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Group Rings $R[G]$ with 4-Generated Ideals When R is an Artinian Ring with the 2-Generator Property

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INTRODUCTION

For the convenience of the reader, let's recall the following facts. We have from the restriction on Krull dimension, $1 \geq dimR[G] = dimR + r$, where r denotes the torsion free rank of G. If $r = 0$, then G must be a finite group. If $r = 1$, then $G \cong Z \oplus H$, where H is a finite abelian group and Z the group of the integers. We will concentrate on the case in which R is Artinian and r = 0, that is, G is a finite abelian group. The cases $n = 2$ and $n = 3$ were considered in [15, Theorem 4.1] and [1], respectively. However, for $n \geq 4$, the problem of when $R[G]$ has the $n-$generator property remains open.

As the problem of determining when a group ring $R[G]$ has the 4−generator property, when R is an Artinian principal ideal ring and G is a finite group is resolved in [2], in this paper, we consider the case where R is an Artinian ring with the 2-generator property.

Rings and groups are taken to be commutative and the groups are written additively. If p is a prime integer, then the $p-$sylow subgroup of the finite abelian group G will be denoted G_p. When I is an ideal of R, we shall use $\mu(I)$ to denote the number of generators in a minimal basis for I. Finally, recall that in a local ring (R, m), if I is $n-$generated, then the n generators of I may be chosen from elements of a given set of generators of I (cf. [13, (5.3), p. 14]).

PROPOSITION 1 *Assume that G is a nontrivial finite $2-$group, (R, M) is an Artinian local ring with the 2-generator property but R is not a principal ideal ring and that $2 \in M$. Then $R[G]$ has the $4-$generator property if and only if*
$G \cong Z/2^iZ$, where
(1) $i \geq 1$ if M^2 is a principal ideal and $M^3 = 0$
(2) $1 \leq i \leq 2$ if M^2 is a principal ideal, $M^3 \neq 0$ and $M^2 \subset (2)$.
(3) $i = 1$ otherwise.

Proof. $\Rightarrow]$ Assume that G is not a cyclic group and $R[G]$ has the $4-$generator property. Then the homomorphic image $R[Z/2Z \oplus Z/2Z]$ does also. Hence N^2 is $4-$generated where $N = (u, v, 1 - X^g, 1 - X^h)$, $M = (u, v)$ and $< g > \oplus < h >= Z/2Z \oplus Z/2Z$. Since $| < g > | = 2$ and $2 \in M$, then $N^2 = (u^2, v^2, uv, u(1 - X^g), v(1 - X^g), u(1 - X^h), v(1 - X^h), (1 - X^g)(1 - X^h))$.

It is easy to see that $(1-X^g)(1-X^h)$ is required as a generator of N^2. Since $M = (u, v)$ is not a principal ideal, it is also easy to verify that $u(1 - X^g), v(1 - X^g), u(1 - X^h)$ and $v(1 - X^h)$ are required as generators of N^2. Therefore N^2 needs more than four generators, a contradiction.

(1) Trivial.

(2) Since M^2 is a principal ideal, one can easily check that M^3 is a principal ideal too. Further, we may assume $M = (2, v)$ since $2 \in M \setminus M^2$. Suppose that $R[Z/8Z]$ has the 4-generator property and let $< g >= Z/8Z$, $M^2 = (\alpha)$, and $M^3 = (\mu)$. We have

$N = (2, v, 1 - X^g)$;

$N^2 = (\alpha, 2(1 - X^g), v(1 - X^g), (1 - X^g)^2)$;

$N^3 = (\mu, \alpha(1 - X^g), 2(1 - X^g)^2, v(1 - X^g)^2, (1 - X^g)^3)$.

Since $M^3 \neq 0$ and $| < g > | > 3$, it is clear that μ and $(1 - X^g)^3$ are required as generators of N^3.

If $\alpha(1 - X^g)$ is a redundant generator of N^3, then by passing to the homomorphic image $R/M^3[< g >]$ and by using [1, Lemma 1.4], we get $\alpha = 8\lambda$ for some $\lambda \in R/M^3$. It follows that $\alpha \in M^3$, whence $M^2 = M^3$, i. e., $M^2 = 0$, a contradiction.

If $2(1 - X^g)^2$ is redundant, then passing to the homomorphic image $R/(4, v)[< g >]$ yields $2(1 - X^g)^2 = \sum_{i=0}^{i=7} a_i X^{ig}(1 - X^g)^3$ where $a_i \in R/(4, v)$. After setting corresponding terms equal, we obtain a system of 8 linear equations in 8 unknowns. After resolving this system, we obtain $2 = 0$ in $R/(4, v)$, i. e., $M = (2, v) = (2^2, v) = (2^3, v) = \cdots = (v)$, since R is Artinian, a contradiction.

If $v(1 - X^g)^2$ is redundant, then passing to the homomorphic image $R/(2, v^2)[< g >]$, yields $v(1 - X^g)^2 \in (1 - X^g)^3 R/(2, v^2)[< g >]$, whence $v(1-X^g)^7 \in (1-X^g)^8 R/(2, v^2)[< g >] = 0$. Therefore $v \in (2, v^2)$ i. e., $M = (2, v) = (2)$, a contradiction. Consequently, N^3 is not $4-$generated.

(3) We consider separately three subcases. <u>*case1*</u> : Assume M^2 is not a principal ideal. It suffices to prove that $R[Z/4Z]$ does not have the $4-$generator property.

Since M and M^2 are not principal ideals and $| < g > | > 3$, it is easily seen that $N^2 = (u^2, v^2, uv, u(1 - X^g), v(1 - X^g), (1 - X^g)^2)$ is not $4-$generated where $M = (u, v)$ and $< g >= Z/4Z$.

<u>*case2*</u>: Assume M^2 is a principal ideal, $M^3 \neq 0$, and $2 \in M^2$. We claim that N^3 is not $4-$generated in $R[Z/4Z]$, where $N = (u, v, 1 - X^g)$ and $< g >= Z/4Z$. Indeed, we have
$N^2 = (\alpha, u(1 - X^g), v(1 - X^g), (1 - X^g)^2)$ and $N^3 = (\alpha u, \alpha v, \alpha(1 - X^g), u(1 - X^g)^2, v(1 - X^g)^2, (1 - X^g)^3)$, where $M^2 = (\alpha)$.

$| < g > | = 4$ implies that $(1 - X^g)^3$ is required as a generator of N^3. If $u(1 - X^g)^2$ is redundant, then passing to the homomorphic image $R/(u^2, v)[< g >]$ yields $u(1-X^g)^2 \in (1-X^g)^3 R/(u^2, v)[< g >]$, whence $u(1 - X^g)^3 \in (1 - X^g)^4 R/(u^2, v)[< g >] \subset 2R/(u^2, v)[< g >]$. Since $2 \in M^2$ and $R/(u^2, v)[< g >]$ is a free $(R/(u^2, v))$-module, then $u \in (u^2, v)$, a contradiction. Likewise for $v(1 - X^g)^2$.

If $\alpha(1 - X^g)$ is a redundant generator of N^3, then passing to the homomorphic image $R/M^3[< g >]$ yields $\alpha(1 - X^g) \in (1 - X^g)^2 R/M^3[< g >]$. By [1, Lemma 1.4] $\alpha = 4\lambda$, for some $\lambda \in R/M^3$. It follows that $\alpha = 0$ in R/M^3, i.e., $M^2 = (\alpha) = 0$, a contradiction.

Since $M^3 \neq 0$, it is clear that N^3 needs more than four generators. Consequently, $R[Z/4Z]$ does not have the $4-$generator property.

<u>*Case3*</u>: Assume M^2 is a principal ideal, $M^3 \neq 0$, $2 \in M \setminus M^2$, and $M^2 \not\subset (2)$. Clearly, M^3 is principal. Further, we may assume $M = (2, v)$, and hence $M^2 = (v^2)$. We claim that $R[Z/4Z]$ does not have the $4-$generator property. Effectively,

Suppose $4 \notin M^4$. It follows from the assumption $M^2 \not\subset (2)$ that $4 \in M^3 \setminus M^4$, and hence $M^3 = (4)$.

