots, n$ it follows that

$$\mathrm{ht}\, m_1 + t_{m_j} \le t_{m_1} + \mathrm{ht}\, m_j$$

$$\mathrm{ht}\, m_1 + t_{m_2} + \cdots + t_{m_n} \le t_{m_1} + \cdots + t_{m_{j-1}} + \mathrm{ht}\, m_j + t_{m_{j+1}} + \cdots + t_{m_n} .$$

Therefore, being $\mathrm{ht}\, m_1 = d_1$ and $t_{m_j} = t_j$ for any $j = 2, \dots, n$, on the basis of Remark 1.7 (b) then

$$\dim(A_1 \otimes \cdots \otimes A_n) \ge d_1 + t_2 + \cdots + t_n .$$

According to Remark 1.7 (b) it follows $\dim(A_1 \otimes \cdots \otimes A_n) = d_1 + t_2 + \cdots + t_n$.

Example 1.9. Let us now give an example of two AF-rings A_1 and A_2 where $\dim(A_1 \otimes A_2)$ does not satisfy the formula of Wadsworth's Theorem.

Let X_1, X_2, X_3 be three indeterminates over k. Let $R_1 = k[X_1, X_2, X_3]_{(X_1)}$ and $R_2 = k[X_1, X_2]$. We consider $A_1 = R_1 \times R_2$ and $A_2 = k[X_1, X_2]_{(X_1)}$. A_1 is an AF-ring so that $\dim A_1 = 2$ and $t(A_1) = 3$; A_2 is an AF-ring so that $\dim A_2 = 1$ and $t(A_2) = 2$. According to Corollary 1.5, knowing $\mathrm{Max}\,(R_1 \times R_2)$ and $\mathrm{Max}\,(A_2)$, it is very easy to calculate that $\dim(A_1 \otimes A_2) = 3$. So $\dim(A_1 \otimes A_2) < t(A_1) + t(A_2) - 1 = 4$.

We will now illustrate a number of applications of Theorem 1.8; we note in particular that we arrive at Wadsworth's Theorem 3.8 regarding AF-domains (see Corollary 1.12).

Corollary 1.10. *Let $A_1, \dots, A_n$ be AF-rings, with $t_i = t(A_i)$ and $d_i = \dim A_i$ such that for any $i = 1, \dots, n$ an $m_i \in \mathrm{Max}\,(A_i)$ with $\mathrm{ht}\, m_i = d_i$ and $t_{m_i} = t_i$ exists. Then*

$$\dim(A_1 \otimes \cdots \otimes A_n) = t_1 + t_2 + \cdots + t_n - \max\{t_i - d_i, 1 \le i \le n\} .$$

Proof. Let $r \in \{1, 2, \dots, n\}$ such that $t(A_r/m_r) = \max\{t(A_i/m_i), 1 \le i \le n\}$; then $\mathrm{ht}\, m_r = d_r$ and for any $j \in \{1, 2, \dots, n\} - \{r\}$, $t_{m_j} = t_j$ and $t(A_j/m_j) \le t(A_r/m_r)$. Thus obtaining the result according to Theorem 1.8.

Corollary 1.11. *Let $A_1, \dots, A_n$ be AF-rings, with $t_i = t(A_i)$ and $d_i = \dim A_i$ such that for any $i = 1, \dots, n$ and for any $M_i \in \mathrm{Max}\,(A_i)$, $t_{M_i} = t_i$. Then*

$$\dim(A_1 \otimes \cdots \otimes A_n) = t_1 + t_2 + \cdots + t_n - \max\{t_i - d_i, 1 \le i \le n\} .$$

Proof. For any $i = 1, \dots, n$ let $m_i \in \mathrm{Max}\,(A_i)$ such that $\mathrm{ht}\, m_i = d_i$; so $\mathrm{ht}\, m_i = d_i$ and $t_{m_i} = t_i$. Then Corollary 1.10 completes the proof.

