COMMUNICATIONS IN ALGEBRA, 26(8), 2559-2589 (1998)

ON THE PRIME SPECTRUM OF COMMUTATIVE SEMIGROUP RINGS

Souad Améziane Hassani
Département de Mathématiques
Faculté des Sciences Saiss
Université S.M. Ben Abdellah
Fès, MOROCCO

David E. Dobbs
Department of Mathematics
University of Tennessee
Knoxville, TN 37996-1300
U.S.A.

Salah-Eddine Kabbaj
Département de Mathématiques et Informatique
Faculté des Sciences Dhar Al-Mehraz
Université S.M. Ben Abdellah
Fès, MOROCCO

Abstract. Let A be an integral domain and S a torsion-free cancellative Abelian semigroup. By analogy with known results on polynomial rings and group rings, results are sought for a number of properties of the semigroup ring $A[S]$. The properties of interest include coequidimensionality, (universal) catenarity, (stably strong) S-domain, and (locally, residually, totally) Jaffard domain. Positive results, leading to new examples of rings with some of the above properties, are obtained in case (the quotient group of) S has rank 1 or S is finitely generated. An example shows that some results do not carry over in case S has rank 2 but is not finitely generated.

1. Introduction. Let A be a commutative ring (with 1) and S an Abelian semigroup. As usual, $A[S]$ denotes the semigroup ring associated to A and S; as a set, $A[S] = \{\sum a_s X^s | a_s \in A$ for each $s \in S$, $a_s = 0$ for all but finitely many $s\}$. Our purpose here is to study several properties related to the order-theoretic structure of Spec($A[S]$), the set of all prime ideals of $A[S]$. These properties include coequidimensionality (in the sense of [BDF1]), catenarity, universal catenarity [BDF1], S- and strong S-domains [Kap], stably strong S-domains [MM], first and second chain conditions [R2], and (locally, residually, totally) Jaffard domains ([ABDFK], [C]). As many of these properties are best studied when $A[S]$ is an integral domain, we often impose the corresponding restrictions on A and S (cf. [G3, Theorem 8.1]), namely that A is an integral domain and that S is torsion-free and cancellative.

One benefit of the above assumptions on S is that S then has an associated quotient group, denoted $\langle S\rangle$, into which S embeds (cf. [G3, p. 6]). It will be convenient to refer to the *rank* of S; by this, we mean the usual (torsion-free) rank of $\langle S\rangle$. If rank$(S) = r$ and $A[S]$ is an integral domain, a fundamental result that we often need is due to Arnold and Gilmer [AG]: $\dim(A[S]) = \dim(A[\langle S\rangle]) = \dim(A[X_1, \ldots, X_r])$. (We use dim to denote Krull dimension; $\dim_v$ denotes valuative dimension.) This result illustrates that it is appropriate to organize a study of the above properties for $A[S]$ by analogy with earlier work on the special cases of group rings and polynomial rings. (Of course, polynomial rings over A are examples of semigroup rings over A, since $A[\mathbb{N}^n] \cong A[X_1, \ldots, X_n]$ for each positive integer n. Here $\mathbb{N} := \{x \in \mathbb{Z} | x \geq 0\}$. Often, it is convenient to assume $0 \in S$ for the ambient semigroup, i.e., that S is a monoid, but interesting technicalities are possible, as in the proofs of Lemmas 2.4 and 2.8, without this hypothesis.)

A considerable literature has developed regarding the transfer between A and $A[X_1, \ldots, X_n]$ of properties such as catenarity, strong S-, $\cdots$, etc.: see [BDF1], [MM], [ABDFK]; we cite such work as needed. The only corresponding work regarding transfer between A and group rings over A is the recent preprint [ACKZ]. The present work depends heavily on [ACKZ], both for its organization and for specific facts, such as those recalled in Remark 2.6(c).

The interplay between $A[S]$, $A[\langle S\rangle]$, and $A[X_1, \ldots, X_r]$ leads to a number of positive results analogous to the work on group rings $A[G]$ in [ACKZ]. While most of [ACKZ] is deveoped for groups G of arbitrary finite rank, our positive results on semigroup rings $A[S]$ are mostly in case rank$(S) = 1$ (in Section 2) or S is finitely generated (in Section 3). The work in Section 2 is developed by means of Proposition 2.2(b) and the above-cited lemmas, which strongly use the hypothesis that rank$(S) = 1$. As explained in Remark 2.3(b), the conclusion of Proposition 2.2(b) was obtained (in a different formulation) by Arnold and Gilmer [AG, Theorem 4.1] for finitely generated S. Thus, it is not surprising that Section 3 relies heavily on results from [AG].

The results of Sections 2 and 3 permit, in Section 4, the development of new examples of universally catenarian, stably strong S-, locally Jaffard, and residually Jaffard domains. Perhaps more importantly, Example 4.5(a) shows that the "rank$(S) = 1$" or "S is finitely generated" hypotheses cannot be deleted from some of the results in Sections 2 and 3; and Example 4.5(b) shows that a key result on group rings from [ACKZ] does not extend to semigroup rings $A[S]$ with rank$(S) = 2$. We conclude that further positive work on $A[S]$ with rank$(S) = 2$ will require fundamentally new insights.

Any unexplained material is standard, as in [G2] or [Kap]. For instance, following [Kap, p. 28] we use LO, GU, INC, and GD to denote the lying-over, going-up, incomparable, and going-down properties, respectively; and $\subset$ denotes proper inclusion.

2. Results for semigroups of rank 1. The results on semigroup rings $A[S]$ in this section are analogues of the transfer results on group rings in [ACKZ] for the case rank$(S) = 1$. We begin with a result which is valid for any finite rank and which generalizes a fundamental result in the dimension theory of polynomial rings (cf. [Kap, Theorem 37]).

Lemma 2.1. *Let A be a commutative ring, n a nonnegative integer, and S a torsion-free cancellative Abelian semigroup such that rank$(S) = n$. Then there does not exist a chain $Q_1 \subset Q_2 \subset \ldots \subset Q_{n+2}$ of $n+2$ distinct prime ideals of $A[S]$ such that $Q_1 \cap A = Q_{n+2} \cap A$.*

Proof. Deny. Put $q := Q_1 \cap A$ and $T := A \backslash q$. As $A[S]_T = A_q[S]$, we may replace A with A_q, and thus assume that A is quasi-local, with maximal ideal q. Since

$$qA[S] \subseteq Q_1 \subset Q_2 \subset \ldots \subset Q_{n+2},$$

$ht(Q_{n+2}/qA[S]) \geq n+1$, and so $(A/q)[S] \cong A[S]/qA[S]$ has (Krull) dimension at least $n+1$. Now, let G be the quotient group of S; that is, $G = \langle S \rangle$, using the notation introduced above. According to the principal result of [AG], $\dim((A/q)[S]) = \dim((A/q)[G])$; and if $\{X_\lambda\}$ is a set of algebraically independent indeterminates whose cardinality is rank(G), then [G1, Corollary 1] yields that $\dim((A/q)[G]) = \dim((A/q)[\{X_\lambda\}])$. Since A/q is a field, $\dim((A/q)[\{X_\lambda\}]) = \text{rank}(G) = \text{rank}(S) = n$. The upshot is that $\dim((A/q)[S]) = n$, the desired contradiction. □

Proposition 2.2. *Let A be a commutative ring and S a torsion-free cancellative Abelian semigroup of rank 1. Then:*

(a) *$ht(p) \leq ht(pA[S]) \leq 2 \cdot ht(p)$ for each $p \in Spec(A)$.*

(b) *If $P \in Spec(A[S])$ and $p := P \cap A$ satisfies $pA[S] \neq P$, then $ht(P) = ht(pA[S]) + 1$.*

Proof. Let $G := \langle S \rangle$, the quotient group of S.

(a) For each $p \in \text{Spec}(A)$, $pA[S] \in \text{Spec}(A[S])$ since the hypotheses on S ensure that $A[S]/pA[S] \cong (A/p)[S]$ is an integral domain [G3, Theorem 8.1]. Since S is torsion-free, so is G [G3, pp. 6-7]. According to [ACKZ, Proposition 2.1(a)], it now follows that $ht(p) \leq ht(pA[G]) \leq 2ht(p)$. Thus, it suffices to show that $ht(pA[S]) = ht(pA[G])$. For this, recall (cf. [AG, p. 300]) that $A[G] = A[S]_T$, where T is the multiplicative subset $T = \{X^s | s \in S\}$ of $A[S]$; then, since $pA[S] \cap T = \emptyset$, we have $ht(pA[S]) = ht((pA[S])A[S]_T) = ht(pA[S]A[G]) = ht(pA[G])$, as desired.

(b) Let $h := ht(p)$. As $ht(pA[S]) \geq h$, we may suppose that $h < \infty$. The proof proceeds by induction on h.

For the induction basis, $h = 0$; that is, p is a minimal prime of A. Since $A[S]_{A \setminus p} \cong A_p[S]$, we may replace A with A_p and, thus, assume that A is quasi-local, with maximal ideal p. Also, as p is minimal and $A[S]/pA[S] \cong (A/p)[S]$, we may replace A with A/p and, thus, assume that A is a field. As $0 = pA[S] \subsetneqq P$, it is enough to show that $ht(P) \leq 1$ and, for this, it suffices to prove that $\dim(A[S]) = 1$. Appealing to [AG] and [G1] as in the proof of Lemma 2.1, we have $\dim(A[S]) = \dim(A[G]) = \dim(A[X]) = 1$, the last two equalities holding since $\mathrm{rank}(G) = \mathrm{rank}(S) = 1$ and A is a field.

For the induction step, $h \geq 1$, and we suppose the assertion for all $Q \in \mathrm{Spec}(A[S])$ with $ht(Q \cap A) \leq h - 1$. Let $m := ht(P)$. As $h < \infty$, it follows from Lemma 2.1 that $m < \infty$. Choose a saturated chain of prime ideals of $A[S]$:

$$Q_0 \subset Q_1 \subset \ldots \subset Q_m = P.$$

The "saturated" condition ensures that $ht(Q_{m-1}) = m - 1$. Put $q := Q_{m-1} \cap A$. Evidently, $q \subseteq P \cap A = p$. We next consider two cases.