In $R[Z/4Z]$, let $I = (4, v^2(1-X^g), 2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$ where $<g>= Z/4Z$. Since $4 \neq 0$ and $|<g>| > 3$, it is easily checked that 4 and $(1-X^g)^3$ are required as generators of I. Moreover, using techniques similar to ones used above, we prove that $v(1-X^g)^2$ must appear in a party of 4 generators extracted from the original set of generators of I. If $v^2(1-X^g)$ is redundant, then passing to the homomorphic image $R/(2)[<g>]$ yields $v^2(1-X^g) \in (1-X^g)^2 R/(2)[<g>]$. By [1, Lemma 1.4], we have $v^2 = 0$ in $R/(2)[<g>]$, i. e., $v^2 \subset (2)$, a contradiction since $M^2 = (v^2) \not\subset (2)$. Therefore $I = (4, v^2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$. Now $2(1-X^g) \in I$, then passing to the homomorphic image $R/(4,v)[<g>]$ yields $2(1-X^g) = \sum_{i=0}^{i=3} a_i X^{ig}(1-X^g)^3$, where $a_i \in R/(4,v)$. After setting corresponding terms equal, we obtain the following equations :

X^o	$a_o - a_1 + 3a_2 - 3a_3 = 2$
X^g	$-3a_o + a_1 - a_2 + 3a_3 = -2$
X^{2g}	$3a_o - 3a_1 + a_2 - a_3 = 0$
X^{3g}	$-a_o + 3a_1 - 3a_2 + a_3 = 0$

This yields $2 = 0$ in $R/(4,v)$, i. e., $M = (2,v) = (v)$, a contradiction. Consequently, I needs more than four generators.

Suppose $4 \in M^4$. Let $M^3 = (\mu)$, if $M^3 \not\subset (2)$, we consider $I = (2, \mu, v^2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$. Since $2 \notin M^2$, $M^3 \not\subset (2)$ and $|<g>| > 3$, it is an easy matter to verify that 2, μ and $(1-X^g)^3$ are required as generators of I. Moreover, using arguments similar to ones used above, it is easy to check that $v^2(1-X^g)$ and $v(1-X^g)^2$ are required as generators of I. Thus I is not 4-generated.
If $M^3 \subset (2)$, then $\mu = 2\lambda$ where $\lambda \in M$ since $2 \in M \setminus M^2$. Therefore $\mu = 4\alpha_1 + 2\alpha_2 v$, where $\alpha_1, \alpha_2 \in R$.

Since $M^3 \neq 0$, $M^2 \not\subset (2)$, M^3 is a principal ideal and $4 \in M^4$, then $M^3 = (2v)$.

On the other hand, $M^3 = (v^3, 2v^2)$. Since R is an Artinian ring and $2v \in M^3$, then $M^3 = (v^3)$, whence there exists λ a unit in R such that $2v = \lambda v^3$. Let $I = (v^3, 2-\lambda v^2, v^2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$. As before, one can easily check that $v(1-X^g)^2$ and $(1-X^g)^3$ are required as generators of I. If $2-\lambda v^2$ is redundant, then passing to the homomorphic image $R[<g>]/((1-X^g)) \simeq R$ yields $2-\lambda v^2 \in (v^3)$, i. e., $2-\lambda v^2 = \beta v^3$ where $\beta \in R$. Hence $v^2 \in (2)$, so that $M^2 = (v^2) \subset (2)$, a contradiction. If v^3 is redundant, then passing to the homomorphic image $R[<g>]/((1-X^g)) \simeq R$, we obtain that $v^3 \in (2-\lambda v^2)$, i. e., $v^3 = \beta(2-\lambda v^2)$ where $\beta \in R$. Since $M^2 = (v^2) \not\subset (2)$, $M^3 \subset (2)$ and λ is a unit, then β is not a unit in R, whence

$$\begin{aligned} v^3 &= (2\beta_1 + v\beta_2)(2-\lambda v^2), \quad \text{where } \beta_1, \beta_2 \in R \\ &= \beta_1(4 - \lambda 2v^2) + \beta_2(2v - \lambda v^3) \\ &= \beta_1(4 - \lambda 2v^2) \end{aligned}$$

$4 \in M^4$ and $2v \in M^3$, then $(4-\lambda 2v^2) \in M^4$, whence $M^3 = (v^3) \subset M^4$, a contradiction since $M^3 \neq 0$. Finally, if $v^2(1-X^g)$ is redundant, then $v^3(1-X^g) \in (v^4, 2v - \lambda v^3, v^2(1-X^g)^2, v(1-X^g)^3) = (v^4, v^2(1-X^g)^2, v(1-X^g)^3)$. By passing to the homomorphic image $R/(v^4)[<g>]$, we obtain that $v^3(1-X^g) \in ((1-X^g)^2)R/(v^4)[<g>]$. By [1, lemma 1.4], we get $v^3 = 4\gamma$ where $\gamma \in R/(v^4)$. Since $4 \in M^4 = (v^4)$, then $v^3 \in (v^4)$, i. e., $M^3 = M^4$, a contradiction ($M^3 \neq 0$). Consequently, I needs more than four generators. Thus, $R[Z/4Z]$ does not have the 4−generator property.

$\Leftarrow$) Now, $R[G]$ is a local ring with maximal ideal $N = (u, v, 1-X^g)$ where u v are the generators of M and g generates the cyclic group G.

Step 1. We claim that N, N^2, N^3, and N^4 are 4−generated. Indeed,

(1) Assume $M^2 = (\alpha)$ is a principal ideal and $M^3 = 0$. Clearly,

$$\begin{aligned} N &= (u, v, 1-X^g) \ ; \\ N^2 &= (\alpha, u(1-X^g), v(1-X^g), (1-X^g)^2) \ ; \\ N^3 &= (1-X^g)N^2 \ \text{ and } ; \\ N^4 &= (1-X^g)^2 N^2. \end{aligned}$$

(2) Assume M^2 is a principal ideal, $M^3 \neq 0$, $M^2 \subset (2)$, and $G = Z/2^i Z$ with $1 \leq i \leq 2$.

Since $M^2 = (\alpha) \subset (2)$, then

$$\begin{aligned}
N &= (2, v, 1-X^g) \ ; \\
N^2 &= (\alpha, 2(1-X^g), v(1-X^g), (1-X^g)^2) \ ; \\
N^3 &= (\mu, \alpha(1-X^g), v(1-X^g)^2, (1-X^g)^3) \text{ where } M^3 = (\mu) \ ; \\
N^4 &= (\alpha^2, \mu(1-X^g), v(1-X^g)^3, (1-X^g)^4).
\end{aligned}$$

(3)<u>Case1</u> Assume M^2 is not a principal ideal and $G = Z/2Z$. Clearly,

$$\begin{aligned}
N &= (u, v, 1-X^g) \ ; \\
N^2 &= (a, b, u(1-X^g), v(1-X^g)) \text{ where } M^2 = (a, b) \ ; \\
N^3 &= (a', b', a(1-X^g), b(1-X^g)) \text{ where } M^3 = (a', b') \ ; \\
N^4 &= (a", b", a'(1-X^g), b'(1-X^g)) \text{ where } M^4 = (a", b").
\end{aligned}$$

(3)<u>Case2</u> Assume $M^2 = (\alpha)$ is a principal ideal, $M^3 \neq 0$, $2 \in M^2$, and $G = Z/2Z$. We verify that

$$\begin{aligned}
N &= (u, v, 1-X^g) \ ; \\
N^2 &= (\alpha, u(1-X^g), v(1-X^g)) \ ; \\
N^3 &= (\alpha u, \alpha v, \alpha(1-X^g)) = \alpha N \ ; \\
N^4 &= \alpha N^2.
\end{aligned}$$

(3) <u>Case3</u> Assume $M^2 = (\alpha)$ is a principal ideal, $M^3 = (\mu) \neq 0$, $2 \in M \setminus M^2$, $M^2 \not\subset (2)$, and $G = Z/2Z$. We easily check that

$$\begin{aligned}
N &= (2, v, 1-X^g) \ ; \\
N^2 &= (\alpha, 2(1-X^g), v(1-X^g)) \ ; \\
N^3 &= (\mu, \alpha(1-X^g)) \ ; \\
N^4 &= (\alpha^2, \mu(1-X^g)).
\end{aligned}$$

Step 2. Let I be an ideal of $R[G]$, we claim that I is 4−generated. Indeed, (1) Assume M^2 is a principal ideal and $M^3 = 0$. Then $N^3 = (1-X^g)N^2$, whence by [12, Lemma 2] $\mu(I) \leq \mu(I + N^2)$.