Corollary 1.12. *Let $A_1, \dots, A_n$ be AF-domains, with $t_i = t(A_i)$ and $d_i = \dim A_i$. Then*

$$\dim(A_1 \otimes \cdots \otimes A_n) = t_1 + t_2 + \cdots + t_n - \max\{t_i - d_i, 1 \le i \le n\} .$$

Corollary 1.13. *Let $A_1, \dots, A_n$ be AF-rings, with $t_i = t(A_i)$ and $d_i = \dim A_i$ such that for any $i = 1, \dots, n$ and for any $P_i \in \mathrm{Min}\,(A_i)$, $t(A_i/P_i) = t_i$. Then*

$$\dim(A_1 \otimes \cdots \otimes A_n) = t_1 + t_2 + \cdots + t_n - \max\{t_i - d_i, 1 \le i \le n\} .$$

Proof. For any $i = 1, \dots, n$ let $M_i \in \mathrm{Max}\,(A_i)$; therefore a $P_i \in \mathrm{Min}\,(A_i)$ such that $\mathrm{ht}\, M_i = \mathrm{ht}\,(M_i/P_i)$ exists. Since every A_i is an AF-ring, according to Lemma 1.3

$$t_{M_i} = t_{P_i} = t(A_i/P_i) = t_i .$$

So the result follows from Corollary 1.10.

Corollary 1.14. *Let $A_1, \dots, A_n$ be equicodimensional AF-rings, with $t_i = t(A_i)$ and $d_i = \dim A_i$. Then*

$$\dim(A_1 \otimes \cdots \otimes A_n) = t_1 + t_2 + \cdots + t_n - \max\{t_i - d_i, 1 \le i \le n\} .$$

Proof. For any $i = 1, \dots, n$ let $m_i \in \mathrm{Max}\,(A_i)$ such that $t_{m_i} = t_i$; let $r \in \{1, 2, \dots, n\}$ such that $t(A_r/m_r) = \max\{t(A_i/m_i), 1 \le i \le n\}$. Then $\mathrm{ht}\, m_r = d_r, t_{m_j} = t_j$ and $t(A_j/m_j) \le t(A_r/m_r)$ for any $j \ne r$. Thus the conditions of Theorem 1.8 are satisfied and the result obtained.

It is known [Gi, Corollary 3.3] that if A is an AF-ring, then

$$\dim(A \otimes A) = \dim_v(A \otimes A) \le \dim A + t(A) = \dim_v A + t(A) .$$

The same result is also obtained in the case of Corollary 1.5. By applying Theorem 1.8 to $A \otimes A$ we obtain:

Corollary 1.15. *Let A be an AF-ring. Then* $\dim(A\otimes A) = \dim A + t(A)$ *if and only if there exist two maximal ideals m and n of A such that* $\operatorname{ht} m = \dim A, t_n = t(A)$ *and* $t(A/n) \le t(A/m)$.

Example 1.16. Let us now offer an example of an AF-ring A such that

$$\dim(A\otimes A) < \dim A + t(A) .$$

Let K be an extension field of k such that $t(K) = 2$. Let $A = K \times k[X]$, where X is an indeterminate over k. The AF-ring A is such that $\dim A = 1$ and $t(A) = 2$. The maximal ideals of A are $(0) \times k[X]$ and $K \times N$ with $N \in \operatorname{Max}(k[X])$; besides $\operatorname{ht}((0)\times k[X]) = 0$ and $t_{(0)\times k[X]} = 2$, $\operatorname{ht}(K\times N) = 1$ and $t_{K\times N} = 1$. It follows, therefore, on the basis of Corollary 1.5 that

$$\dim(A\otimes A) = 2 < \dim A + t(A) = 3 .$$

We will conclude this section by giving an example which requires the following technical result.

Lemma 1.17. *Let A be an AF-ring such that two prime ideals p and q with $t_p \neq t_q$ exist. Then for any AF-ring B, $A\otimes B$ is not the tensor product of a finite number of AF-domains.*

Proof. Suppose that $A\otimes B = D_1 \otimes \cdots \otimes D_n$, where D_i is an AF-domain for $i = 1,\ldots,n$. Then $P, Q \in \operatorname{Spec}(A\otimes B)$ such that $P\cap A = p$ and $Q \cap A = q$ exist. Therefore, on the basis of Lemma 1.1, it follows that

$$\operatorname{ht} P + t(A\otimes B/P) = t_p + t_{p'}$$

where $p' = P\cap B$; besides, according to Corollary 1.2

$$\operatorname{ht} P + t(D_1\otimes\cdots\otimes D_n/P) = t(D_1\otimes\cdots\otimes D_n) = t(A) + t(B) .$$

Therefore, $t_p = t(A)$; in the same way it follows that $t_q = t(A)$, which is impossible.