If $q = p$, then $pA[S] = qA[S] \subseteq Q_{m-1} \subsetneqq P$, so that Lemma 2.1 gives $Q_{m-1} = pA[S]$, whence $ht(P) = m = ht(Q_{m-1}) + 1 = ht(pA[S]) + 1$, as desired. In the remaining case, $q \subsetneqq p$. It suffices to prove that $ht(P) \leq ht(pA[S]) + 1$, as the reverse inequality is clear. Since $q \subset p$, we have that $ht(q) \leq ht(p) - 1 = h - 1$ and $ht(qA[S]) \leq ht(pA[S]) - 1$. Then, by the induction hypothesis, $ht(Q_{m-1}) \leq ht(qA[S]) + 1$, and so $ht(P) = m = ht(Q_{m-1}) + 1 \leq ht(qA[S]) + 2 \leq ht(pA[S]) + 1$. $\square$

Remark 2.3. (a) Let $n \geq 2$ be a positive integer. Then the analogue of Proposition 2.2(b) is false for torsion-free cancellative Abelian semigroups of rank n. To see this, let $S := \mathbb{N}^n = \mathbb{N} \oplus \cdots \oplus \mathbb{N}$ (n summands), let A be a one-dimensional Noetherian integral domain, and let p be a nonzero prime ideal of A. Then there exist $P, Q \in \operatorname{Spec}(A[S])$ such that $P \cap A = p = Q \cap A$, $pA[S] \subsetneqq P$, and $ht(P) < ht(Q) = ht(pA[S]) + n$.

For a proof, observe that $A[S] = A[X_1, \ldots, X_n]$. Since A is Noetherian, it follows from [Kap, Theorem 149] and the Hilbert Basis Theorem that $ht(pA[S]) = ht(pA[X_1, \ldots, X_n]) = ht(pA[X_1, \ldots, X_{n-1}]) = \cdots = ht(p) = 1$. Put $P := (p, X_1)A[X_2, \ldots, X_n]$. Since $X_1 \in P \backslash pA[S]$, we have $pA[S] \subsetneqq P$ (and $P \cap A[S] = p$). Since A is Noetherian, it follows from [Kap, Theorem 149] that $ht(p, X_1) = ht(p)+1 = 2$. Thus, again invoking [Kap, Theorem 149] and the Noetherianness of A, we have $ht(P) = ht((p, X_1)A[X_2, \ldots, X_{n-1}]) = \cdots = ht(p, X_1) = 2 < 1 + n = ht(pA[S]) + n$. Finally, to find a suitable Q, observe via [Kap, Theorem 149] that since A_p is one-dimensional Noetherian, $A[S]_{A\backslash p} = A_p[S] = A_p[X_1, \ldots, X_n]$ has dimension $1 + n$; it suffices to choose $W \in \operatorname{Spec}(A[S]_{A\backslash p})$ with $ht(W) = 1 + n$ and put $Q := f^{-1}(W)$, where f denotes the canonical ring-homomorphism $A[S] \to A[S]_{A\backslash p}$.

(b) Let A be a commutative ring, S a torsion-free cancellative Abelian semigroup, $P \in \operatorname{Spec}(A[S])$, and $p := P \cap A$. A reformulation of Proposition 2.2(b) states that if $\operatorname{rank}(S) = 1$, then $ht(P) = ht(pA[S]) + ht(P/pA[S])$. According to [AG, Theorem 4.1], the same conclusion holds if S is finitely generated, but not in general [AG, pp. 311-312]. (In the counter-example of Arnold and Gilmer, one begins with rationally independent positive real numbers $\lambda_1, \ldots, \lambda_n$, for some $n > 1$, and then lets S denote the additive semigroup of all nonnegative real numbers of the form $i_1\lambda_1 + \cdots + i_n\lambda_n$, with each $i_j \in \mathbb{Z}$. Of course, $\operatorname{rank}(S) \neq 1$ and S is not finitely generated.) Thus, our emphasis in Section 2 may be viewed as our pursuit of consequences of a property of polynomial rings which, although not valid when generalized to the context of finitely generated semigroups, does remain valid in the rank 1 context.

Adapting terminolgy introduced in [BDF1] for integral domains, we say that a commutative ring is *coequidimensional* in case all its maximal ideals have the same height. Theorem 2.5 characterizes the coequidimensional semigroup rings in case $\mathrm{rank}(S) = 1$. First, we give a preparatory result. Its proof is included for lack of a convenient reference.

Lemma 2.4. *Let A be a commutative ring and S a torsion-free cancellative Abelian semigroup of rank 1. If there exists $t \in S\backslash\{0\}$ such that $-t \in S$, then S is a group. If no such t exists, then there exists a subsemigroup T of S such that $T \cong \mathbb{N}\backslash\{0\}$ and $A[S]$ is integral over $A[T]$.*

Proof. Let $G := \langle S \rangle$. Suppose first that $-t \in S$ for some $t \in S\backslash\{0\}$. It is enough to show that if $x \in S$, then $-x \in S$. Now, $\mathrm{rank}(G/\mathbb{Z}t) = \mathrm{rank}(G) - \mathrm{rank}(\mathbb{Z}t) = \mathrm{rank}(S) - \mathrm{rank}(\mathbb{Z}) = 1 - 1 = 0$, where we have used the torsion-free condition on S (equivalently on G [G3, pp. 6-7]) to conclude that $\mathbb{Z}t \cong \mathbb{Z}$ has rank 1. Hence, $G/\mathbb{Z}t$ is a torsion group. Thus, given $x \in S$, there exist $n \in \mathbb{N}\backslash\{0\}$ and $m \in \mathbb{Z}$ such that $nx = mt$. As $x \neq 0$ without loss of generality, the torsion-free hypothesis ensures that $m \neq 0$. If $n = 1$, then $x = mt$ and so $-x = (-m)t = m(-t) \in (\mathbb{N}\backslash\{0\})t \cup (\mathbb{N}\backslash\{0\})(-t) \subseteq S$. If $n > 1$, then $-x = (n-1)x - nx = (n-1)x + (-m)t \in S + S \subseteq S$.

Next, suppose that $-t \notin S$ for each $t \in S\backslash\{0\}$. Choose $y \in S\backslash\{0\}$ and put $T := (\mathbb{N}\backslash\{0\})y$. Since S is torsion-free, $T \cong \mathbb{N}\backslash\{0\}$. Arguing as above, we see that $G/\mathbb{Z}y$ is a torsion group. It follows that $A[S]$ is integral over $A[\mathbb{Z}y \cap S]$ (cf. [G3, p. 151]). Thus, it suffices to show that $\mathbb{Z}y \cap S = T$. Of course, $\mathbb{Z}y \cap S \supseteq T$. For the reverse inclusion, consider $z = ky \in S$, with $k \in \mathbb{Z}$. As S is torsion-free, $z \neq 0$ and so $k \neq 0$. If $k < 0$, then $t := z \in S\backslash\{0\}$ satisfies $-t = (-k)y \in T \subseteq S$, contrary to hypothesis. Therefore, $k > 0$, whence $z \in (\mathbb{N}\backslash\{0\})y = T$, as desired. $\square$

Theorem 2.5. *Let A be a commutative ring and S a torsion-free cancellative Abelian semigroup of rank 1. Then $A[S]$ is coequidimensional if and only if $A[X]$ is coequidimensional.*

Proof. [ACKZ, Corollary 2.6] dispatches the case in which S is a group. Thus, without loss of generality, S is not a group; by Lemma 2.4, $A[S]$ is integral over $A[T]$ for some subsemigroup T of S such that $T \cong \mathbb{N}\backslash\{0\}$. Observe that $A[T] \cong A[X]$; it will be convenient to identify $A[T]$ with $A[X]$. As usual, put $G := \langle S \rangle$.

Suppose, first, that $A[S]$ is coequidimensional, with $e := \dim(A[S])$. As in the proof of Lemma 2.1, it follows from [AG] and [G1] that $e = \dim(A[G]) = \dim(A[X])$, the last equality holding since $\operatorname{rank}(G) = 1$. It suffices to show that if M is a maximal ideal of $A[X]$, then $ht(M) = e$. Let $h := ht(M)$. If $h = \infty$, then it follows from LO and GU that for each positive integer n, there exists a prime ideal N of $A[S]$ such that $N \cap A[X] = M$ and $ht(N) \geq n$. By integrality, N is a maximal ideal and so, since $A[S]$ is coequidimensional, $e = ht(N)$. As n was arbitrary, $e = \infty$; that is, $ht(M) = e$ if $h = \infty$. Thus, without loss of generality, $h < \infty$. Then, by [Kap, Theorem 46], it follows from GU and INC that $ht(W) = h$ for some $W \in \operatorname{Spec}(A[S])$ such that $W \cap A[X] = M$. As above, W is a maximal ideal and $e = ht(W)$. It follows that $ht(M) = e$ if $h < \infty$, thus completing the proof of the "only if" assertion.