Since N^2 is 4−generated, we may assume $N^2 \subset I$. Let $x \in I \setminus N^2$. Then $\mu\left(\frac{N}{(x)}\right) = \mu(N) - 1 = 3 - 1 = 2$, so that $\frac{N}{(x)} = (\overline{u}, \overline{v})$ or $\frac{N}{(x)} = (\overline{u}, \overline{1-X^g})$ or $\frac{N}{(x)} = (\overline{v}, \overline{1-X^g})$, where $N = (u, v, 1-X^g)$.

If $\frac{N}{(x)} = (\overline{u}, \overline{v})$, then $(N/(x))^2$ is 2−generated since $M^2 = (u^2, v^2, uv)$ is 2−generated. By [12, Theorem 1, 1⇐ 6], $R[G]/(x)$ has the 2−generator property. Hence I is 4−generated.

If $\frac{N}{(x)} = (\overline{u}, \overline{1-X^g})$, then $\left(\frac{N}{(x)}\right)^2 = \frac{N^2+(x)}{(x)} \subseteq \frac{I}{(x)}$. We consider separately two cases :

Assume $\left(\frac{N}{(x)}\right)^2 \subset \frac{I}{(x)}$. Choose $z \in I$ such that $\overline{z} \in \frac{I}{(x)} \setminus \left(\frac{N}{(x)}\right)^2$. We have

$$\begin{aligned}
\mu\left(\frac{N}{(x,z)}\right) &= \mu\left(\frac{N/(x)}{(\overline{z})}\right) \\
&\leq \mu\left(\frac{N}{(x)}\right) - 1 \\
&\leq 2 - 1 = 1.
\end{aligned}$$

Consequently, $\left(\frac{R[G]}{(x,z)}\right)$is a principal ideal ring, so that $\left(\frac{I}{(x,z)}\right)$ is a principal ideal, whence I is 4−generated.

Assume $\left(\frac{N}{(x)}\right)^2 = \frac{N^2+(x)}{(x)} = \frac{I}{(x)}$. Then $I = N^2 + (x)$, where $N^2 = (\alpha, u(1-X^g), v(1-X^g), (1-X^g)^2)$ and $M^2 = (\alpha)$. Since $x \in N$, $x = \lambda u + \mu v + \gamma(1-X^g)$ for some $\lambda, \mu, \gamma \in R[G]$. Moreover, we may assume that γ is not a unit. Hence there exist $\lambda', \mu', \gamma' \in R[G]$ such that $x = \lambda' u + \mu' v + \gamma'(1-X^g)^2$. Clearly, since $x \notin N^2$, λ' or μ', say λ' is a unit. Since $I = N^2 + (x)$ we may choose $x = u + \beta v$ for some $\beta \in R[G]$ then $x(1-X^g) = u(1-X^g) + \beta v(1-X^g)$ therefore $I = (\alpha, v(1-X^g), (1-X^g)^2, x)$ which is 4−generated.

Likewise, for $\frac{N}{(x)} = (\overline{v}, \overline{1-X^g})$.

From now on, (3) case 1, (3) case 2, and (3) case 3 refer to the three subcases considered in the proof of the "only if" assertion (3).

We first handle (2) and (3) case 1 simultaneous. We have $\mu(N^4) \leq 4$, then by [2, Lemma 4], $\mu(I) \leq \mu(I+N^3)$. Since N^3 is 4−generated, we may assume that $N^3 \subset I$.

<u>*CaseI*</u>: Suppose there exists $x \in I \setminus N^2$, then $\mu\left(\frac{N}{(x)}\right) = \mu(N) - 1 = 3 - 1 = 2$, therefore $\frac{N}{(x)} = (\overline{u}, \overline{v})$ or $\frac{N}{(x)} = (\overline{u}, \overline{1-X^g})$ or $\frac{N}{(x)} = (\overline{v}, \overline{1-X^g})$, here $N = (u, v, 1-X^g)$).

If $\frac{N}{(x)} = (\overline{u}, \overline{v})$, using arguments similar to ones used above, we can check that I is 4−generated.

If $\frac{N}{(x)} = (\overline{v}, \overline{1-X^g})$, $\left(\frac{N}{(x)}\right)^3 = \frac{N^3+(x)}{(x)} \subseteq \frac{I}{(x)}$. We consider separately two cases:

If $\left(\frac{N}{(x)}\right)^3 \subset \frac{I}{(x)}$, the proof is similar to that one given in the proof of [2, proposition 3] (see pages 8,9)

If $\left(\frac{N}{(x)}\right)^3 = \frac{N^3+(x)}{(x)} = \frac{I}{(x)}$, then $I = N^3 + (x)$.

(2) $I = N^3 + (x) = (x, \mu, \alpha(1-X^g), v(1-X^g)^2, (1-X^g)^3)$.
$x \in N = (2, v, 1-X^g)$ then $x = 2\lambda + \beta v + \gamma(1-X^g)$ for some $\lambda, \beta, \gamma \in R[G]$. Moreover, we may assume that γ is not a unit. Hence there exist $\lambda', \beta', \gamma' \in R[G]$, with λ' or β' is a unit such that $x = 2\lambda' + \beta' v + \gamma'(1-X^g)^2$.
If β' is a unit, then $v \in (2, x, 1-X^g)$. Therefore $v(1-X^g)^2 \in (2(1-X^g)^2, x(1-X^g)^2, (1-X^g)^3) \subset (4(1-X^g), x, (1-X^g)^3) \subset (\alpha(1-X^g), x, (1-X^g)^3)$ (see [2, page 6]. Consequently, $I = N^3 + (x) = (x, \mu, \alpha(1-X^g), (1-X^g)^3)$.
If β' is not a unit, then λ' is a unit because $x \notin N^2$. Now, $x(1-X^g) = 2\lambda'(1-X^g) + \beta' v(1-X^g)^2 + \gamma'(1-X^g)^3$ then $2(1-X^g) \in I$. Since $M^2 = (\alpha) \subset (2)$, $I = (x, \mu, 2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$. Finally, since λ' is a unit, $I = (x, \mu, v(1-X^g)^2, (1-X^g)^3)$.

(3)<u>*case1*</u>: M^2 is not a principal ideal and $< g >= Z/2Z$. We are in the situation where $\left(\frac{N}{(x)}\right)^3 = \frac{I}{(x)}$. We have

$$\begin{aligned}
\frac{N}{(x)} &= (\overline{v}, \overline{1-X^g}) \text{ and };\\
\left(\frac{N}{(x)}\right)^2 &= \left(\overline{v^2}, \overline{v(1-X^g)}, \overline{(1-X^g)^2}\right);\\
\left(\frac{N}{(x)}\right)^3 &= \left(\overline{v^3}, \overline{v^2(1-X^g)}, \overline{v(1-X^g)^2}, \overline{(1-X^g)^3}\right);\\
&= \left(\overline{v^3}, \overline{v^2(1-X^g)}, \overline{2v(1-X^g)}, \overline{4(1-X^g)}\right);\\
&= \left(\overline{v^3}, \overline{a(1-X^g)}, \overline{b(1-X^g)}\right) \text{ where } M^2 = (a, b).
\end{aligned}$$

Thus, $\left(\frac{N}{(x)}\right)^3$ is 3-generated, and hence so is $\frac{N}{(x)}$. It follows that I is 4−generated.

The argument is similar if $\frac{N}{(x)} = (\overline{u}, \overline{1-X^g})$.

<u>*CaseII*</u>: $(N^3 \subset) I \subseteq N^2$. In this case, we claim that there exists $x \in I \setminus N^3$ such that $\mu\left(\left(\frac{N}{(x)}\right)^3\right) \leq 3$. Indeed,

(2) We have

$$\begin{aligned}
N &= (2, v, 1-X^g) ;\\
N^2 &= (\alpha, 2(1-X^g), v(1-X^g), (1-X^g)^2) ;\\
N^3 &= (\mu, \alpha(1-X^g), v(1-X^g)^2, (1-X^g)^3).
\end{aligned}$$

Let $x \in I \setminus N^3$, $x = a_x\alpha + b_x 2(1-X^g) + c_x v(1-X^g) + d_x(1-X^g)^2$ for some $a_x, b_x, c_x, d_x \in R[G]$, where at least one of a_x, b_x, c_x, d_x is a unit.