Example 1.18. For each positive integer n, two AF-rings A_1 and A_2 exists such that
a) $\dim(A_1\otimes A_2) = n$;
b) $A_1\otimes A_2$ is not the tensor product of a finite number of AF-domains;
c) if a not finitely generated separable extension of k exists, then neither A_1 nor A_2 is a finite direct product of AF-domains.

a) and b). Let K be a separable extension of k. Consider

$$V_1 = K(X)[Y]_{(Y)} = K(X) + M_1 \text{ (with } M_1 = YV_1) ;$$

V_1 is a one-dimensional valuation domain of $K(X,Y)$; consider $V = K(Y)[X]_{(X)} = K(Y)+ M$ and

$$V_2 = K[Y]_{(Y)} + M = K + M_2 ;$$

V_2 is a two-dimensional valuation domain of K(X,Y). Since V_1 and V_2 are incomparable, by [N, Theorem 11.11] $T = V_1\cap V_2$ is a two-dimensional Prüfer domain with only two maximal ideals, m_1 and m_2, such that $T_{m_1} = V_1$ and $T_{m_2} = V_2$. Let $I = m_1m_2$ and $R = T/I$. R is a zero-dimensional ring with only two prime ideals, $p_1 = m_1/I$ and $p_2 = m_2/I$. Furthermore, $t(R/p_1) = 1$ and $t(R/p_2) = 0$. Now according to Corollary 1.6,

$$\dim(R\otimes R[X_1,\ldots,X_n]) = \dim((R\otimes R)[X_1,\otimes,X_n]) = \dim(R\otimes R)+n = t(R)+n = 1+n .$$

Besides, according to Lemma 1.17, $R\otimes R[X_1,\otimes,X_n]$ is not the tensor product of a finite number of AF-domains; so it suffices to consider $A_1 = R$ and $A_2 = R[X_1,\ldots,X_{n-1}]$.

c) Now assume K as not being finitely generated over k. Therefore, $K\otimes K$ is reduced [ZS, Theorem 39], zero-dimensional [S, Theorem 3.1] and is not Noetherian [V, Theorem 11]. Therefore, $\operatorname{Spec}(K\otimes K)$ is infinite [V, Lemma 0]. Now let us consider $A = K\otimes R$; since A is an integral extension of R, it is zero-dimensional. Furthermore, two prime ideals of A, P_1 and P_2 such that $P_1\cap R = p_1$ and $P_2 \cap R = p_2$ with $t(A/P_1) = 1$ and $t(A/P_2) = 0$ exist. Since K is the quotient field of R/p_2 and $\operatorname{Spec}(K\otimes K)$ is infinite, by [W, Proposition 3.2] $\operatorname{Spec}(A) = \operatorname{Min}(A)$ is infinite. Thus A is not a finite direct product of AF-domains and the same holds for $A[X_1,\ldots,X_n]$. Now, according to Corollary 1.6,

$$\dim(A\otimes A[X_1,\ldots,X_n]) = \dim((A\otimes A)[X_1\ldots,X_n]) = \dim(A\otimes A)+n = t(A)+n = 1+n .$$

Furthermore, according to Lemma 1.17, $A\otimes A[X_1,\otimes,X_n]$ is not the tensor product of a finite number of AF-domains; therefore, it suffices to consider $A_1 = A$ and $A_2 = A[X_1,\ldots,X_{n-1}]$.

2. Tensor products of AF-rings and locally Jaffard rings

We will now present this section's main theorem.

Theorem 2.1. *Let A be an AF-ring and B a locally Jaffard ring. Then $A\otimes B$ is a locally Jaffard ring.*

In order to prove this theorem the following premise is necessary.