For the converse, suppose that $A[X]$ is coequidimensional, with $d := \dim(A[X])$. It suffices to show that if N is a maximal ideal of $A[S]$, then $ht(N) = d$. By integrality, $M := N \cap A[X]$ is a maximal ideal of $A[X]$, whence $ht(M) = d$, since $A[X]$ is coequidimensional. Put $p := N \cap A$. As $pA[S]$ is not a maximal ideal, $pA[S] \subsetneqq N$, and so Proposition 2.2(b) yields that $ht(N) = ht(pA[S]) + 1$. Similarly, $ht(M) = ht(pA[X]) + 1$. As in the proof of Proposition 2.2(a), we have $ht(pA[S]) = ht(pA[G])$. Thus, it suffices to prove that $ht(pA[G]) = ht(pA[X])$. Now, by the general reasoning preceding the statement of [ACKZ, Proposition 2.1], there exists an indeterminate Y such that $B := A[Y] \subseteq A[G]$ is an integral extension, $A[G]$ is a free B-module, $pA[G] \cap B = pB$, and $ht(pA[G]) = ht(pB)$. As $ht(pB) = ht(pA[X])$, the proof is complete. □

Remark 2.6. (a) Let n be a positive integer and, as usual, let $\mathbb{Q}^n := \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$ (n summands) and $\mathbb{Q}^+ := \{x \in \mathbb{Q} | x \geq 0\}$. Notice, as an application of Theorem 2.5, that the semigroup ring $\mathbb{Z}[\mathbb{Q}^+]$ is coequidimensional; indeed, $\text{rank}(\mathbb{Q}^+) = 1$ and it is known that $\mathbb{Z}[X]$ is coequidimensional. As another application of Theorem 2.5, we claim that if K is a field, then the semigroup ring $A := K[\mathbb{Q}^n \oplus \mathbb{Q}^+]$ is coequidimensional. Indeed, since $A \cong K[\mathbb{Q}^n][\mathbb{Q}^+]$ (cf. [G3, Theorem 7.1]) and $\text{rank}(\mathbb{Q}^+) = 1$, Theorem 2.5 reduces the assertion to proving that $B := K[\mathbb{Q}^n][X]$ is coequidimensional. As $B \cong K[X][\mathbb{Q}^n]$ and $\mathbb{Q}^n$ is torsion-free and of finite rank, [ACKZ, Corollary 2.6] reduces the assertion to the well-known fact that $K[X][X_1, \dots, X_n] \cong K[Y_1, \dots, Y_{n+1}]$ is coequidimensional, thus proving the claim.

(b) Example 4.5 will show that one cannot delete the hypothesis that $\text{rank}(S) = 1$ in Theorem 2.5.

(c) The final step in the proof of Theorem 2.5 appealed to the general reasoning preceding the statement of [ACKZ, Proposition 2.1]. For the ease of later references, it is convenient to restate that material for arbitrary finite rank n, as follows. If A is a commutative ring and G is a torsion-free Abelian group of rank n, then there exist indeterminates $Y_1, \dots, Y_n$ such that $B := A[Y_1, \dots, Y_n] \subseteq A[G]$ is an integral ring extension and $A[G]$ is a free B-module. In particular, the extension $B \subseteq A[G]$ satisfies LO, GU, INC, and GD; $PA[G] \cap B = P$ for each $P \in \text{Spec}(B)$; and $ht(Q) = ht(Q \cap B)$ for each $Q \in \text{Spec}(A[G])$.

Recall from [Kap, p. 26] that an integral domain A is called an S-domain in case $ht(PA[X]) = 1$ for all $P \in \text{Spec}(A)$ such that $ht(P) = 1$. It was shown in [FK, Proposition 2.1] that $A[X]$ is an S-domain for any integral domain A. This result is generalized in Theorem 2.7.

Theorem 2.7. *If A is an integral domain and S is a torsion-free cancellative Abelian semigroup of rank 1, then $A[S]$ is an S-domain.*

Proof. Let $G := \langle S \rangle$. By [ACKZ, Theorem 3.1], we may assume that S is not a group; that is, $G \neq S$. Hence, by Lemma 2.4, there exists an indeterminate X such that the ring extension $A[X] \subseteq A[S]$ is integral. (More precisely, $A[T] \subseteq A[S]$ is integral for some subsemigroup T of S such that $T \cong \mathbb{N}\backslash\{0\}$; it is convenient here, and in similar arguments later, to identify $A[T] = A[X]$.) The assertion follows if A is integrally closed, by an application of [MM, Corollary 4.10], as we have recalled that $A[X]$ is an S-domain and $A[X]$ inherits the property of being integrally closed from A.

For the general case, consider $P \in \mathrm{Spec}(A[S])$ such that $ht(P) = 1$; our task is to show that $ht(PA[S][Z]) = 1$, where Z is an indeterminate over $A[S]$. Put $p := P \cap A$. As in the proof of Proposition 2.2(a), $pA[S] \in \mathrm{Spec}(A[S])$. We next consider two subcases.

Suppose, first, that $pA[S] = P$. As in the proof of Proposition 2.2(a), $ht(pA[S]) = ht(pA[G])$; and by the reasoning recalled in Remark 2.6(c), $ht(pA[G]) = ht(pA[Y])$. Thus, $ht(pA[Y]) = ht(P) = 1$. Since $A[Y]$ is an S-domain, it now follows that $ht(pA[Y]A[Z]) = 1$. Viewing $A[G] = A[S]_T$ as in the proof of Proposition 2.2(a), we have $pA[G][Z] = pA[S][Z]_T$, whence

$$ht(PA[S][Z]) = ht(pA[S][Z]) = ht(pA[S][Z]_T) = ht(pA[G][Z]).$$

Now, recall from Remark 2.6(c) that $A[Y] \subseteq A[G]$ is an integral ring extension making $A[G]$ free as an $A[Y]$-module. These conditions are inherited by the extension $A[Y][Z] \subseteq A[G][Z]$. Since $pA[G][Z] \cap A[Y][Z] = pA[Y][Z]$ by faithful flatness, it follows via GD and INC that $ht(pA[G][Z]) = ht(pA[Y][Z])$. Therefore,

$$ht(PA[S][Z]) = ht(pA[Y][Z]) = ht(pA[Y]A[Z]) = 1,$$

completing the proof of the first subcase.

Suppose, finally, that $pA[S] \neq P$. By Proposition 2.2(b), $ht(P) = ht(pA[S])+1$. As $ht(P) = 1$, we have $ht(pA[S]) = 0$, whence $p = 0$ (since $A[S]$ is an integral domain). Consider $K := A_{A\setminus\{0\}}$, the quotient field of A. Since K is integrally closed, $K[S]$ is an S-domain, by the first case treated above. As $P \cap A = p = 0$, we have $PA[S][Z] \cap A = 0$, whence

$$\begin{aligned} ht(PA[S][Z]) &= ht(PA[S][Z]_{A\setminus\{0\}}) = ht(PA[S]_{A\setminus\{0\}}A[S][Z]_{A\setminus\{0\}}) \\ &= ht(PK[S]K[S][Z]) = 1, \end{aligned}$$

the last equality holding since $ht(PK[S]) = ht(P) = 1$ and $K[S]$ is an S-domain. The proof is complete. $\square$

Recall from [Kap, p. 26] that an integral domain A is called a strong S-domain in case A/P is an S-domain for each $P \in \operatorname{Spec}(A)$; equivalently, in case $ht(P_2A[X]/P_1A[X]) = 1$ for all prime ideals $P_1 \subset P_2$ of A such that $ht(P_2/P_1) = 1$. Before studying semigroup rings which are strong S-domains, we need the following result.

Lemma 2.8. *Let A be a commutative ring, S a torsion-free cancellative Abelian semigroup of rank 1, and $P \in Spec(A[S])$. Let $T := \{X^s | s \in S\}$, $T' := T\setminus\{X^0\}$, and $p := P \cap A$. If $P \cap T \neq \emptyset$, then $P = (p, \{X^s | s \in S\setminus\{0\}\}) = (p, T')$.*

Proof. Choose $X^t \in P \cap T$. If $-t \in S$, then $1 = X^tX^{-t} \in PA[S] = P$, a contradiction. Thus, $-t \notin S$, and so S is not a group. Put $G := \langle S \rangle$. As in the proof of Lemma 2.4, $\operatorname{rank}(G/\mathbb{Z}t) = 0$; that is, $G/\mathbb{Z}t$ is a torsion group. Thus, given $s \in S\setminus\{0\}$, there exist $n \in \mathbb{N}\setminus\{0\}$ and $m \in \mathbb{Z}$ such that $ns = mt$. If $-m \in \mathbb{N}\setminus\{0\}$, then $-ns = (-m)t \in S\setminus\{0\}$ and $ns \in S\setminus\{0\}$, a contradiction to Lemma 2.4, since S is not a group. Thus, $m \in \mathbb{N}\setminus\{0\}$. Hence

$(X^s)^n = X^{ns} = X^{mt} = (X^t)^m \in P$ and, since P is a prime ideal, $X^s \in P$. We thus have a chain of prime ideals $pA[S] \subseteq (p, \{X^s | s \in S\backslash\{0\}) \subseteq (p, T') \subseteq P$ in $A[S]$. (Of course, (p, T') is prime since $A[S]/(p, T') \cong A/p$ is an integral domain.) Now, $X^t \in (p, T')\backslash pA[S]$ since $1 \notin p$, and so $pA[S] \subsetneqq (p, T') \subseteq P$. Lemma 2.1 ensures that $ht(P/pA[S]) = 1$, whence $P = (p, T')$. ◻

Theorem 2.9. *Let A be an integral domain and S a torsion-free cancellative Abelian semigroup of rank 1. Put $G = \langle S \rangle$. Then $A[S]$ is a strong S-domain if and only if $A[G]$ is a strong S-domain.*

Proof. Put $T := \{X^s | s \in S\}$. Since $A[G] = A[S]_T$, it follows from [MM, Corollary 2.4] that if $A[S]$ is a strong S-domain, then $A[G]$ is a strong S-domain.

Conversely, suppose that $A[G]$ is a strong S-domain, and consider prime ideals $Q_1 \subset Q_2$ of $A[S]$ such that $ht(Q_2/Q_1) = 1$; our task is to show that $ht(Q_2A[S][Y]/Q_1A[S][Y]) = 1$. If $Q_2 \cap T = \emptyset$, the primes $Q_1A[S]_T \subset Q_2A[S]_T$ of $A[G]$ satisfy $ht(Q_2A[S]_T/Q_1A[S]_T) = 1$ and so, since $A[G]$ is a strong S-domain, $ht(Q_2A[S]_T[Y]/Q_1A[S]_T[Y]) = 1$; i.e., $ht(Q_2A[S][Y]_T/Q_1A[S][Y]_T) = 1$ and the assertion follows. Thus, without loss of generality, $Q_2 \cap T \neq \emptyset$.