If a_x is a unit, then $\overline{\alpha} \in \left(\overline{2(1-X^g)}, \overline{v(1-X^g)}, \overline{(1-X^g)^2}\right)$, whence $\overline{\mu} \in \left(\overline{\alpha(1-X^g)}, \overline{2(1-X^g)^2}, \overline{v(1-X^g)^2}\right) \subseteq \left(\overline{\alpha(1-X^g)}, \overline{(1-X^g)^3}, \overline{v(1-X^g)^2}\right)$. So that $\left(\frac{N}{(x)}\right)^3 = \frac{N^3+(x)}{(x)} = \left(\overline{\alpha(1-X^g)}, \overline{(1-X^g)^3}, \overline{v(1-X^g)^2}\right)$.

If c_x is a unit, then $\overline{v(1-X^g)} \in \left(\overline{2(1-X^g)}, \overline{\alpha}, \overline{(1-X^g)^2}\right)$, whence $\overline{v(1-X^g)^2} \in \left(\overline{2(1-X^g)^2}, \overline{\alpha(1-X^g)}, \overline{(1-X^g)^3}\right) \subseteq \left(\overline{\alpha(1-X^g)}, \overline{(1-X^g)^3}\right)$. Therefore $\left(\frac{N}{(x)}\right)^3 = \frac{N^3+(x)}{(x)} = \left(\overline{\mu}, \overline{\alpha(1-X^g)}, \overline{(1-X^g)^3}\right)$.

If d_x is a unit, then $\overline{(1-X^g)^2} \in \left(\overline{2(1-X^g)}, \overline{\alpha}, \overline{v(1-X^g)}\right)$, whence $\overline{v(1-X^g)^2} \in \left(\overline{\mu}, \overline{\alpha(1-X^g)}\right)$. Hence

$$\left(\frac{N}{(x)}\right)^3 = \frac{N^3+(x)}{(x)} = \left(\overline{\mu}, \overline{\alpha(1-X^g)}, \overline{(1-X^g)^3}\right).$$

Otherwise, for each $x \in I \setminus N^3$, a_x, c_x, and d_x are not units, Necessarily, b_x is a unit. It follows that $2(1-X^g) \in I \setminus N^3$.

$$\begin{aligned}\left(\frac{N}{(2(1-X^g))}\right)^3 &= \frac{N^3+(2(1-X^g))}{(2(1-X^g))} \\ &= \frac{(\mu, \alpha(1-X^g), v(1-X^g)^2, (1-X^g)^3, 2(1-X^g))}{(2(1-X^g))}.\end{aligned}$$

Since $M^2 = (\alpha) \subset (2)$, then

$$\left(\frac{N}{(2(1-X^g))}\right)^3 = \left(\overline{\mu}, \overline{v(1-X^g)^2}, \overline{(1-X^g)^3}\right).$$

(3) <u>case1</u>: We have

$$\begin{aligned} N &= (u, v, 1-X^g); \\ N^2 &= (a, b, u(1-X^g), v(1-X^g)) \text{ where } M^2 = (a,b); \\ N^3 &= (a', b', a(1-X^g), b(1-X^g)) \text{ where } M^3 = (a', b'). \end{aligned}$$

Let $x \in I \setminus N^3$. Clearly, $x = a_x a + b_x b + c_x u(1-X^g) + d_x v(1-X^g)$ for some $a_x, b_x, c_x, d_x \in R[G]$, where at least one of a_x, b_x, c_x, d_x is a unit. In each case, one may verify that $\mu\left(\left(\frac{N}{(x)}\right)^3\right) \leq 3$ (Assume $a \in \{u^2, uv\}$ and $b = v^2$).

We get

$$\begin{aligned}\mu\left(\frac{I}{(x)}\right) &\leq \mu\left(\frac{I}{(x)} + \left(\frac{N}{(x)}\right)^2\right) = \mu\left(\frac{I+N^2}{(x)}\right) \quad \text{by [2, Lemma 4] ;} \\ &= \mu\left(\frac{N^2}{(x)}\right) \quad \text{since } I \subseteq N^2 \text{ ;} \\ &\leq 3 \quad \text{since } x \in N^2 \setminus N^3 \text{ and } N^2 \text{ is 4-generated .}\end{aligned}$$

Consequently, I is 4−generated.

(3) <u>cases 2 and 3</u>: We have $\mu(N^3) \leq 3$, then by [2, Lemma 4], $\mu(I) \leq \mu(I+N^2)$.
Since N^2 is 4−generated, we can assume that $N^2 \subset I$. We ape the proof of (1) (page 7) to reach the desired conclusion when $\left(\frac{N}{(x)}\right)^2 \subset \frac{I}{(x)}$. Otherwise, $I = N^2 + (x)$ is 4−generated because in (3) cases 2 and 3, N^2 is 3−generated. ◇

PROPOSITION 2 *Assume G is a non trivial finite 3−group, (R, M) is an Artinian local ring with the 2-generator property but R is not a principal ideal ring, and that $3 \in M$. Then $R[G]$ has the 4−generator property if and only if*

(a) G is a cyclic group.

(b_1) When M^2 is a principal ideal and $M^3 \neq 0$ then

(α_1) If $3 \in M^2$, then $G \cong Z/3Z$ and M^3 is a principal ideal.

(α_2) If $3 \in M \setminus M^2$, then $G \cong Z/3^iZ$ with $1 \leq i \leq 2$, moreover, if $9 \in M^3$ then $G \cong Z/3Z$.

(b_2) When M^2 is not a principal ideal, then $3 \notin M^2$, $G \cong Z/3Z$, moreover, if $M^3 \neq 0$ and $M^2 \not\subset (3)$ then M^3 is a principal ideal and

(θ_1) If $9 \in M^2 \setminus M^3$ then $M^3 \subset (9)$.

(θ_2) If $9 \in M^3$ then $M^3 = 3M^2$.

Proof. ⇒] (a) Assume that G is not a cyclic group and $R[G]$ has the 4−generator property. Necessarily, the homomorphic image $R[Z/pZ \oplus Z/pZ]$ does also, when p = 3. Then N^2 is 4−generated, where $N = (u, v, 1-X^g, 1-X^h)$, $M = (u, v)$, and $< g > \oplus < h > = Z/pZ \oplus Z/pZ$.

$N^2 = (u^2, v^2, uv, u(1-X^g), v(1-X^g), u(1-X^h), v(1-X^h), (1-X^g)(1-X^h), (1-X^g)^2, (1-X^h)^2)$.

Since $|< g >| = 3$, via [1, Lemma 1.4], it is easy to verify that N^2 needs more than four generators. Thus $G = Z/p^m Z$, with $m \geq 1$.

(b_1) Assume $M^2 = (\alpha)$ is a principal ideal and $M^3 \neq 0$.

(α_1) Suppose $p = 3 \in M^2$. If $G = Z/p^m Z$ with $m > 1$, we claim that N^3 is not 4−generated in $R[Z/p^m Z]$ where $N = (u, v, 1-X^g)$, $M = (u, v)$, and $< g >= Z/p^m Z$.
We have $N^2 = (\alpha, u(1-X^g), v(1-X^g), (1-X^g)^2)$ and $N^3 = (\alpha u, \alpha v, \alpha(1-X^g), u(1-X^g)^2, v(1-X^g)^2, (1-X^g)^3)$.
By [1, Lemma 1.7], $u(1-X^g)^2$ and $v(1-X^g)^2$ are required as generators of N^3.
Since $| < g > | > 3$ it is clear that $(1-X^g)^3$ is required as generator of N^3.

If $\alpha(1-X^g)$ is a redundant generator of N^3, then passing to the hommorphic image $R/M^3[< g >]$, yields $\alpha(1-X^g) \in (1-X^g)^2 R/M^3[< g >]$. By [1, Lemma 1.4] $\alpha = \lambda p^m$ for some $\lambda \in R/M^3$. It follows that $\alpha = 0$ in R/M^3. That is, $M^2 = (\alpha) = 0$, a contradiction. Further, αu or αv is required as a generator of N^3, since $M^3 \neq 0$.

Now, suppose M^3 is not a principal ideal and $G = Z/3Z$.