Lemma 2.2. *Let A be an AF-ring and B any ring; let Q be any prime ideal of $T = A\otimes B$, and let $p = Q\cap A$, $q = Q\cap B$. Then*

$$\operatorname{ht} Q + t(T/Q) = t_p + \operatorname{ht} q[X_1,\ldots,X_{t_p}] + t(B/q) .$$

Proof. By localizing, we may assume that p and q are maximal ideals in local rings. Let $\overline{B} = B/q$ and let $\overline{Q}$ be the image of Q in $A\otimes\overline{B}$; let $\overline{Q}_0 \subsetneq \overline{Q}_1 \subsetneq \cdots \subsetneq \overline{Q}_h = \overline{Q}$ be a chain of prime ideals of $A\otimes\overline{B}$ such that $h = \operatorname{ht}\overline{Q}$. Then $\overline{Q}_0\cap A = p_0$ is a minimal prime of A and $\overline{Q}_0 \cap \overline{B} = 0\overline{B}$. Let $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_h = Q$ be the chain of inverse images in $A\otimes B$. As in the case of demonstration given in Theorem 1.4, $\operatorname{ht} Q_0 \ge \operatorname{ht} q[X_1,\ldots,X_{t_p}]$. Since $\overline{B}$ is an AF-domain, according to [W, Remark 1 (b) p. 398] it follows that

$$\operatorname{ht}\overline{Q} + t((A\otimes\overline{B})/\overline{Q}) = t(\overline{B}) + \operatorname{ht} p[X_1,\ldots,X_{t(\overline{B})}] + t(A/p) = t_p + t(B/q) .$$

Therefore

$$\operatorname{ht} Q \geq \operatorname{ht} Q_0 + \operatorname{ht}(Q/Q_0) \geq \operatorname{ht} Q_0 + \operatorname{ht} \overline{Q} \geq \operatorname{ht} q[X_1, \dots, X_{t_p}] + t_p + t(B/q) - t(T/Q) .$$

Then

$$\operatorname{ht} Q + t(T/Q) \geq t_p + \operatorname{ht} q[X_1, \dots, X_{t_p}] + t(B/q) .$$

On the other hand, let $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_s = Q$ be a chain of prime ideals of T such that $s = \operatorname{ht} Q$. Therefore, a finitely generated k-algebra D_1 contained in A such that $t(A/p) = t(D_1/p_1)$, where $p_1 = p \cap D_1$, and

$$Q_0 \cap (D_1 \otimes B) \subsetneq Q_1 \cap (D_1 \otimes B) \subsetneq \cdots \subsetneq Q_s \cap (D_1 \otimes B) = Q \cap (D_1 \otimes B) ,$$

exists. It is $t(D_1) \leq t(A)$. By choosing $g_1, \dots, g_r \in T$ such that, setting $T''' = (D_1 \otimes B)[g_1, \dots, g_r]$ and $Q''' = Q \cap T'''$, it emerges $t(T''') = t(T)$ and $t(T'''/Q''') = t(T/Q)$. This gives rise to a finitely generated k-algebra D where $D_1 \subseteq D \subseteq A$, where, if we set $T'' = D \otimes B$ and $Q'' = Q \cap T''$, then we obtain $t(T'') = t(T)$ and $t(T''/Q'') = t(T/Q)$. So $t(D) = t(A) = t_p$. According to Noether's normalization Lemma [M, Lemma 2 p.262] $z_1, \dots, z_{t_p} \in D$ which are algebraically independent over k, so that D is integral over $C = k[z_1, \dots, z_{t_p}]$ exist. Let $T' = C \otimes B$; since $D \otimes B$ is integral over $C \otimes B$, distinct primes of $D \otimes B$ in a chain contract to distinct primes of $C \otimes B$. Thus

$$Q_0 \cap (C \otimes B) \subsetneq Q_1 \cap (C \otimes B) \subsetneq \cdots \subsetneq Q_s \cap (C \otimes B) = Q \cap (C \otimes B) = Q' .$$

Then $\operatorname{ht} Q \leq \operatorname{ht} Q'$ and $t(T'/Q') = t(T/Q)$. So $\operatorname{ht} Q + t(T/Q) \leq \operatorname{ht} Q' + t(T'/Q')$; since $T' = C \otimes B$, according to [W, Remark 1 (b) p. 398] it follows that

$$\operatorname{ht} Q' + t(T'/Q') = t_p + \operatorname{ht} q[X_1, \dots, X_{t_p}] + t(B/q) .$$

Consequently

$$\operatorname{ht} Q + t(T/Q) \leq t_p + \operatorname{ht} q[X_1, \dots, X_{t_p}] + t(B/q) .$$

