Commutative Algebra: Recent Advances in Commutative Rings, Integer-Valued Polynomials, and Polynomial Functions
Springer, New York, pp. 1–24, 2014

Weak global dimension of Prüfer-like rings

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Abstract In 1969, Osofsky proved that a chained ring (i.e., local arithmetical ring) with zero divisors has infinite weak global dimension; that is, the weak global dimension of an arithmetical ring is 0, 1, or ∞. In 2007, Bazzoni and Glaz studied the homological aspects of Prüfer-like rings, with a focus on Gaussian rings. They proved that Osofsky's aforementioned result is valid in the context of coherent Gaussian rings (and, more generally, in coherent Prüfer rings). They closed their paper with a conjecture sustaining that "the weak global dimension of a Gaussian ring is 0, 1, or ∞." In 2010, the authors of [3] provided an example of a Gaussian ring which is neither arithmetical nor coherent and has an infinite weak global dimension. In 2011, the authors of [1] introduced and investigated the new class of fqp-rings which stands strictly between the two classes of arithmetical rings and Gaussian rings. Then, they proved the Bazzoni-Glaz conjecture for fqp-rings. This paper surveys a few recent works in the literature on the weak global dimension of Prüfer-like rings making this topic accessible and appealing to a broad audience. As a prelude to this, the first section of this paper provides full details for Osofsky's proof of the existence of a module with infinite projective dimension on a chained ring. Numerous examples -arising as trivial ring extensions- are provided to illustrate the concepts and results involved in this paper.

Key words: Weak global dimension; arithmetical ring; fqp-ring; Gaussian ring; Prüfer ring; semihereditary ring; quasi-projective module; trivial extension.

Mathematics Subject Classification13F05, 13B05, 13C13, 16D40, 16B50

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1 Introduction

All rings considered in this paper are commutative with identity element and all modules are unital. Let R be a ring and M an R-module. The weak (or flat) dimension (resp., projective dimension) of M, denoted $\mathrm{w.dim}_R(M)$ (resp., $\mathrm{p.dim}_R(M)$), measures how far M is from being a flat (resp., projective) module. It is defined as follows: Let n be an integer ≥ 0. We have $\mathrm{w.dim}_R(M) \leq n$ (resp., $\mathrm{p.dim}_R(M) \leq n$) if there is a flat (resp., projective) resolution

$$0 \to E_n \to E_{n-1} \to \ldots \to E_1 \to E_0 \to M \to 0.$$

If n is the least such integer, $\mathrm{w.dim}_R(M) = n$ (resp., $\mathrm{p.dim}_R(M) = n$). If no such resolution exists, $\mathrm{w.dim}_R(M) = \infty$ (resp., $\mathrm{p.dim}_R(M) = \infty$). The weak global dimension (resp., global dimension) of R, denoted by $\mathrm{w.gl.dim}(R)$ (resp., $\mathrm{gl.dim}(R)$), is the supremum of $\mathrm{w.dim}_R(M)$ (resp., $\mathrm{p.dim}_R(M)$), where M ranges over all (finitely generated) R-modules. For more details on all these notions, we refer the reader to [6, 13, 23].

A ring R is called coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R [13]. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, and semihereditary rings.

Gaussian rings belong to the class of Prüfer-like rings which has recently received much attention from commutative ring theorists. A ring R is called Gaussian if for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$ where $c(f)$, the content of f, is the ideal of R generated by the coefficients of f [25]. The ring R is said to be a chained ring (or valuation ring) if its lattice of ideals is totally ordered by inclusion; and R is called arithmetical if $R_{\mathfrak{m}}$ is a chained ring for each maximal ideal m of R [11, 18]. Also R is called semihereditary if every finitely generated ideal of R is projective [8]; and R is Prüfer if every finitely generated regular ideal of R is projective [7, 16]. In the domain context, all these notions coincide with the concept of Prüfer domain. Glaz, in [15], constructs examples which show that all these notions are distinct in the context of arbitrary rings. More examples, in this regard, are provided via trivial ring extensions [1, 3].