By [1, Lemma 1.7] and the fact that M^3 is not a principal ideal, we can easily check that $\alpha u, \alpha v, u(1-X^g)^2 and v(1-X^g)^2$ are required as generators of N^3, then, if N^3 is 4-generated, necessarly, $N^3 = (\alpha u, \alpha v, u(1-X^g)^2, v(1-X^g)^2)$. Further $|< g >| = 3$ and $3 \in M^2$, $\alpha(1-X^g) \notin (\alpha u, \alpha v, u(1-X^g)^2, v(1-X^g)^2)$. Then N^3 needs more than four generators.

(α_2) Suppose $p = 3 \in M \setminus M^2$. Let's show that N^3 is not 4−generated in $R[Z/p^3 Z]$.
We have:

$N = (p, v, 1-X^g)$ and $N^3 = (p\alpha, v\alpha, \alpha(1-X^g), p(1-X^g)^2, v(1-X^g)^2, (1-X^g)^3)$.

Since $| < g > | = p^3 > 3$ and $M = (p, v)$ is not a principal ideal, by [1, Lemma 1.4 and Lemma 1.7], $\alpha(1-X^g), p(1-X^g)^2, v(1-X^g)^2$, and $(1-X^g)^3$ are required as generators of N^3. Furthermore, since $M^3 \neq 0$, it is clear that N^3 needs more than four generators. It follows that $G = Z/p^i Z$ with $1 \leq i \leq 2$, as desired.

Suppose in addition that $p^2 = 9 \in M^3$. Using the arguments similar to ones used above, it is easy to verify that $p(1-X^g)^2, v(1-X^g)^2$, and $(1-X^g)^3$ must appear in a party of four generators extracted from the original set of generators of N^3. Furthermore, if $\alpha(1-X^g)$ is redundant, then $\alpha(1-X^g) \in (p\alpha, \alpha v, p(1-X^g)^2, v(1-X^g)^2, (1-X^g)^3)$, whence passing to the homomorphic image $R/M^3[< g >]$, we get $\alpha(1-X^g) \in (1-X^g)^2 R/M^3[< g >]$. By [1, Lemma 1.4], $\alpha = \lambda p^2 = 0$ for some $\lambda \in R/M^3$, a contradiction ($M^3 \neq 0$). Thus, $G = Z/pZ$, as desired.

(b_2) Assume M^2 is not a principal ideal. one may easily show that N^2 is not 4−generated neither if $< g >= Z/9Z$ nor if $3 \in M^2$ and $< g >= Z/3Z$. Necessarily, $3 \in M \setminus M^2$ and $< g >= Z/3Z$.

Set $p = 3$. Assume in addition $M^3 \neq 0$ and $M^2 \not\subset (p)$. we claim that M^3 is a principal ideal. Deny. Let $N = (p, v, 1-X^g)$ and $< g >= Z/pZ$. Clearly, $N^3 = (a', b', a(1-X^g), b(1-X^g), p(1-X^g)^2, v(1-X^g)^2, (1-X^g)^3)$, where $M^2 = (a, b)$ and $M^3 = (a', b') = (p^3, p^2 v, pv^2, v^3)$. Further, $M^2 = (a, b) = (v^2, p^2, pv)$, since $M^2 \not\subset (p)$, we can take $a = v^2$ and $b \in \{p^2, pv\}$. Then $N^3 = (a', b', (1-X^g)^3, v^2(1-X^g), v(1-X^g)^2)$.

Since M^3 is not a principal ideal, by [1, Lemma 1.4 and Lemma 1.7], a',b', $v^2(1-X^g)$, and $v(1-X^g)^2$ are required as generators of N^3. Since N^3 is 4-generated, then $(1-X^g)^3 = -3X^g(1-X^g) \in (a', b', v^2(1-X^g), v(1-X^g)^2)$ (Here p = 3). By passing to the homomorphic image $R/(v)[< g >]$, we obtain that $3 \in (27, v)$. It follows that $M = (3, v) = (v)$ since R is Artinian, a contradiction. Consequently, $M^3 = (\mu)$ is a principal ideal.

Let $I = (v^3, b, v^2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$.

Since $| < g > | > 3$ and $b \notin M^3$ (R Artinian and M^2 not principal), it is clear that b and $(1-X^g)^3$ are required as generators of I.

If $v(1-X^g)^2$ is redundant, then by passing to the homomorphic image $R/M^2[< g >]$, and by using [1, Lemma 1.7], we get $v = \lambda p$ for some $\lambda \in R/M^2$. Hence, $v \in (p, v^2)$. Therefore $M = (p, v) = (p, v^2) = \cdots = (p)$, a contradiction.

If $v^2(1-X^g)$ is redundant, then passing to the homomorphic image $R/(v^3, b)[< g >]$ yields $v^2(1-X^g) \in (1-X^g)^2 R/(v^3, b)[< g >]$. By [1, Lemma 1.4], we have $v^2 = \lambda p$ for some $\lambda \in R/(v^3, b)$. Similarly, $(p, v^2) = (p, v^3) = \cdots = (p)$, whence $M^2 = (p^2, v^2, pv) \subset (p)$, a contradiction.

Thus $I = (b, v^2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$. Further, $v^3 \in I$, by passing to the homomorphic image $R[< g >]/(1-X^g) \cong R$, we obtain that $v^3 \in (b)$.

(θ_1) If $p^2 \in M^2 \setminus M^3$, we may assume $b = p^2$.

$M^3 = (p^3, p^2v, pv^2, v^3)$. Since $pv \in (p^2, v^2)$ then $pv^2 \in (v^3, p^2v)$. Therefore $M^3 \subset (p^2)$, as desired.

(θ_2) If $p^2 \in M^3$, we may assume $b = pv$. We have $M^2 = (v^2, pv) = vM$ so that $M^3 = v^2M = (v^3, pv^2)$. Since $v^3 \in (b) = (pv)$ and $pv \notin M^3$, then $v^3 \in (p^2v, pv^2)$. Therefore $M^3 = (pv^2, p^2v) = p(v^2, pv) = pM^2$, as desired.

$\Leftarrow$) Now, $R[G]$ is a local ring with maximal ideal $N = (u, v, 1 - X^g)$ where u and v are the generators of M and g is a generator of the cyclic group G.

Step 1. We claim that N, N^2, N^3, and N^4 are 4−generated. Indeed,

(b_1) Assume $M^2 = (\alpha)$ is a principal ideal. If $M^3 = 0$, then the proof is straightforward (see the case p = 2).

In the sequel, we suppose $M^3 \neq 0$.

α_1) Assume $3 \in M^2$, $G = Z/3Z$, and $M^3 = (\mu)$ is a principal ideal. We easily check that

$$\begin{aligned} N &= (u, v, 1 - X^g) \ ; \\ N^2 &= (\alpha, u(1 - X^g), v(1 - X^g), (1 - X^g)^2) \ ; \\ N^3 &= (\mu, \alpha(1 - X^g), u(1 - X^g)^2, v(1 - X^g)^2) \ ; \\ N^4 &= (\alpha^2, \mu(1 - X^g), \alpha(1 - X^g)^2). \end{aligned}$$

(α_2) Assume $p = 3 \in M \setminus M^2$ and $G \cong Z/p^iZ$, $1 \leq i \leq 2$. Since M^2 is a principal ideal, it is easy to verify that $M^3 = (\mu)$ is a principal ideal.