Proof of theorem. Let $T = A \otimes B$, let $Q \in \operatorname{Spec}(T)$ and let $p = Q \cap A$, $q = Q \cap B$. For any $n \geq 0$ let $T' = T[X_1, \dots, X_n]$; $T' \cong A[X_1, \dots, X_n] \otimes B$ and $A[X_1, \dots, X_n]$ is an AF-ring by [W, Corollary 3.2]; therefore, on the basis of the previous Lemma

$$\operatorname{ht} Q[X_1, \dots, X_n] + t(T'/Q[X_1, \dots, X_n]) = t_{p'} + \operatorname{ht} q[X_1, \dots, X_{t_{p'}}] + t(B/q)$$

where $p' = Q[X_1, \dots, X_n] \cap A[X_1, \dots, X_n] = p[X_1, \dots, X_n]$. Furthermore, since A is an AF-ring,

$$t_{p'} = t(A[X_1, \dots, X_n]_{p'}) = \operatorname{ht} p' + t(A[X_1, \dots, X_n]/p[X_1, \dots, X_n]) = t_p + n .$$

Since B is a locally Jaffard ring,

$$\operatorname{ht} Q[X_1, \dots, X_n] + t(T'/Q[X_1, \dots, X_n]) = n + t_p + \operatorname{ht} q + t(B/q) .$$

Besides,

$$\operatorname{ht} Q[X_1, \dots, X_n] + t(T'/Q[X_1, \dots, X_n]) = \operatorname{ht} Q[X_1, \dots, X_n] + n + t(T/Q) .$$

Therefore, by applying the previous Lemma to Q, it follows that

$$\operatorname{ht} Q[X_1, \dots, X_n] = t_p + \operatorname{ht} q + t(B/q) - t(T/Q) = \operatorname{ht} Q .$$

Consequently $A \otimes B$ is a locally Jaffard ring.

Lemma 2.3. *Let A be an AF-ring and B any ring; let $P \in \operatorname{Spec}(A)$. Then for any $r \geq 1$*

$$D(t_P, \operatorname{ht} P, B[X_1, \dots, X_r]) = D(t_P + r, \operatorname{ht} P + r, B) .$$

Proof. Since $A_P \otimes B[X_1, \dots, X_r] \cong A_P[X_1, \dots, X_r] \otimes B$, according to Theorem 1.4 and Remark 1.7 (a), it emerges that

$$D(t_P, \operatorname{ht} P, B[X_1, \dots, X_r]) = \max\{D(t_{P'}, \operatorname{ht} P', B) \mid P' \in \operatorname{Spec}(A_P[X_1, \dots, X_r])\} .$$

Since A_P is a locally Jaffard ring, $\dim A_P[X_1, \dots, X_r] = \operatorname{ht} P + r$; therefore, for any $P' \in \operatorname{Spec}(A_P[X_1, \dots, X_r])$ it follows that

$$D(t_{P'}, \operatorname{ht} P', B) \leq D(t_P + r, \operatorname{ht} P + r, B)$$

and therefore that

$$D(t_P, \operatorname{ht} P, B[X_1, \dots, X_r]) \leq D(t_P + r, \operatorname{ht} P + r, B) .$$

Furthermore, by letting $M' = (PA_P, X_1, \dots, X_r)$, it follows that $\operatorname{ht} M' = \operatorname{ht} P + r$ and $t_{M'} = t_P + r$; therefore

$$D(t_P + r, \operatorname{ht} P + r, B) \leq D(t_P, \operatorname{ht} P, B[X_1, \dots, X_r]) .$$

Proposition 2.4. *Let A be an AF-ring and B any ring. Then for any $r \geq \dim_v B - 1$*

$$\begin{aligned}\dim_v(A \otimes B) &= \max\{D(t_p + r, \operatorname{ht} p + r, B) \mid p \in \operatorname{Spec}(A)\} - r = \\ &= \max\{\operatorname{ht} q[X_1, \dots, X_r] + \min(t_p, \operatorname{ht} P + t(B/q) \mid p \in \operatorname{Spec}(A) \text{ and } q \in \operatorname{Spec}(B)\} .\end{aligned}$$

Proof. Let $r \geq \dim_v B - 1$. Since, according to [C, Proposition 1. ii)], $B[X_1, \dots, X_r]$ is a locally Jaffard ring, according to Theorem 2.1 $A \otimes B[X_1, \dots, X_r]$ is a Jaffard ring. Therefore by Theorem1.4 and Lemma 2.3

$$\begin{aligned}\dim_v(A \otimes B[X_1, \dots, X_r]) &= \dim(A \otimes B[X_1, \dots, X_r]) \\ &= \max\{D(t_p, \operatorname{ht} p, B[X_1, \dots, X_r]) \mid p \in \operatorname{Spec}(A)\} \\ &= \max\{D(t_p + r, \operatorname{ht} p + r, B) \mid p \in \operatorname{Spec}(A)\} .\end{aligned}$$