By the proof of Lemma 2.8, S is not a group and so, by Lemma 2.4, there exists an indeterminate X such that $A[X] \subseteq A[S]$ is an integral ring extension; that proof shows that we may choose $X = X^s$ for some $s \in S\backslash\{0\}$. Put $p_2 := Q_2 \cap A$. We see via Lemma 2.8 that $Q_2 = (p_2, \{X^s | s \in S\backslash\{0\}\})$. Put $P_i = Q_i \cap A[X]$ for $i = 1, 2$. We claim that $ht(P_2/P_1) = 1$.

Of course, $P_1 \neq P_2$ by INC. If the claim fails, consider a chain of prime ideals $P_1 \subsetneqq P \subsetneqq P_2$ in $A[X]$. Now, $X \in Q_2 \cap A[X] = P_2$. It follows that $P_2 = (p_2, X)$ by [Kap, Theorem 37] (or Lemma 2.1). Next, since the extension $A[X] \subseteq A[S]$ satisfies GU (and LO), there exist prime ideals $Q_1 \subset Q \subset Q_2'$ of $A[S]$ such that $Q_1 \cap A[X] = P_1$, $Q \cap A[X] = P$, and $Q_2' \cap A[X] = P_2$.

Notice that $Q_2' \cap T \neq \emptyset$. (Indeed, Lemma 2.4 allows us to pick $X = X^t$ for some $t \in S \backslash \{0\}$ at the outset, so that $X \in Q_2' \cap T$.) Hence, it follows from Lemma 2.8 that $Q_2' = (p_2, \{X^s | s \in S \backslash \{0\}\})$, and so $Q_2' = Q_2$. Thus $1 = ht(Q_2/Q_1) = ht(Q_2'/Q_1) \geq 2$: this contradiction establishes the claim that $ht(P_2/P_1) = 1$.

Since $A[G]$ is a strong S-domain, [ACKZ, Proposition 3.2(a)] gives that $A[X]$ is a strong S-domain. Hence, $ht(P_2A[X][Y]/P_1A[X][Y]) = 1$. Since the ring extension $A[X][Y] \subseteq A[S][Y]$ is integral and $Q_iA[S][Y] \cap A[X][Y] = P_iA[X][Y]$, it follows via INC that $ht(Q_2A[S][Y]/Q_1A[S][Y]) = 1$ (cf. [MM, Lemma 4.1]). $\square$

As in [MM], we say that an integral domain A is a stably strong S-domain in case $A[X_1, \ldots, X_n]$ is a strong S-domain for each positive integer n. We next study some semigroup rings which are stably strong S-domains.

Corollary 2.10. *Let A be an integral domain, S a torsion-free cancellative Abelian semigroup of rank 1, and $G := \langle S \rangle$. Then $A[S]$ is a stably strong S-domain if and only if $A[G]$ is a stably strong S-domain.*

Proof. Since $A[U][X_1, \ldots, X_n] = A[X_1, \ldots, X_n][U]$ for any semigroup U, an application of Theorem 2.9 competes the proof. $\square$

Corollary 2.11. *Let A be an integral domain and S a torsion-free cancellative Abelian semigroup of rank 1. Then:*

(a) *Suppose that $dim(A) \leq 1$. Then $A[S]$ is a strong S-domain if and only if $A[X]$ is a strong S-domain.*

(b) *If A is Noetherian, then $A[S]$ is a stably strong S-domain.*

Proof. (b) Let $G := \langle S \rangle$. By [ACKZ, Proposition 3.4(b)], $A[G]$ is a stably strong S-domain. Hence, without loss of generality, $S \neq G$; that is, S is not a group. Thus, by Lemma 2.4, there exists an indeterminate X such that the ring extension $A[X] \subseteq A[S]$ is integral. By Hilbert Basis Theorem, $A[X]$ is Noetherian, and so the assertion now follows from [MM, Proposition 4.20].

(a) Any stably strong S-domain is a strong S-domain. (Indeed, any factor domain of a strong S-domain is a strong S-domain: cf. [MM, p. 261].) Thus, if $\dim(A) = 0$ (that is, if A is a field), the assertion in (a) follows from (b). Hence, without loss of generality, $\dim(A) = 1$. Then, with $G := \langle S \rangle$, [ACKZ, Proposition 3.4(a)] yields that $A[X]$ is a strong S-domain if and only if $A[G]$ is a strong S-domain. An application of Theorem 2.9 completes the proof. □

As in [BDF1], a commutative ring A is called locally finite-dimensional (in short, LFD) if $ht(P) < \infty$ for each $P \in \operatorname{Spec}(A)$.

Remark 2.12. It is convenient next to record the following result. If A is an LFD commutative ring and S is a torsion-free cancellative Abelian semigroup of finite rank n, then $A[S]$ is LFD. For a proof, consider $P \in \operatorname{Spec}(A[S])$ and put $p := P \cap A$. To show that $ht(P) < \infty$, it is enough to consider the height of $PA[S]_{A \setminus p} = PA_p[S]$, and so we may suppose that A is quasi-local, with unique maximal ideal p. Hence, $\dim(A) = ht(p) < \infty$. With $G := \langle S \rangle$, we may appeal to [AG] and [G1, Corollary 1], as in the proof of Lemma 2.1, to show that $ht(P) \leq \dim(A[S]) = \dim(A[G]) = \dim(A[X_1, \ldots, X_n]) < \infty$, as asserted. □

Using terminology from [BDF1], we say that an integral domain A is catenarian if, for each pair $P \subsetneq Q$ of prime ideals of A, all saturated chains of primes from P to Q have a common finite length. Evidently, if A is a catenarian integral domain, then A is LFD.

Theorem 2.13. *Let A be an integral domain, S a torsion-free cancellative Abelian semigroup of rank 1, and $G := \langle S \rangle$. Then $A[S]$ is catenarian if and only if $A[G]$ is catenarian.*

Proof. As the catenarian property is preserved by localization, the "only if" assertion is immediate.

Conversely, suppose that $A[G]$ is catenarian. Since $A[G]$ is then LFD by the above comment, it follows easily that A is LFD and so, by Remark 2.12, $A[S]$ is LFD. Thus, it suffices to show that if $Q_1 \subset Q_2$ are prime ideals of $A[S]$ such that $ht(Q_2/Q_1) = 1$, then $ht(Q_2) = ht(Q_1) + 1$. As usual, put $T := \{X^s | s \in S\}$. If $Q_2 \cap T = \emptyset$, view matters in $A[S]_T = A[G]$, where $ht(Q_2A[G]/Q_1A[G]) = ht(Q_2/Q_1) = 1$; as $A[G]$ is catenarian, $ht(Q_2A[G]) = ht(Q_1A[G]) + 1$, that is, $ht(Q_2) = ht(Q_1) + 1$, since $ht(Q_iA[G]) = ht(Q_i)$. Thus, we may suppose that $Q_2 \cap T \neq \emptyset$. We see, as in the proof of Lemma 2.8, that S is not a group and $Q_2 = (p_2, \{X^s | s \in S \setminus \{0\}\})$, where $p_2 := Q_2 \cap A$. By Lemma 2.4, there exists an indeterminate X such that $A[X] \subseteq A[S]$ is an integral ring extension. Put $P_i := Q_i \cap A[X]$. As in the proof of Theorem 2.9, $ht(P_2/P_1) = 1$ and so, since [ACKZ, Proposition 3.2(b)] ensures that $A[X]$ inherits catenarity from $A[G]$, we have $ht(P_2) = ht(P_1) + 1$. By adjusting X as in the proof of Theorem 2.9, we have $P_2 = (p_2, X)$ and so $ht(P_2) = ht(p_2A[X]) + 1$. Moreover, since $p_2A[S] \neq Q_2$, Proposition 2.2(b) yields that $ht(Q_2) = ht(p_2A[S]) + 1$. Put $p_1 := Q_1 \cap A$. Observe that $p_1A[S] \subseteq Q_1 \subset Q_2$. We next consider two cases.

Suppose, first, that $p_1A[S] = Q_1$. As $p_1 \subseteq p_2$, we have $Q_1 \subseteq p_2A[S] \subsetneqq Q_2$ and so, since $ht(Q_2/Q_1) = 1$, it follows that $Q_1 = p_2A[S]$. Then $ht(Q_2) = ht(p_2A[S]) + 1 = ht(Q_1) + 1$, as desired.

In the remaining case, $p_1A[S] \neq Q_1$. Therefore, by Proposition 2.2(b), $ht(Q_1) = ht(p_1A[S]) + 1$. Thus, it suffices to show that $ht(p_2A[S]) = ht(p_1A[S]) + 1$. It is helpful to note that if $p \in \mathrm{Spec}(A)$, then $ht(pA[S]) = ht(pA[S]_T) = ht(pA[G]) = ht(pA[X])$, the final equality holding by Remark 2.6(c). Hence

$$\begin{aligned} ht(p_2A[S]) &= ht(p_2A[X]) = ht(P_2) - 1 = ht(P_1) \\ &= ht(p_1A[X]) + 1 = ht(p_1A[S]) + 1, \end{aligned}$$

to complete the proof. □

As in [BDF1], we say that an integral domain A is universally catenarian if $A[X_1, \dots, X_n]$ is catenarian for each positive integer n. Important examples of universally catenarian domains include arbitrary one-dimensional Noetherian integral domains (cf. [R1, (2.6)], [BDF1, Corollary 6.4]) and LFD Prüfer domains (cf. [BDF1, Theorem 6.2]).

Corollary 2.14. *Let A be an integral domain, S a torsion-free cancellative Abelian semigroup of rank 1, and $G := \langle S \rangle$. Then $A[S]$ is universally catenarian if and only if $A[G]$ is universally catenarian.*

Proof. As in the proof of Corollary 2.10, we need only note $A[U][X_1, \dots, X_n] = A[X_1, \dots, X_n][U]$ and apply the preceding theorem. □

Corollary 2.15. *For a Noetherian integral domain A, the following conditions are equivalent:*

(1) *$A[S]$ is universally catenarian for each torsion-free cancellative Abelian semigroup S of rank 1;*
(2) *$A[G]$ is universally catenarian for each torsion-free Abelian group G;*
(3) *$A[S]$ is universally catenarian for some torsion-free cancellative Abelian semigroup S of rank 1;*
(4) *$A[G]$ is universally catenarian for some torsion-free Abelian group G of rank 1;*
(5) *A is universally catenarian.*

Proof. It is trivial that (1) $\Rightarrow$ (3). Moreover, Corollary 2.14 yields that (3) $\Leftrightarrow$ (4) and that (2) $\Rightarrow$ (1). Since (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) by [ACKZ, Theorem 3.3], the proof is complete. □

Recall that a commutative ring A of (Krull) dimension d is said to satisfy the first chain condition (in short, f.c.c.) in case each maximal chain of prime ideals in A has length d. If A is a finite-dimensional integral domain, then A satisfies the f.c.c. if and only if A is coequidimensional and catenarian (cf. [R2, Remark (1.2.2), (1.2.3)]).