Suppose $p^2 = 9 \in M^2 \setminus M^3$. Clearly $M^2 = (p^2)$.

$$\begin{aligned} 1 &= (1 - X^g + X^g)^{p^2} \\ &= \sum_{i=o}^{i=p^2} \binom{p^2}{i} (1 - X^g)^i X^{(p^2 - i)g} \\ &= 1 + p^2(1 - X^g)X^{(p^2-1)g} + \frac{p^2(p^2 - 1)}{2}(1 - X^g)^2 X^{(p^2-2)g} \\ &\quad + (1 - X^g)^3 \left(\sum_{i=3}^{i=p^2} \binom{p^2}{i} (1 - X^g)^{(i-3)} X^{(p^2-i)g} \right). \end{aligned}$$

Then $p^2(1 - X^g) \in (p^2(1 - X^g)^2, (1 - X^g)^3) \subset ((1 - X^g)^3)$. Therefore,

$$\begin{aligned} N &= (p, v, 1 - X^g) \ ; \\ N^2 &= (p^2, p(1 - X^g), v(1 - X^g), (1 - X^g)^2) \ ; \\ N^3 &= (\mu, p(1 - X^g)^2, v(1 - X^g)^2, (1 - X^g)^3) \ ; \\ N^4 &= (p^4, \mu(1 - X^g), p(1 - X^g)^3, v(1 - X^g)^3, (1 - X^g)^4). \end{aligned}$$

We have $M^3 = (\mu) \subset M^2 = (p^2)$, whence $\mu \in (p^3, p^2v)$ since $p^2 \notin M^3$. Therefore $\mu(1 - X^g) \in (p^3(1 - X^g), p^2v(1 - X^g)) \subset (p(1 - X^g)^3, v(1 - X^g)^3)$. It results that

$N^4 = (p^4, p(1 - X^g)^3, v(1 - X^g)^3, (1 - X^g)^4)$

Suppose $9 \in M^3$ and $G = Z/3Z$. Clearly,

$$\begin{aligned} N &= (3, v, 1 - X^g) \ ; \\ N^2 &= (\alpha, v(1 - X^g), (1 - X^g)^2) \ ; \\ N^3 &= (\mu, \alpha(1 - X^g), v(1 - X^g)^2, 3(1 - X^g)) \ ; \\ N^4 &= (\alpha^2, \mu(1 - X^g), \alpha(1 - X^g)^2, 3(1 - X^g)^2). \end{aligned}$$

(b_2) Set $p = 3$. Assume $M^2 = (a, b)$ is not a principal ideal, $p \notin M^2$, and $G = Z/pZ$. Clearly,

$$\begin{aligned} N &= (p, v, 1 - X^g) \ ; \\ N^2 &= (a, b, v(1 - X^g), (1 - X^g)^2). \end{aligned}$$

If $M^3 = 0$, then

$$\begin{aligned} N^3 &= (a(1 - X^g), b(1 - X^g), v(1 - X^g)^2, (1 - X^g)^3) = (1 - X^g)N^2; \\ N^4 &= (1 - X^g)^2 N^2 \ \text{(Recall that } p(1 - X^g) \in (1 - X^g)^3\text{)}. \end{aligned}$$

In the sequel, we suppose $M^3 \neq 0$.

If $M^2 \subset (p)$, then $M^2 = (p^2, pv)$ ($p \notin M^2$), whence $N^3 = (p^3, p^2v, v(1 - X^g)^2, (1 - X^g)^3)$ and $N^4 = (p^4, p^3v, v(1 - X^g)^3, (1 - X^g)^4)$, since $p(1 - X^g) \in ((1 - X^g)^2)$.

Now, assume $M^2 \not\subset (p)$ and $M^3 = (\mu)$ is a principal ideal. We may assume $a = v^2$ and $b \in \{pv, p^2\}$.

It is easily seen that $N^3 = (\mu, v^2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$. It remains to show that N^4 is 4-generated.

If $p^2 \in M^3$ and $M^3 = pM^2$, then $M^4 = p^2M^2 \subset M^5$, whence $M^4 = 0$. Therefore $N^4 = (\mu(1-X^g), v^2(1-X^g)^2, v(1-X^g)^3, (1-X^g)^4)$.

If $p^2 \in M^2 \setminus M^3$ and $M^3 \subset (p^2)$, since M^3 is a principal ideal, it is easy to verify that $(M^4 = (\gamma))$ is a principal ideal. So that $N^4 = (\gamma, \mu(1-X^g), v^2(1-X^g)^2, v(1-X^g)^3, (1-X^g)^4)$.

Since $p(1-X^g) \in ((1-X^g)^2)$ and $\mu \in (p^3, p^2v)$ $(M^3 = (\mu) \subset (p^2))$, then $\mu(1-X^g) \in (v(1-X^g)^3, (1-X^g)^4)$. Therefore $N^4 = (\gamma, v^2(1-X^g)^2, v(1-X^g)^3, (1-X^g)^4)$.

Step 2. Let I be an ideal of $R[G]$, we claim that I is 4−generated. Indeed,

If $M^3 = 0$, then the proof is similar to the one given for p = 2.

If $M^3 \neq 0$, as in the proof of Proposition 1 (cases (2) and (3) <u>*case*1</u>), we may assume $N^3 \subset I$.

<u>*CaseI*</u>: Suppose that there exists $x \in I \setminus N^2$. Via the proof of Proposition 1, it suffices to consider the case $I = N^3 + (x)$.

(b_1) Assume $M^2 = (\alpha)$ is a principal ideal and $M^3 \neq 0$.

(α_1) We got from step 1 that $N^3 = (\mu, \alpha(1-X^g), u(1-X^g)^2, v(1-X^g)^2)$. Since $x \in N = (u, v, 1-X^g)$, $x = \lambda u + \beta v + \gamma(1-X^g)$ for some $\lambda, \beta, \gamma \in R[G]$, where λ or β or γ is a unit. If γ is a unit, $\dfrac{N}{(x)} = (\bar{u}, \bar{v})$. We conclude in the same way as in the case p = 2 step 2 page 7.

If γ is not a unit, necessarily, λ or β is a unit, say λ. Clearly, $u \in (x, v, 1-X^g)$, then $u(1-X^g)^2 \in (x, v(1-X^g)^2, 3(1-X^g)) \subset (x, v(1-X^g)^2, \alpha(1-X^g))$, since $|<g>| = 3$ and $3 \in M^2$. Therefore $I = (x, \mu, v(1-X^g)^2, \alpha(1-X^g))$.

(α_2) Assume $p = 3 \in M \setminus M^2$ and $p^2 \in M^2 \setminus M^3$. We got from step1 that $N^3 = (\mu, p(1-X^g)^2, v(1-X^g)^2, (1-X^g)^3)$. Since $x \in N = (p, v, 1-X^g)$, $x = \lambda p + \beta v + \gamma(1-X^g)$, where λ or β or γ is a unit ($x \notin N^2$). In each case, it is easy to verify that $I = N^3 + (x)$ is 4−generated.

Now, assume $9 \in M^3$ and $G = Z/3Z$. If $M^2 \subset (3)$, we are done via [1, Proposition 2.1]. Let's suppose $M^2 \not\subset (3)$. We have $N^3 = (\mu, v^2(1-X^g), v(1-X^g)^2, 3(1-X^g))$. Since $x \in N \setminus N^2$, $x = 3\lambda + \beta v + \gamma(1-X^g)$ for some $\lambda, \beta, \gamma \in R[G]$, with λ or β or γ is a unit. The cases in which β or γ is a unit are straightforward.

We assume then that β and γ are not units. Then $x = 3\lambda' + \beta' v^2 + \mu' v(1-X^g) + \gamma'(1-X^g)^2$ for some $\lambda', \beta', \mu', \gamma' \in R[G]$. Clearly, λ' is a unit. Therefore $x(1-X^g) = 3\lambda'(1-X^g) + \beta' v^2(1-X^g) + \mu' v(1-X^g)^2 + \gamma'(1-X^g)^3$.

If μ' or β' is a unit, it is easy to see that $I = (x, \mu, v^2(1-X^g), 3(1-X^g))$ or $I = (x, \mu, v(1-X^g)^2, 3(1-X^g))$.

If μ' and β' are not units, since $I = N^3 + (x)$, we can take $x = 3\lambda' + \gamma'(1-X^g)^2$. Furthermore, if γ' is not a unit, we may take $x = 3$, whence $I = (3, \mu, v^2(1-X^g), v(1-X^g)^2)$. If γ' is a unit, then $(1-X^g)^2 \in (3, x)$, hence $v(1-X^g)^2 \in (3v, x) \subset (\mu, x)$ since $3v \in M^3 = (\mu)$ (Recall M^2 is a principal ideal and $M^2 \not\subset (3)$). Thus, $I = (x, \mu, v^2(1-X^g), 3(1-X^g))$.

(b_2) Assume M^2 is not a principal ideal, $p = 3 \notin M^2$ and $G = Z/pZ$.

If $M^2 \subset (p)$, we have $N^3 = (p^3, p^2v, v(1-X^g)^2, (1-X^g)^3)$. Since, $p(1-X^g) \in ((1-X^g)^3)$, we easily show that $I = N^3 + (x)$ is 4-generated.