From this it follows that $\dim_v(A \otimes B) = \max\{D(t_p + r, \operatorname{ht} p + r, B) \mid p \in \operatorname{Spec}(A)\} - r = \max\{\operatorname{ht} q[X_1, \dots, X_r] + \min(t_p, \operatorname{ht} p + t(B/q)) \mid p \in \operatorname{Spec}(A) \text{ and } q \in \operatorname{Spec}(B)\}$.

Corollary 2.5. *Let A be an AF-domain with $t = t(A)$ and $d = \dim A$ and let B be any ring. Then for any $r \geq \dim_v B - 1$*

$$\begin{aligned}\dim_v(A \otimes B) &= D(t + r, d + r, B) - r = \\ &= \max\{\operatorname{ht} q[X_1, \dots, X_r] + \min(t, d + t(B/q)) \mid q \in \operatorname{Spec}(B)\} .\end{aligned}$$

Corollary 2.6. *Let A be an AF-domain with $t = \mathrm{t}(A)$ and B a ring such that $\dim_v B \leq t+1$. Then $A \otimes B$ is a Jaffard ring.*

Corollary 2.7. *Let A be an AF-domain with $t = \mathrm{t}(A)$ and B a Jaffard ring such that $B[X]$ is a locally Jaffard ring. Then $A \otimes B$ is a Jaffard ring.*

Proof. If $t = 0$, then $A \otimes B$ is an integral extension of B; since B is a Jaffard ring, according to [J, Proposition 4, p. 58] $A \otimes B$ is a Jaffard ring. Assume that $t \geq 1$; $A \otimes B[X]$ is a Jaffard ring according to Theorem 2.1; furthermore, according to [J, Theorem 2 p. 60], $\dim_v(A \otimes B[X]) = \dim_v(A \otimes B) + 1$; so according to [W, Theorem 3.7]

$$\begin{aligned}\dim_v(A \otimes B) &= \dim_v(A \otimes B[X]) - 1 = \dim(A \otimes B[X]) - 1 = \\ &= \max\{\mathrm{ht}\, Q[X_1, \ldots, X_t] + \min(t, d + t(B[X]/Q)) \mid Q \in \mathrm{Spec}\,(B[X])\} - 1 = \\ &= \max\{\mathrm{ht}\, q[X] + \min(t+1, d+1+\mathrm{t}(B/q)) \mid q \in \mathrm{Spec}\,(B)\} - 1 = \\ &= D(t+1, d+1, B) - 1 = D(t, d, B) = \dim(A \otimes B)\ .\end{aligned}$$

In conclusion, $A \otimes B$ is a Jaffard ring.

Remark 2.8. The example 3.2 of [ABDFK] is an example of a Jaffard, not locally Jaffard ring B, where $B[X]$ is a locally Jaffard ring.

Example 2.9. The result of Theorem 2.1 is the best-possible one: the tensor product of an AF-domain and a Jaffard ring is not necessarily a Jaffard ring.

It is possible to deduce the following example from [ABDFK]. Let Z_1, Z_2, Z_3, Z_4 be four indeterminates over k. Let $L = k(Z_1, Z_2, Z_3, Z_4)$. Let

$$V_1 = k(Z_1, Z_2, Z_3)[Z_4]_{(Z_4)} = k(Z_1, Z_2, Z_3) + M_1$$

V_1 is a one-dimensional valuation ring of L, with maximal ideal $M_1 = Z_4 V_1$. Let V' be a one-dimensional valuation overring of $k(Z_4)[Z_2, Z_3]$ of the form $V' = k(Z_4) + M'$. Let $V'_2 = k[Z_4]_{(Z_4)} + M' = k + M'_2$, where $M'_2 = Z_4 k[Z_4]_{(Z_4)} + M'$. V'_2 is a two- dimensional valuation ring. Let $V = k(Z_2, Z_3, Z_4)[Z_1]_{(Z_1)} = k(Z_2, Z_3, Z_4) + M$, with $M = Z_1 V$; let $M_2 = M'_2 + M$ and

$$V_2 = V'_2 + M = k + M_2\ .$$

V_2 is a three-dimensional valuation ring.