Corollary 2.16. *Let A be a finite-dimensional integral domain, S a torsion-free cancellative Abelian semigroup of rank 1, and $G := \langle S \rangle$. Then:*

(a) *$A[S]$ satisfies the f.c.c. if and only if $A[G]$ satisfies the f.c.c.*

(b) *Suppose, in addition, that A is Noetherian and universally catenarian. Then the following conditions are equivalent:*

(1) $A[S]$ satisfies the f.c.c.;

(2) $A[G]$ satisfies the f.c.c.;

(3) $A[X]$ satisfies the f.c.c.

Proof. For (a), the above remark allows us to consider transfer of the "co-equidimensional" and "catenarian" properties. The former is handled by Theorem 2.5 and [ACKZ, Corollary 2.6]; the latter, by Theorem 2.13. Then (b) follows from (a) and [ACKZ, Corollary 3.7]. □

Remark 2.17. Let K be a field and, as usual, $\mathbb{Q}^+ := \{x \in \mathbb{Q} | x \geq 0\}$. As $K[X]$ satisfies the f.c.c., Corollary 2.16(b) yields that the semigroup ring $K[\mathbb{Q}^+]$ satisfies the f.c.c. We see, in the same way, that $\mathbb{Z}[\mathbb{Q}^+]$ also satisfies the f.c.c. For another application of Corollary 2.16(b), we have that $K[S]$ and $\mathbb{Z}[S]$ each satisfy the f.c.c., where $S := \frac{1}{2}\mathbb{N} + \frac{1}{3}\mathbb{N} = \{x \in \mathbb{Q} |$ there exist $m, n \in \mathbb{N}$ such that $x = m/2 + n/3\}$.

Recall that an integral domain A is said to satisfy the second chain condition (in short, s.c.c.) if B satisfies the f.c.c. for each integral domain $B \supseteq A$ such that B is integral over A.

Proposition 2.18. *Let A be an integral domain, S a torsion-free cancellative Abelian semigroup of rank 1, and $G := \langle S \rangle$. Then the following conditions are equivalent:*

(1) *$A[S]$ satisfies the s.c.c.;*

(2) *$A[G]$ satisfies the s.c.c.;*

(3) *$A[X]$ satisfies the s.c.c.*

Proof. By [ACKZ, Proposition 3.9], (2) $\Leftrightarrow$ (3). Thus we may suppose that S is not a group and so, by Lemma 2.4, there exists an indeterminate X such that the extension $A[X] \subseteq A[S]$ is integral. As the "s.c.c." property is inherited by the smaller partner of an integral extension (cf. [R2, Remark 1.3.4]), (1) $\Rightarrow$ (3). Finally, (3) $\Rightarrow$ (1) since "integrality" is transitive. $\square$

Remark 2.19. Let K be a field, n a positive integer and, as usual, $\mathbb{N} := \{x \in \mathbf{Z} | x \geq 0\}$. As an application of Proposition 2.18, we see that the semigroup ring $K[\mathbb{N} \oplus \mathbb{Z}]$ satisfies the s.c.c. Indeed, $K[\mathbb{N} \oplus \mathbb{Z}] \cong K[\mathbb{N}][\mathbb{Z}]$ ($\cong K[X][\mathbf{Z}]$) by [G3, Theorem 7.1] and so an appeal to either Propposition 2.18 or [ACKZ, Proposition 3.9] reduces the assertion to the classical fact that $K[X, Y]$ satisfies the s.c.c. $\square$

Let A be an integral domain, with $d := \dim(A) < \infty$. As in [ABDFK], A is called a Jaffard domain if d is also the valuative dimension of A. This concept is related to some properties studied above by the following nonreversible implications noted in [ABDFK, Section 0] (for finite-dimensional A):

A is universally catenarian $\Rightarrow$ A is a stably strong S-domain $\Rightarrow$

A satisfies the altitude inequality formula $\Rightarrow$ A is a Jaffard domain.

To motivate Theoerm 2.20, observe the following consequence of [ABDFK, Corollary 1.18]. If A is an integral domain, S a torsion-free cancellative Abelian semigroup of rank 1, and $G := \langle S \rangle$, then: $A[S]$ is a Jaffard domain $\Leftrightarrow$ $A[G]$ is a Jaffard domain $\Leftrightarrow$ $A[X]$ is a Jaffard domain.

The analogues studied in Theorem 2.20 depend on the following concepts introduced in [C]. Let A be an integral domain. We say that A is locally Jaffard if A_P is a Jaffard domain for each $P \in \mathrm{Spec}(A)$; that A is residually Jaffard if A/P is a Jaffard domain for each $P \in \mathrm{Spec}(A)$; and that A is totally Jaffard if A_P is residually Jaffard for each $P \in \mathrm{Spec}(A)$ or, equivalently, if A/P is locally Jaffard for each $P \in \mathrm{Spec}(A)$. It is well known (as a consequence of [K, Lemme 1.4]) that an LFD integral domain A is locally Jaffard if and only if A satisfies the altitude inequality formula.

Theorem 2.20. *Let A be a finite-dimensional integral domain and S a torsion-free cancellative Abelian semigroup of rank 1. Then:*

(a) *$A[S]$ is locally Jaffard if and only if $A[X]$ is locally Jaffard.*

(b) *$A[S]$ is residually Jaffard if and only if $A[X]$ is residually Jaffard.*

(c) *If $A[S]$ is totally Jaffard, then $A[X]$ is totally Jaffard.*

Proof. Let $G := \langle S \rangle$. By [ACKZ, Theorem 4.1], we may suppose that $S \neq G$; that is, S is not a group. By Remark 2.4, there exists an indeterminate X such that the ring extension $A[X] \subsetneq A[S]$ is integral. Applying the results on transfer of the "locally Jaffard," "residually Jaffard," and "totally Jaffard" properties for integral extensions in [ACKZ, Lemma 4.3], we immediately obtain (b), (c), and the "only if" assertion in (a). Thus, it remains only to prove that if $A[X]$ is locally Jaffard, then $A[S]$ is locally Jaffard.

By the usual combination of [AG] and [G1], observe that $A[S]$ is finite-dimensional, hence LFD. Therefore, by [K, Lemme 1.4], it suffices to prove that $ht(P[Y]) = ht(P)$ for all $P \in \mathrm{Spec}(A[S][X_1, \dots, X_n])$ and all positive integers n. As $A[S][X_1, \dots, X_n] \cong A[X_1, \dots, X_n][S]$ canonically, we may view $P \in \mathrm{Spec}(A[X_1, \dots, X_n][S])$. For convenience, put

$$A' := A[X_1, \dots, X_n], \;\; p := P \cap A', \text{ and } p' := P[Y] \cap A'[Y] = (P \cap A')[Y] = p[Y].$$

By the reformulation of Proposition 2.2(b) noted in Remark 2.3(b), we have that $ht(P) = ht(pA'[S]) + ht(P/pA'[S])$ and $ht(P[Y]) = ht(p'A'[S]) + ht(P[Y]/p'A'[S])$, where we have identified $A'[S][Y]$ with $A'[Y][S]$.

Next, if Z is an indeterminate, note that

$$ht(pA'[S]) = ht(pA'[G]) = ht(p[Z]),$$

with the first equality holding as in the proof of Proposition 2.2(a) and the second holding via Remark 2.6(c). Similarly,

$$ht(p'[S]) = ht(p'[G]) = ht(p'[Z]) = ht(p[Y][Z]).$$

We claim that $ht(pA'[S]) = ht(p'[S])$. To prove the claim, it suffices to show that $ht(p[Z]) = ht(p[Y][Z])$. As $p[Y][Z] = p[Z][Y]$, [K, Lemme 1.4] reduces our task to showing that $A'[Z] \cong A[Z][X_1, \dots, X_n]$ satisfies the altitude inequality formula. Now, since $A[Z] \cong A[X]$ is locally Jaffard, $A[Z]$ satisfies the altitude inequality formula and so, by [C, Proposition 1(ii)], $A[Z][X_1, \dots, X_n]$ also satisfies the altitude inequality formula. Thus, the claim has been established. To complete the proof, it is enough to show that $ht(P/pA'[S]) = ht(P[Y]/p'A'[S])$.

View $P/pA'[S]$ inside $(A'/p)[S] \cong A'[S]/pA'[S]$ and, similarly, $P[Y]/p'A'[S] = P[Y]/pA'[S][Y] \subseteq (A'/p)[S][Y]$. As these prime ideals are each disjoint from the multiplicatively closed set $T := (A'/p)\backslash\{0\}$, their heights are not affected by localizing at T. With K denoting the quotient field of A'/p, we thus have $(P/pA'[S])_T \subseteq (A'/p)[S]_T = K[S]$ and, similarly, $(P[Y]/p'A'[S])_T \subseteq K[S][Y]$. Hence, it suffices to prove that if $q \in \mathrm{Spec}(K[S])$, then $ht(q) = ht(q[Y])$. Now, by Corollary 2.11 (or by combining Corollary 2.15 with [BDF1, Theorem 2.4]), $K[S]$ is a stably strong S-domain. In particular, $K[S]$ is a strong S-domain, and so one way to finish the proof is to appeal to [Kap, Theorem 39]. For an alternate finish, using that $K[S]$ is a stably strong S-domain, use [K, Théorème 1.6] to see that $K[S]$ satisfies the altitude inequality formula; then, an application of [K, Lemme 1.4] completes the proof. $\square$

3. Results for finitely generated semigroups. This section is devoted to the semigroup rings $A[S]$, where A is an integral domain and S is a finitely generated torsion-free cancellative Abelian semigroup. It is clear that if S can be generated by n elements, then $A[S] \cong A[X_1, \dots, X_n]/P$ for some prime ideal P such that $P \cap A = 0$.