If $M^2 \not\subset (p)$, we have $N^3 = (\mu, v^2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$. Similarly, $x = \lambda p + \beta v + \gamma(1-X^g)$ for some $\lambda, \beta, \gamma \in R[G]$, with λ or β or γ is a unit. We can assume that β and γ are not units, hence, there exist $\lambda', \beta', \gamma', \delta' \in R[G]$ such that $x = \lambda' p + \beta' v^2 + \gamma' v(1-X^g) + \delta'(1-X^g)^2$, where λ' is a unit.

If β' or γ' is a unit, it is easy to verify that $I = (x) + N^3$ is 4−generated.

If β' and γ' are not units, since $I = (x) + N^3$, we can suppose that $x = \lambda' p + \delta'(1-X^g)^2$, whence $p^2 \in (x, p(1-X^g)^2) \subset (x, (1-X^g)^3)$ and $pv \in (x, v(1-X^g)^2)$. Furtheremore, under the present hypotheses, one may check that $M^3 = (\mu) \subset (b)$ where $b \in \{p^2, pv\}$. Hence, $\mu \in (x, v(1-X^g)^2, (1-X^g)^3)$. Therefore $I = (x, v^2(1-X^g), v(1-X^g)^2, (1-X^g)^3)$.

<u>*CaseII*</u> Suppose $(N^3 \subset) I \subseteq N^2$. Using step1 and arguments similar to ones used above, we show that there exists $x \in I \setminus N^3$ such that $\mu\left(\left(\dfrac{N}{(x)}\right)^3\right) \leq 3$.

Actually, it remains to handle the following case : Assume M^2 is not principal, $p = 3 \notin M^2$, $<g> = Z/pZ$, $M^3 = (\mu)$ is a nonzero principal ideal, and $M^2 \not\subset (p)$. we got by step1 that $N^2 = (v^2, b, v(1-X^g), (1-X^g)^2)$, and $N^3 = (\mu, v^2(1-X^g), v(1-X^g)^2, (1-$

$X^g)^3)$, where $b \in \{p^2, pv\}$. Let $x \in I \setminus N^3$, $x = a_x v^2 + b_x b + c_x v(1 - X^g) + d_x (1 - X^g)^2$ for some $a_x, b_x, c_x, d_x \in R[G]$, with a_x or b_x or c_x or d_x is a unit.

If a_x or c_x or d_x is a unit, easily we check that $\mu\left(\left(\frac{N}{(x)}\right)^3\right) \leq 3$.

Otherwise, since b_x is a unit, $x \notin N^3$, and hence $b \in (x) + N^3$. Therefore $N^3 + (x) = N^3 + (b)$. Since $b \notin N^3$ and $M^3 = (\mu) \subset (b)$, then $\mu\left(\frac{N}{(b)}\right)^3 = \mu\left(\frac{N^3 + (b)}{(b)}\right) \leq 3$.

By the same proof for $p = 2$, we claim that I is 4−generated. $\diamondsuit$

PROPOSITION 3 *Assume that G is a non trivial finite p−group, (R, M) is an Artinian local ring with the 2-generator property but R is not a principal ideal ring and that $p \in M$. Then $R[G]$ has the 4−generator property if and only if*

(a) G is a cyclic group.

(b_1) When M^2 is a principal ideal and $M^3 \neq 0$ then

(α_1) If $p \in M^2$, then $G \cong Z/pZ$, $p \notin M^3$, and M^3 is a principal ideal.

(α_2) If $p \in M \setminus M^2$, then $G \cong Z/p^iZ$ with $1 \leq i \leq 2$, moreover, if $p^2 \in M^3$ then $G \cong Z/pZ$ and either $M^2 \subset (p)$ or $M^3 \subset (p)$.

(b_2) When M^2 is not a principal ideal, then $p \notin M^2$, $G \cong Z/pZ$, moreover, if $M^3 \neq 0$ and $M^2 \not\subset (p)$ then M^3 is a principal ideal and

(θ_1) If $p^2 \in M^2 \setminus M^3$ then $M^3 \subset (p^2)$.

(θ_2) If $p^2 \in M^3$ then $M^3 = pM^2$.

Proof of Proposition 3. It is almost similar to the proof of Proposition 2. Here the main fact is that $|< g >| = p > 3$. The remaining two cases are : (α_1) and (α_2) when $p^2 \in M^3$.

$\Rightarrow$] (α_1) Assume $p \in M^2$. If M^3 is not a principal ideal or $p \in M^3$ or $G = Z/p^mZ$ with $m > 1$, by the same proof given for Proposition 2 (α_1) we verify that N^3 is not 4−generated in $R[Z/p^mZ]$ where $N = (u, v, 1 - X^g)$, $M = (u, v)$, and $< g > = Z/p^mZ$.

(α_2) Assume $p \in M \setminus M^2$ and $p^2 \in M^3$. Necessarily, $< g > = Z/pZ$. Let's suppose $M^2 \not\subset (p)$ and $M^3 \not\subset (p)$. Let $I = (p, \mu(1 - X^g), \alpha(1 - X^g)^2, v(1 - X^g)^3, (1 - X^g)^4)$.

Since $p \in M \setminus M^2$ and $|< g >| > 4$, then p and $(1 - X^g)^4$ are required as generators of I.

If $v(1 - X^g)^3$ is redundant then by passing to the homomorphic image $R/(\alpha, p)[< g >]$, we obtain that $v(1 - X^g)^3 \in (1 - X^g)^4 R/(\alpha, p)[< g >]$, and whence $v(1 - X^g)^{p-1} = 0$ in $R/(\alpha, p)[< g >]$. Therefore $M = (p, v) = (p, v^2) = \cdots\cdots = (p)$, a contradiction.

If $\mu(1 - X^g)$ is a redundant generator then by passing to the homomorphic image $R/(p)[< g >]$, we obtain that $\mu(1 - X^g) \in (1 - X^g)^2 R/(p)[< g >]$. By [1, Lemma 1.4], we get $\mu = \lambda p$ for some $\lambda \in R/(p)$. Hence $M^3 = (\mu) \subset (p)$, a contradiction.

If $\alpha(1 - X^g)^2$ is redundant, then by passing to the homomorphic image $R/(p, \mu)[< g >]$, we obtain that $\alpha(1 - X^g)^2 \in (1 - X^g)^3 R/(p, \mu)[< g >]$. By [1, Lemma 1.7], we get $\alpha = \lambda p$ for some $\lambda \in R/(p, \mu)$, whence $v^2 \in (p, v^3)$. Hence $(p, v^2) = (p, v^3) = \cdots = (p)$, so that $M^2 = (v^2, p^2, pv) \subset (p)$, a contradiction.

Consequently, I is not 4−generated.

$\Leftarrow$) Now, we know that $R[G]$ is a local ring with maximal ideal $N = (u, v, 1 - X^g)$, where u and v are the generators of M and g is a generator of the cyclic group G.

Step : 1. We claim that N, N^2, N^3, and N^4 are 4−generated. Indeed,

α_1) Assume $p \in M^2$, $G = Z/pZ$, $p \notin M^3$, and M^3 is a principal ideal. Necessarily, $M^2 = (p)$.

Since $p(1 - X^g) \in ((1 - X^g)^3)$, we get

$$\begin{aligned} N &= (u, v, 1 - X^g) \ ; \\ N^2 &= (p, u(1 - X^g), v(1 - X^g), (1 - X^g)^2) \ ; \\ N^3 &= (\mu, u(1 - X^g)^2, v(1 - X^g)^2, (1 - X^g)^3) \ ; \\ N^4 &= (p^2, u(1 - X^g)^3, v(1 - X^g)^3, (1 - X^g)^4). \end{aligned}$$

(α_2) Assume $p \in M \setminus M^2$, $p^2 \in M^3$, $G \cong Z/pZ$, and either $M^2 \subset (p)$ or $M^3 \subset (p)$. We have

$$\begin{aligned} N &= (p, v, 1 - X^g) \ ; \\ N^2 &= (\alpha, v(1 - X^g), (1 - X^g)^2) \text{ where } M^2 = (\alpha) \ ; \\ N^3 &= (\mu, \alpha(1 - X^g), v(1 - X^g)^2, (1 - X^g)^3) \text{ where } M^3 = (\mu) \ ; \\ N^4 &= (\alpha^2, \mu(1 - X^g), \alpha(1 - X^g)^2, v(1 - X^g)^3, (1 - X^g)^4). \end{aligned}$$

If $M^2 \subset (p)$ then $\alpha(1-X^g) \in (p^2(1-X^g), pv(1-X^g)) \subset (v(1-X^g)^3, (1-X^g)^4)$, whence $N^4 = (\alpha^2, \mu(1-X^g), v(1-X^g)^3, (1-X^g)^4)$.