We now wish to demonstrate that V_1 and V_2 are incomparable. If not, it would follow from the one-dimensionality of V_1 that $V_2 \subset V_1$. Then we would have $V_1 = (V_2)_M$. We would have that M is a divided prime ideal of V_2. Then $Z_4 V_1 = M_1 = M(V_2)_M$. Thus $1 = Z_4 Z_4^{-1} \in MV = M$, which is a contradiction. Since V_1 and V_2 both have quotient field $k(Z_1, Z_2, Z_3, Z_4)$, we can now see from [N, Theorem 11.11] that $S = V_1 \cap V_2$ is a three-dimensional Prüfer domain with only two maximal ideals, m_1 and m_2, such that $S_{m_1} = V_1$ and $S_{m_2} = V_2$. Let $F = k(Z_1)$, $f : V_1 \longrightarrow k(Z_1, Z_2, Z_3)$ be the natural ring homomorphism and $D = f^{-1}(F) = F + M_1$. Let $g : S \longrightarrow S/m_1 \cong V_1/m_1 \cong k(Z_1, Z_2, Z_3)$ be the natural ring homomorphism and $B = g^{-1}(F)$. It follows that $B = D \cap S = D \cap V_2$ and $\dim B = \dim S = 3$. Furthermore, according to [ABDFK, Theorem 2.11], it follows that

$$\dim_v B = \max\{\dim_v S,\ \dim_v F + \dim_v S_{m_1} + \mathrm{t}(S/m_1 : F)\} = 3\ .$$

Thus, B is a Jaffard ring. Since $B = D \cap V_2$ and V_1, V_2 are incomparable, it follows that $B_{n_1} = D$ and $B_{n_2} = V_2$, where $\{n_1, n_2\} = \mathrm{Max}\,(B)$. Moreover, $\mathrm{ht}\, n_1[X_1, \ldots, X_s] = \mathrm{ht}\, n_1 B_{n_1}[X_1, \ldots, X_s] = \mathrm{ht}\, M_1[X_1, \ldots, X_s]$. Since V_1 is a Jaffard ring, by [A, Theorem 1.7] it follows that $\mathrm{ht}_{D[X_1,\ldots,X_s]} M_1[X_1, \ldots, X_s] = \mathrm{ht}_{V_1} M_1 + \inf(s, 2)$. Thus, $\mathrm{ht}\, n_1 = 1$, $\mathrm{ht}\, n_1[X_1] = 2$ and $\mathrm{ht}\, n_1[X_1, X_2] = 3$; $\mathrm{t}(B/n_1) = \mathrm{t}(D/M_1) = 1$ and $\mathrm{t}(B/n_2) = \mathrm{t}(V_2/M_2) = 0$. Let $A = k(X)$. According to Theorem 1.4,

$$\dim(A \otimes B) = D(\mathrm{t}(A), 0, B) = \max\{\mathrm{ht}\, q[X_1] + \min(1, \mathrm{t}(B/q)) \mid q \in \mathrm{Spec}\,(B)\}\ .$$

For $q = n_1$, it is $\mathrm{ht}\, n_1[X_1] + \min(1, \mathrm{t}(B/n_1)) = 2 + 1 = 3$; for $q = n_2$, it is $\mathrm{ht}\, n_2[X_1] + \min(1, \mathrm{t}(B/n_2)) = \mathrm{ht}\, n_2 = 3$ and $\mathrm{ht}\, q[X_1] + \min(1, \mathrm{t}(B/q)) \leq 3$ for every prime ideal of B contained in n_2. Consequently, $\dim(A \otimes B) = 3$. On the basis of Corollary 2.5,

$$\dim_v(A \otimes B) = \max\{\mathrm{ht}\, q[X_1, X_2] + \min(1, \mathrm{t}(B/q)) \mid q \in \mathrm{Spec}\,(B)\}$$

for $q = n_1$, $\mathrm{ht}\, n_1[X_1, X_2] + \min(1, \mathrm{t}(B/n_1)) = 3 + 1 = 4$. Therefore $\dim_v(A \otimes B) = 4 \neq \dim(A \otimes B)$. In conclusion $A \otimes B$ is not a Jaffard ring.

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