Proposition 3.1. *Let A be an integral domain and S a finitely denerated torsion-free cancellative Abelian semigroup. Then $A[S]$ is a stably strong S-domain (resp. universally catenarian) if and only if A is a stably strong S-domain (resp., universally catenarian).*

Proof. The classes of stably strong S-domains and of universally catenarian domains are each stable with respect to localization, factor domains, and adjunction of finitely many indeterminates (cf. [MM, Theorem 3.2], [BDF1, Corollary 3.3]). Writing $A[S]$ as $A[X_1, \dots, X_n]/P$ as above, we thus obtain the "if" assertions. For the converses, it suffices to observe that A is a homomorphic image of a ring of fractions of $A[S]$. □

Proposition 3.2. *Let A be a finite-dimensional integral domain. Then $A[S]$ is coequidimensional for each finitely generated torsion-free cancellative Abelian semigroup S if and only if $A[X_1, \dots, X_n]$ is coequidimensional for each positive integer n.*

Proof. The "only if" assertion is trivial because $A[\mathbb{N}^n] \cong A[X_1, \dots, X_n]$.

Conversely, for S as in the statement, we have $A[S] \cong A[X_1, \dots, X_n]/P$, with P prime such that $P \cap A = 0$ and $A[X_1, \dots, X_k]$ coequidimensional for each k. Put $G := \langle S \rangle$ and $r := \text{rank}(S) = \text{rank}\, G$. Since S is finitely generated, so is G, whence $r < \infty$. Without loss of generality, $S \neq 0$, and so $r \geq 1$. It suffices to show that if M is a maximal ideal of $A[S]$, then $ht(M) = \dim(A[S])$. Put $m := M \cap A$. Note that under the standard A-algebra isomorphism $A[S] \cong A[X_1, \dots, X_n]/P$, we have $M \cong N/P$ for some maximal ideal N of $A[X_1, \dots, X_n]$, whence $N \cap A = M \cap A = m$. Now, since $A[X_1, \dots, X_n]$ is coequidimensional and finite-dimensional (this is the only place we use the hypothesis that $\dim(A) < \infty$), it follows from (the second assertion in) [AG, Corollary 4.3] that m is a maximal ideal of A. Also, since $A[X_1, \dots, X_r]$ is coequidimensional, $\dim(A[X_1, \dots, X_r]) = ht(Q)$, where Q is any maximal ideal containing $mA[X_1, \dots, X_r]$. Now, as m is maximal,

$ht(Q) = ht(mA[X_1, \ldots, X_r]) + r$ (cf. Lemma 2.1). Since [AG] and [G1] combine, as in the proof of Lemma 2.1, to show that $\dim(A[S]) = \dim(A[G]) = \dim(A[X_1, \ldots, X_r])$, the upshot is that $\dim(A[S]) = ht(mA[X_1, \ldots, X_r]) + r$. It remains only to prove that $ht(M) = ht(mA[X_1, \ldots, X_r]) + r$.

By (the first assertion in) [AG, Corollary 4.3], $ht(M) = ht(mA[S]) + r$, and so it suffices to show that $ht(mA[S]) = ht(mA[X_1, \ldots, X_r])$. For this, note that $ht(mA[S]) = ht(mA[G])$ as in the proof of Proposition 2.2(a); and $ht(mA[G]) = ht(mA[X_1, \ldots, X_r])$ by Remark 2.6(c). □

It may be of some technical value to record that the preceding proof actually established the following statement. Let A be a finite-dimensional integral domain and S an n-generated torsion-free cancellative Abelian semigroup of rank r. If both $A[X_1, \ldots, X_n]$ and $A[X_1, \ldots, X_r]$ are coequidimensional, then so is $A[S]$. (In the preceding assertion, if A is also assumed to be a stably strong S-domain, then we may delete the hypothesis that $A[X_1, \ldots, X_r]$ is coequidimensional. Indeed, since $r \leq n$, the coequidimensionality of $A[X_1, \ldots, X_r]$ would follow from that of $A[X_1, \ldots, X_n]$. More generally, we have the following result, of independent interest. If A is a strong S-domain such that $A[X]$ is coequidimensional, then A is coequidimensional. This may be proved by using the fact [Kap, Theorem 39] that if a maximal ideal M of A has $ht(M) = d$, then (M, X) is a maximal ideal of $A[X]$ with height $d + 1$.)

Corollary 3.3. *Let A be a finite-dimensional integral domain. Then $A[S]$ satisfies the f.c.c. (first chain condition) for each finitely generated torsion-free cancellative Abelian semigroup S if and only if $A[X_1, \ldots, X_n]$ satisfies the f.c.c. for each positive integer n.*

Proof. Recall that a finite-dimensional domain satisfies the f.c.c. if and only if it is coequidimensional and catenarian. Moreover, by the usual combination of [AG] and [G1], each of the semigroup rings (and polynomial rings) mentioned in the statement is finite-dimensional. The statement now follows easily from Propositions 3.1 and 3.2, in light of the following two observations. First, if each $A[S]$ of the above type is catenarian, then it is universally catenarian, for $A[S][X_1,\dots,X_n] \cong A[S][\mathbb{N}^n] \cong A[S \oplus \mathbb{N}^n]$ and $S \oplus \mathbb{N}^n$ is also a semigroup of the above type. Second, if A is universally catenarian, then so is $A[X_1,\dots,X_n]$ for each positive integer n. $\square$

Example 4.5 will show that one cannot delete the hypothesis that S is finitely generated in Proposition 3.2 or Corollary 3.3.

Remark 3.4. As $\mathbb{Z}$ is universally catenarian and $\mathbb{Z}[X_1,\dots,X_n]$ is known to be coequidimensional for each positive integer n, $\mathbb{Z}[X_1,\dots,X_n]$ satisfies the f.c.c. Therefore, by Corollary 3.3, $\mathbb{Z}[S]$ satisfies the f.c.c. for each finitely generated torsion-free cancellative Abelian semigroup S. Similarly, if K is any field and S is as above, we may verify that $K[S]$ satisfies the f.c.c. Indeed, any such $K[S]$ satisfies the s.c.c., by [BCL, Theorem C], since $K[S] \cong K[X_1,\dots,X_n]/P$ is an affine domain.

A word is in order regarding the techniques of proof in the final two results in this section. In dealing with finitely generated semigroups, we cannot automatically use results developed for the rank 1 case, such as Lemma 2.4, Lemma 2.8, and Proposition 2.2(b). Recall that Proposition 2.2(b) was reformulated in Remark 2.3(b) to state that $ht(P) = ht(pA[S]) + ht(P/pA[S])$ for A, S, P and p as in the hypothesis of Proposition 2.2(b). According to [AG, Theorem 4.1], the same conclusion holds for A, S as in this section, and so it will be exploited below. As in earlier proofs, we shall also exploit [AG], [G1] and Remark 2.6(c).

Theorem 3.5. *If A is an integral domain and S is a nonzero finitely generated torsion-free cancellative Abelian semigroup, then $A[S]$ is an S-domain.*

Proof. We show that if $P \in \mathrm{Spec}(A[S])$ and $ht(P) = 1$, then $ht(P[X]) = 1$.

Put $p := P \cap A$. Evidently, $pA[S] \in \mathrm{Spec}(A[S])$ and $pA[S] \subseteq P$. There are two cases.

Suppose, first, that $pA[S] = P$. Put $G := \langle S \rangle$ and $r = \mathrm{rank}(S) = \mathrm{rank}(G)$. As in the proof of Proposition 2.2(a), $ht(pA[S]) = ht(pA[G])$; and by Remark 2.6(c), $ht(pA[G]) = ht(pA[X_1, \dots, X_r])$. In particular, $ht(pA[X_1, \dots, X_r]) = ht(P) = 1$. As [FK, Proposition 2.1] ensures that $A[X_1, \dots, X_n]$ is an S-domain, $ht(pA[X_1, \dots, X_r][X]) = 1$. Now, by setting $T := \{X^s | s \in S\}$ as usual, we have $ht(P[X]) = ht(pA[S][X]) = ht(pA[S][X]_T) = ht(pA[S]_T[X]) = ht(pA[G][X]) = ht(pA[G][\mathbb{N}]) = ht(pA[G \oplus \mathbb{N}])$. As $\langle G \oplus \mathbb{N} \rangle = G \oplus \mathbb{Z}$, it follows from the proof of Proposition 2.2(a) that $ht(pA[G \oplus \mathbb{N}]) = ht(pA[G \oplus \mathbb{Z}])$. Moreover, since $\mathrm{rank}(G \oplus \mathbb{Z}) = \mathrm{rank}(G) + \mathrm{rank}(\mathbb{Z}) = r + 1$, Remark 2.6(c) yields that $ht(pA[G \oplus \mathbb{Z}]) = ht(pA[X_1, \dots, X_{r+1}])$, which is the same as $ht(pA[X_1, \dots, X_r][X]) = 1$. Thus, $ht(P[X]) = 1$, as desired.