If $M^3 \subset (p)$. Since $p \in M \setminus M^2$, $\mu \in (p^2, pv)$. Further, $p^2(1-X^g) \in ((1-X^g)^4)$ and $pv(1-X^g) \in (v(1-X^g)^3)$. Then $\mu(1-X^g) \in (v(1-X^g)^3, (1-X^g)^4)$. Therefore $N^4 = (\alpha^2, \alpha(1-X^g)^2, v(1-X^g)^3, (1-X^g)^4)$.

Step : 2 Let I be an ideal of $R[G]$, we claim that I is 4−generated. As in Proposition 1, we may assume that $N^3 \subset I$.

<u>*CaseI*</u>: Suppose that there exists $x \in I \setminus N^2$. As above, it suffices to consider the case $I = N^3 + (x)$.

(α_1) By step1, it is easily seen that $I = N^3 + (x)$ is 4−generated.

(α_2) Assume $p \in M \setminus M^2$, $p^2 \in M^3$, $G \cong Z/pZ$, and either $M^2 \subset (p)$ or $M^3 \subset (p)$. By step1, $N^3 = (\mu, \alpha(1-X^g), v(1-X^g)^2, (1-X^g)^3)$. Since $x \in N \setminus N^2$ then $x = \lambda p + \beta v + \gamma(1-X^g)$ for some $\lambda, \beta, \gamma \in R[G]$, with λ or β or γ is a unit. We can assume that β and γ are not units. Therefore $p \in (x, v, 1-X^g)$.

If $M^2 \subset (p)$, $\alpha \in (p^2, pv)$. Hence $\alpha(1-X^g) \in (p^2(1-X^g), pv(1-X^g)) \subset ((1-X^g)^3)$. So that $I = (x, \mu, v(1-X^g)^2, (1-X^g)^3)$.

If $M^3 \subset (p)$, $x = \lambda' p + \beta' \alpha + \gamma' v(1-X^g) + \delta'(1-X^g)^2$ for some $\lambda', \beta', \gamma', \delta' \in R[G]$. Clearly λ' is a unit ($x \notin N^2$).

If β' or γ' is a unit, we verify that I is 4−generated.

Otherwise, since $I = N^3 + (x)$, we can suppose that $x = \lambda' p + \delta'(1-X^g)^2$. By hypothesis, $M^3 = (\mu) \subset (p)$. Then $\mu = \theta p$ for some $\theta \in M$ ($p \notin M^2$), hence $x\theta = \lambda'\mu + \delta'\theta(1-X^g)^2$. Therefore $\mu \in (x, v(1-X^g)^2, p(1-X^g)^2) \subset (x, v(1-X^g)^2, (1-X^g)^3)$. Consequently, $I = (x, \alpha(1-X^g), v(1-X^g)^2, (1-X^g)^3)$.

<u>*CaseII*</u>: Suppose $(N^3 \subseteq) I \subset N^2$. The proof is the same as in Proposition 2. $\Diamond$

THEOREM. *Let R be an Artinian ring with the 2−generator property and let G be a finite abelian group. Then $R[G]$ has the 4−generator property if and only if $R = R_1 \oplus R_2 \oplus \cdots \oplus R_s$ where, for each j, (R_j, M_j) is a local Artinian ring with the 2−generator property subject to :*

(I) *Assume R_j is a field of characteristic $p \neq 0$.*

(α) when $p = 2$, then G_p is a homomorphic image of $Z/2Z \oplus Z/2Z \oplus Z/2^iZ$ or $Z/4Z \oplus Z/2^iZ$ where $i > 0$

(β) when $p = 3$, then G_p is a homomorphic image of $Z/3Z \oplus Z/3^iZ$ where $i > 0$

(γ) when $p > 3$, then G_p is a cyclic group.

(II) *Assume (R_j, M_j) is a principal ideal ring which is not a field, and p a prime integer such that p divides Ord(G) and $p \in M_j$, then*

(α) Assume $p = 2$,

A) (i) $G_p \cong Z/2Z \oplus Z/2^iZ$ with $i > 1$

(ii) when $M_j^2 \neq 0$, then $G_p \cong Z/2Z \oplus Z/2Z$.

B) (i) G_p is a cyclic group

(ii) When $M_j^4 \neq 0$, then

(a) $G_p \cong Z/2^iZ$, where $1 < i < 2$, if $2 \in M_j^2$

(b) $G_p \cong Z/2^iZ$, where $1 < i < 3$, if $2 \in M_j \setminus M_j^2$.

(β) Assume $p = 3$,

A) $G_p \cong Z/3Z \oplus Z/3Z$, $3 \in M_j \setminus M_j^2$ and $M_j^2 = 0$.

B) (i) G_p is a cyclic group

(ii) When $M_j^4 \neq 0$, then

(a) $G_p \cong Z/3Z$, if $3 \in M_j^2$

(b) $G_p \cong Z/3^iZ$, where $1 < i < 3$, if $3 \in M_j \setminus M_j^2$.

(γ) Assume $p > 3$,

(i) G_p is a cyclic group

(ii) If $M_j^4 \neq 0$, then $p \notin M_j^4$ and

(a) $G_p \cong Z/pZ$, if $p \in M_j^2$

(b) $G_p \cong Z/p^iZ$, where $1 < i < 3$, if $p \in M_j \setminus M_j^2$.

(III) *Assume (R_j, M_j) has the 2−generator property but is not a principal ideal ring and p a prime integer such that p divides Ord(G) and $p \in M_j$, then*

(α) Assume $p = 2$,

$G_p \cong Z/2^iZ$,

(1) $i \geq 1$ if M_j^2 is a principal ideal and $M_j^3 = 0$.

(2) $1 \leq i \leq 2$ if M_j^2 is a principal ideal, $M_j^3 \neq 0$, and $M^2 \subset (2)$.

(3) $i = 1$ otherwise.

(β) Assume $p = 3$,

(a) G_p is a cyclic group

(b_1) When M_j^2 is a principal ideal and $M_j^3 \neq 0$ then

(α_1) If $3 \in M_j^2$, then $G_p \cong Z/3Z$ and M_j^3 is a principal ideal.

(α_2) If $3 \in M_j \setminus M_j^2$, then $G_p \cong Z/3^iZ$ with $1 \leq i \leq 2$, moreover, if $9 \in M_j^3$ then $G_p \cong Z/3Z$.

(b_2) When M_j^2 is not a principal ideal, then $3 \notin M_j^2$, $G_p \cong Z/3Z$, moreover, if

$M_j^3 \neq 0$ and $M_j^2 \not\subset (3)$ then M_j^3 is a principal ideal and

(θ_1) If $9 \in M_j^2 \setminus M_j^3$ then $M_j^3 \subset (9)$.

(θ_2) If $9 \in M_j^3$ then $M_j^3 = 3M_j^2$.

(γ) Assume $p > 3$,

(a) G_p is a cyclic group

(b) (b_1) When M_j^2 is a principal ideal and $M_j^3 \neq 0$ then

(α_1) If $p \in M_j^2$, then $G_p \cong Z/pZ$, $p \notin M_j^3$, and M_j^3 is a principal ideal.

(α_2) If $p \in M_j \setminus M_j^2$ then $G_p \cong Z/p^iZ$ with $1 \leq i \leq 2$, moreover, if $p^2 \in M^3$,

then $G_p \cong Z/pZ$ and either $M_j^2 \subset (p)$ or $M_j^3 \subset (p)$

(b_2) When M_j^2 is not a principal ideal, then $p \notin M_j^2$, $G_p \cong Z/pZ$, moreover, if

$M_j^3 \neq 0$ and $M_j^2 \not\subset (p)$ then M_j^3 is a principal ideal and

(θ_1) If $p^2 \in M_j^2 \setminus M_j^3$ then $M_j^3 \subset (p^2)$.

(θ_2) If $p^2 \in M_j^3$ then $M_j^3 = pM_j^2$.

Proof. We appeal to [2, Theorem], Propositions 1, 2, and 3, and similar techniques used in the proof of [1, Theorem]. $\diamondsuit$

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