In the remaining case, $pA[S] \subsetneqq P$. As $A[S]$ is an integral domain and $ht(P) = 1$, it follows that $pA[S] = 0$, and so $p = 0$. Let K be the quotient field of A. Then $ht(P[X]) = ht(PA[S][X]_{A \setminus \{0\}}) = ht(P_{A \setminus \{0\}} A_{A \setminus \{0\}}[S][X]) = ht(P_{A \setminus \{0\}} K[S][X])$. Observe that $P_{A \setminus \{0\}} K[S] = P_{A \setminus \{0\}}$ is a height 1 prime ideal of $K[S]$. Thus, it suffices to show that $K[S]$ is an S-domain. In fact, since S is finitely generated, $K[S] \cong K[X_1, \dots, X_n]/Q$ for some positive integer n and prime ideal Q; by Hilbert Basis Theorem, $K[S]$ is Noetherian and, hence, a (stably strong) S-domain. $\square$

Theorem 3.6. *Let A be a finite-dimensional integral domain and S a finitely generated torsion-free cancellative Abelian semigroup, with $r := \mathrm{rank}(S)$. Then $A[S]$ is locally Jaffard if and only if $A[X_1, \dots, X_r]$ is locally Jaffard.*

Proof. By following the strategy enunciated prior to the statement of Theorem 3.5, one need only rework the proof of Theorem 2.20. In doing so, the reader may find the following four comments to be helpful. First, the "locally Jaffard" property is preserved by localization at arbitrary multiplicative sets (cf. [ABDFK, Proposition 1.5]). Thus, if $A[S]$ is locally Jaffard, then so is $A[G]$, and by [ACKZ, Theorem 4.1(a)], so is $A[X_1, \dots, X_r]$. Second, to restate part of the strategy, avoid the appeal to Remark 2.3(b) by citing [AG, Theorem 4.1]. Third, replace Z with $Z_1, \dots, Z_r$, as in Remark 2.6(c). Finally, show that $K[S]$ is a stably strong S-domain by arguing as at the close of the proof of Theorem 3.5. □

4. Examples and counterexamples.

The material in this section serves two purposes. Examples 4.1-4.4 and 4.5(b) give some semigroup rings which are new examples of the following types of integral domains: universally catenarian, stably strong S-, locally Jaffard, residually Jaffard. On the other hand, Example 4.5(a) shows that one cannote delete the hypotheses "rank(S) = 1" or "S is finitely generated" in Theorem 2.5, Corollary 2.16, Proposition 3.2 and Corollary 3.3. Example 4.5(a) also illustrates that some results on group rings $A[\langle S\rangle]$, notably [ACKZ, Corollaries 2.6 and 3.7], do not generalize to semigroup rings $A[S]$.

Example 4.1. Let $V = F + M$ be a nontrivial discrete rank 1 valuation domain with maximal ideal M, where F is a field; let k be a proper subfield of F such that $[F : k] < \infty$; and put $A := k + M$. (For instance, let K be a field, consider $V = K(X)[Y]_{(Y)} = K(X) + M$ with $M = YV$, let $k = K(X^2)$ and put $A = K(X^2) + YK(X)[Y]_{(Y)}$.) Then $A[\mathbb{Q}^+]$ is universally catenarian.

Proof. Since $\mathbb{Q}^+$ is a torsion-free cancellative Abelian semigroup of rank 1, Corollary 2.15 shows that it is enough to prove that A is Noetherian and universally catenarian. Now, by standard facts about the classical $D + M$ construction (cf. [G2]), A is Noetherian, since V is Noetherian, k is a field and $[F : k] < \infty$; also, $\dim(A) = \dim(k) + \dim(V) = 0 + 1 = 1$. Finally, to verify that A is universally catenarian, it suffices to note that A is a one-dimensional Noetherian integral domain, by [R1, (2.6)]; or that A is a Cohen-Macaulay ring, by [M, Theorem 31]; or that A is a going-down domain whose integral closure (namely, V) is an LFD Prüfer domain, by [BDF1, Theorem 6.2]; or that a pullback result such as [ADKM, Corollary 2.3] may be applied. $\square$

The above argument shows that in Example 4.1, $\mathbb{Q}^+$ may be replaced, more generally, by any torsion-free cancellative Abelian semigroup of rank 1. We have chosen the above specificity to make clearer the relations between the construction in Example 4.1 and those in Examples 4.2-4.4. Before developing the latter examples, we make two additional remarks about Example 4.1.

First, the ring $A[\mathbb{Q}^+]$ in Example 4.1 does not satisfy the f.c.c. Indeed, although $A[\mathbb{Q}^+]$ is (universally) catenarian by Example 4.1, $A[\mathbb{Q}^+]$ is not coequidimensional. To see this, it suffices, by Theorem 2.5, to show that $A[U]$ is not coequidimensional, where U is an indeterminate over A. Now, since A is one-dimensional Noetherian, $\dim(A[U]) = 2$, and so it is enough to find a height 1 maximal ideal of $A[U]$. This is done via [Kap, Theorem 24] for, since A is a G-domain, some maximal ideal N of $A[U]$ satisfies $N \cap A = 0$ and then, necessarily, $ht(N) = 1$ (cf. [Kap, Theorem 37]).

Second, we address the claim that the ring $A[\mathbb{Q}^+]$ in Example 4.1 is a "new example" of a universally catenarian domain. Specifically, we show that [BDF2, Theorem 1] does not apply to $A[\mathbb{Q}^+]$. The result in question asserts that any LFD stably strong S-going-down domain must be universally catenarian. Now, in view of what was shown in Example 4.1, $A[\mathbb{Q}^+]$ *is* an LFD stably strong S-domain. Accordingly, we must show that $A[\mathbb{Q}^+]$ is not a going-down domain. If not, then since $\mathbb{N}$ is a subsemigroup of $\mathbb{Q}^+$, $A[\mathbb{N}]$ is (isomorphic to) a factor domain of $A[\mathbb{Q}^+]$, and so $A[\mathbb{N}]$ is a going-down domain by [D2, Remarks 2.11 and 3.2(a), (b)]. It follows from [D1, Theorem

2.2] that if Z is an indeterminate, then $A[Z] \cong A[\mathbb{N}]$ is a treed domain, a contradiction.

The next result analyzes further Nagata's example of a two-dimensional Noetherian integral domain A which is not universally catenarian.

Example 4.2. Let K be a field and $W := K[X,Y]_{(X-1,Y)}$. Let (V,P) be a Noetherian valuation overring of $K[X,Y]$ such that $P \cap K[X,Y] = (X,Y)$ and $V/P \cong K$. Put $T := V \cap W$; and let M, N denote the two maximal ideals of T. Put $A := K + (M \cap N)$. Then $A[\mathbb{Q}^+]$ is a stably strong S-domain which is neither Noetherian nor catenarian.

Proof. It is well known (cf. [N, Theorem 11.11], [Kap, Theorem 107]) that since V and W are incomparable, T has exactly two maximal ideals, which may be labeled M, N so that $T_M = V$ and $T_N = W$. Since T is quasi-semilocal and its localizations at its maximal ideals are both Noetherian, it follows from [N, (E1.1), p. 203] that T is Noetherian. Moreover, $T/M \cong T_M/MT_M = V/P = K$ and, similarly, $T/N \cong K$, so that T/M and T/N are each (trivially) finite-dimensional field extensions of K. Accordingly, one may apply [N, (E2.1), p. 204] to show that A is a (Noetherian) local ring. Now, use Remark 2.6(c) to find an indeterminate Z such that there is an integral ring extension $A[Z] \subseteq A[\mathbb{Q}]$. Since $A[Z]$ inherits the Noetherian property from A, [MM, Proposition 4.20] yields that $A[\mathbb{Q}]$ is a stably strong S-domain. As $\langle \mathbb{Q}^+ \rangle = \mathbb{Q}$, Corollary 2.10 then shows that $A[\mathbb{Q}^+]$ is also a stably strong S-domain, as asserted. Finally, [G3, Theorem 20.7] ensures that $A[\mathbb{Q}^+]$ is not Noetherian, since $\mathbb{Q}^+$ is not a finitely generated semigroup; and since it is known that $A[\mathbb{Q}]$ is not catenarian [ACKZ, Example 5.3(c)], an application of Theorem 2.13 yields that $A[\mathbb{Q}^+]$ is not catenarian. $\square$

Example 4.3. If K is a field and $A := K[X_1] + XK(X_1, X_2)[X]_{(X)}$, then $A[\mathbb{Q}^+ \oplus \mathbb{Z}]$ is locally Jaffard but not residually Jaffard.

Proof. A can be obtained via a classical $D+M$ construction, as follows. Take $(V, M) := K(X_1, X_2)[X]_{(X)} = K(X_1, X_2) + M$, with $M = XV$; then $A = K[X_1] + M$. By the standard facts about the $D+M$ construction (cf. [BG, Theorem 2.1], [ABDFK, Theorem 2.11]), $\dim(A) = \dim(K[X_1]) + \dim(V) = 1+1 = 2$ and $\dim_v(A) = \dim_v(K[X_1]) + \dim_v(V) + \text{t.d.}(K(X_1, X_2)/K(X_1)) = 1+1+1 = 3$. Thus, A is not a Jaffard domain. As A is (isomorphic to) a factor domain of $A[Y_1, Y_2]$, it follows that $A[Y_1, Y_2]$ is not residually Jaffard. Thus, to prove that $A[\mathbb{Q}^+ \oplus \mathbb{Z}]$ is not residually Jaffard, [ACKZ, Lemma 4.3(b)] shows that it is enough to find (algebraically independent) indeterminates Y_1, Y_2 such that there is an integral ring extension $A[Y_1, Y_2] \subseteq A[\mathbb{Q}^+ \oplus \mathbb{Z}]$.

Now, since $\mathbb{Q}^+$ is of rank 1 and is not a group, Lemma 2.4 may be applied, yielding an indeterminate Y_1 such that $A[Y_1] \subseteq A[\mathbb{Q}^+]$ is an integral extension. Thus, if Y_2 is an indeterminate which is algebraically independent of Y_1 with respect to A, we infer an integral ring extension $A[Y_1, Y_2] \subseteq A[\mathbb{Q}^+][Y_2]$. Put $B := A[\mathbb{Q}^+]$. It suffices to find an integral ring extension $B[Y_2] \subseteq A[\mathbb{Q}^+ \oplus \mathbb{Z}]$. To this end, identify $A[\mathbb{Q}^+ \oplus \mathbb{Z}] \cong A[\mathbb{Q}^+][\mathbb{Z}] = B[\mathbb{Z}] \cong B[T, T^{-1}]$ for a suitable indeterminate T. As noted in [ACKZ], $U := T + T^{-1}$ is indeterminate over B and the ring extension $B[U] \subseteq B[T, T^{-1}]$ is integral (essentially since $T^2 - UT + 1 = 0$). Hence, $B[Y_2] \cong B[U] \subseteq B[T + T^{-1}] \cong A[\mathbb{Q}^+ \oplus \mathbb{Z}]$ may be viewed as an integral ring extension. In this way, we have the desired integral ring extension $A[Y_1, Y_2] \subseteq A[\mathbb{Q}^+ \oplus \mathbb{Z}]$ and so, as explained above, $A[\mathbb{Q}^+ \oplus \mathbb{Z}]$ is not residually Jaffard.

It remains to show that $A[\mathbb{Q}^+ \oplus \mathbb{Z}]$ is locally Jaffard. By [C, Proposition 1(ii)], $A[Y_1, \ldots, Y_d]$ is locally Jaffard for all integers $d \geq \dim_v(A) - 1 = 2$; in particular, $A[Y_1, Y_2]$ is locally Jaffard. It follows from [ACKZ, Theorem 4.1(a)] that $A[Y_1][\mathbb{Z}] \cong A[\mathbb{Z}][Y_1]$ is locally Jaffard. Hence, by Theorem 2.20(a), so is $A[\mathbb{Z}][\mathbb{Q}^+] \cong A[\mathbb{Q}^+ \oplus \mathbb{Z}]$. □

We turn to a construction whose properties are more fully developed than those in Example 4.3. For the reader's convenience, we give a self-contained development of Example 4.4, which begins by aping the beginning of Example 4.3.

Example 4.4. Let K be a field and $A := K + YK(X)[Y]_{(Y)}$. Then $A[\mathbb{Q}^+]$ is locally Jaffard; but $A[\mathbb{Q}^+]$ is neither residually Jaffard, a strong S-domain, nor catenarian.

Proof. A can be obtained via a classical $D+M$ construction, as follows. Take $(V, M) := K(X)[Y]_{(Y)} = K(X) + M$, with $M = YV$; then $A = K + M$. By the standard facts about the $D + M$ construction (cf. [BG, Theorem 2.1], [ABDFK, Proposition 2.5(a)]), we have that A is quasi-local one-dimensional and $\dim_v(A) = \dim_v(V) + \text{t.d.}(K(X)/K) = 1+1 = 2$. Thus, A is not a locally Jaffard domain; but, by [C, Proposition 1(ii)], $A[X]$ is locally Jaffard. Hence, by Theorem 2.20(a), $A[\mathbb{Q}^+]$ is also locally Jaffard, as asserted. However, since a factor domain of $A[X]$ (namely, A) is not a Jaffard domain, $A[X]$ is not residually Jaffard, and so by Theorem 2.20(b), $A[\mathbb{Q}^+]$ is not residually Jaffard. Finally, since $\langle \mathbb{Q}^+ \rangle = \langle \mathbb{Q} \rangle$ and [ACKZ, Example 5.4(c), (d)] showed that $A[\mathbb{Q}]$ is neither a strong S-domain nor catenarian, Theorems 2.9 and 2.13 yield the corresponding conclusions for $A[\mathbb{Q}^+]$. $\square$

We close with an example having many interesting features. Indeed, Example 4.5 studies additional properties of a construction due to Arnold and Gilmer [AG], which has already been mentioned in Remark 2.3(b). In the first paragraph of this section, we have explained the "counterexample" role played by Example 4.5(a) relative to four results in this paper and two results in [ACKZ]. However, Example 4.5(b) has an additional purpose, namely to give a new class of residually Jaffard domains. As a companion for a result [ACKZ, Theorem 4.1(b)] on group rings $A[G]$ and for the above result, Theorem 2.20(b), on semigroup rings $A[S]$ where $\text{rank}(S) = 1$, Example 4.5(b) identifies a residually Jaffard semigroup ring $A[S]$, such that $\text{rank}(S) = 2$ and S is not finitely generated. Thus, together with Example 4.3, Example

4.5(b) gives reason to focus a future study of semigroup rings $A[S]$ on the case of $\operatorname{rank}(S) = 2$.

Example 4.5. Let K be a field and for some positive integer $n \geq 2$, let $\lambda_1, \dots, \lambda_n$ be rationally independent positive real numbers. Let S denote the additive semigroup of all nonnegative real numbers of the form $i_1\lambda_1 + \cdots + i_n\lambda_n$, with each $i_j \in \mathbb{Z}$. Then:

(a) S is a torsion-free cancellative Abelian semigroup of rank n, S is not a group, and S is not finitely generated. For each positive integer m, $K[X_1, \dots, X_m]$ is coequidimensional and catenarian, and hence satisfies the f.c.c. However, $K[S]$ is not coequidimensional and so does not satisfy the f.c.c.

(b) Assume, in addition, that $n = 2$. Then $K[S]$ is residually Jaffard.

Proof. (a) As $\langle S \rangle = \mathbb{Z}^n$, $\operatorname{rank}(S) = n$. The other assertions about S were noted earlier. Now, by the usual combination of [G1] and [AG], $\dim(K[S]) = \dim(K[X_1, \dots, X_n]) = n > 1$, although [AG, p. 311] produces a maximal ideal P of $K[S]$ such that $ht(P) = 1$. Thus, $K[S]$ is not coequidimensional, and so $K[S]$ does not satisfy the f.c.c. The assertions about $K[X_1, \dots, X_m]$ are classical (cf. [S, Proposition 15, p. III-23]).

(b) We show that $B := K[S]/P$ is a Jaffard domain for each $P \in \operatorname{Spec}(K[S])$. This is clear if $P = 0$, for [ABDFK, Corollary 1.18] reduces the matter to observing that $K[X_1, X_2]$ is (Noetherian and so) a Jaffard domain. The assertion is also clear if $ht(P) = 2$, for B is then a field.

It remains to consider the case $ht(P) = 1$. Without loss of generality, P is not a maximal ideal, and so $\dim(B) = 1$. By [ABDFK, Theorem 1.10], it suffices to prove that B is an S-domain; i.e., that $ht(Q[X]/P[X]) = 1$ for each prime ideal $Q \supsetneqq P$ of $K[S]$. Now, $Q[X]$ is not a maximal ideal of $K[S][X]$, and the usual combination of [G1] and [AG] yields that $K[S][X] \cong K[S \oplus \mathbb{N}] \cong K[X][S]$ is three-dimensional, whence $ht(Q[X]) \leq 2$. Hence, $ht(Q[X]/P[X]) \leq 1$. However, the reverse inequality also holds since $P[X] \subset Q[X]$ are distinct prime ideals. The proof is complete. $\square$

ACKNOWLEDGMENTS

The third-named author warmly thanks the University of Tennessee, Knoxville for its hospitality during his visit in the Spring of 1997.

REFERENCES

[ACKZ] S.Améziane, D.L. Costa, S. Kabbaj and S. Zarzuela, *On the spectrum of the group ring*, submitted for publication.

[ABDFK] D.F. Anderson, A. Bouvier, D.E. Dobbs, M. Fontana and S. Kabbaj, *On Jaffard domains*, Exposition. Math. 6 (1988), 145–175.

[ADKM] D.F. Anderson, D.E. Dobbs, S. Kabbaj and S.B. Mulay, *Universally catenarian domains of $D+M$ type*, Proc. Amer. Math. Soc. 104 (1988), 378–384.

[AG] J.T. Arnold and R. Gilmer, *The dimension theory of commutative semigroup rings*, Houston J. Math. 2 (1976), 299–313.

[BG] E. Bastida and R. Gilmer, *Overrings and divisorial ideals of rings of the form $D+M$*, Michigan Math. J. 20 (1973), 79–95.

[BCL] J.W. Brewer, D.L. Costa and E.L. Lady, *Prime ideals and localization in commutative group rings*, J. Algebra 34 (1975), 300–308.

[BDF1] A. Bouvier, D.E. Dobbs and M. Fontana, *Universally catenarian integral domains*, Advances Math. 72 (1988), 211–238.

[BDF2] A. Bouvier, D.E. Dobbs and M. Fontana, *Two sufficient conditions for universal catenarity*, Comm. Algebra 15 (1987), 861–872.

[C] P.-J. Cahen, *Constructions B, I, D et anneaux localement ou résiduellement de Jaffard*, Arch. Math. 54 (1990), 125–141.

[D1] D.E. Dobbs, *On going-down for simple overrings, II*, Comm. Algebra 1 (1974), 439–458.

[D2] D.E. Dobbs, *Divided rings and going-down*, Pacific J. Math. 67 (1976), 353-363.

[FK] M. Fontana and S. Kabbaj, *On the Krull and valuative dimension of $D + XD_s[X]$ domains*, J. Pure Appl. Algebra 63 (1990), 231–245.

[G1] R. Gilmer, *A two-dimensional non-noetherian factorial ring*, Proc. Amer. Math. Soc. 44 (1974), 25–30.

[G2] R. Gilmer, *Multiplicative ideal theory*, Dekker, New York, 1972.

[G3] R. Gilmer, *Commutative semigroup rings*, Univ. Chicago Press, Chicago, 1984.

[K] S. Kabbaj, *La formule de la dimension pour les S-domaines forts universels*, Boll. Un. Mat. Ital. Série VI 5 (1986), 145–161.

[Kap] I. Kaplansky, *Commutative rings*, rev. ed., Univ. Chicago Press, Chicago, 1974.

[MM] S. Malik and J.L. Mott, *Strong S-domains*, J. Pure Appl. Algebra 28 (1983), 249–264.

[M] H. Matsumura, *Commutative algebra*, Benjamin, New York, 1970.

[N] M. Nagata, *Local rings*, Interscience, New York, 1962.

[R1] L.J. Ratliff, Jr., *On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals, II*, Amer. J. Math. 92 (1970), 99–144.

[R2] L.J. Ratliff, Jr., *Chain conjectures in ring theory*, Lecture Notes Math., vol. 647, Springer-Verlag, Berlin, 1978.

[S] J.-P. Serre, *Algèbre locale: multiplicités*, Lecture Notes Math., vol. 11, Springer-Verlag, Berlin, 1965.

Received: May 1997