The following diagram of implications puts the notion of Gaussian ring in perspective within the family of Prüfer-like rings [4, 5, 1]:

Semihereditary ring
$\Downarrow$
Ring with weak global dimension ≤ 1
$\Downarrow$
Arithmetical ring
$\Downarrow$
fqp-Ring
$\Downarrow$

$$\begin{array}{c}\text{Gaussian ring}\\ \Downarrow \\ \text{Prüfer ring}\end{array}$$

In 1969, Osofsky proved that a local arithmetical ring (i.e., chained ring) with zero divisors has infinite weak global dimension [22]. In view of [13, Corollary 4.2.6], this results asserts that the weak global dimension of an arithmetical ring is 0, 1, or ∞.

In 2007, Bazzoni and Glaz proved that if R is a coherent Prüfer ring (and, a fortiori, a Gaussian ring), then $\mathrm{w.gl.dim}(R)$ = 0, 1, or ∞ [5, Proposition 6.1]. And also they proved that if R is a Gaussian ring admitting a maximal ideal $\mathfrak{m}$ such that the nilradical of the localization R_m is a nonzero nilpotent ideal. Then $\mathrm{w.gl.dim}(R)$ = ∞ [5, Theorem 6.4]. At the end of the paper, they conjectured that "the weak global dimension of a Gaussian ring is 0, 1, or ∞" [5]. In two preprints [9, 10], Donadze and Thomas claim to prove this conjecture (see the end of Section 3).

In 2010, the authors of [3] proved that if $(A,\mathfrak{m})$ is a local ring, E is a nonzero $\frac{A}{\mathfrak{m}}$-vector space, and $R := A \ltimes E$ is the trivial extension of A by E, then:

- R is a total ring of quotients and hence a Prüfer ring.
- R is Gaussian if and only if A is Gaussian.
- R is arithmetical if and only if $A := K$ is a field and $\dim_K E = 1$.
- $\mathrm{w.gl.dim}(R) \gneqq 1$. If, in addition, $\mathfrak{m}$ admits a minimal generating set, then $\mathrm{w.gl.dim}(R) = \infty$.

As an application, they provided an example of a Gaussian ring which is neither arithmetical nor coherent and has an infinite weak global dimension [3, Example 2.7]; which widened the scope of validity of the above conjecture beyond the class of coherent Gaussian rings.

In 2011, the authors of [1] investigated the correlation of fqp-rings with well-known Prüfer conditions; namely, they proved that the class of fqp-rings stands between the two classes of arithmetical rings and Gaussian rings [1, Theorem 3.1]. They also examined the transfer of the fqp-property to trivial ring extensions in order to build original examples of fqp-rings. Also they generalized Osofsky's result (mentioned above) and extended Bazzoni-Glaz's result on coherent Gaussian rings by proving that the weak global dimension of an fqp-ring is equal to 0, 1, or ∞ [1, Theorem 3.11]; and then they provided an example of an fqp-ring that is neither arithmetical nor coherent [1, Example 3.9].

Recently, several papers have appeared in the literature investigating the weak global dimension of various settings subject to Prüfer conditions. This survey paper plans to track and study these works dealing with this topic from the very origin; that is, 1969 Osofsky's proof of the existence of a module with infinite projective dimension on a local arithmetical ring. Precisely, we will examine all main results published in [1, 3, 5, 14, 22].

Our goal is to make this topic accessible and appealing to a broad audience; including graduate students. For this purpose, we present complete proofs of all main results via ample details and simplified arguments along with exact references. Further, numerous examples -arising as trivial ring extensions- are provided to illustrate the concepts and results involved in this paper. We assume familiarity with the basic tools used in the homological aspects of commutative ring theory, and any unreferenced material is standard as in [2, 6, 8, 13, 17, 19, 23, 27].

2 Weak global dimension of arithmetical rings

In this section, we provide a detailed proof for Osofsky's Theorem that the weak global dimension of an arithmetical ring with zero divisors is infinite. In fact, this result enables one to state that the weak global dimension of an arithmetical ring is 0, 1, or ∞. We start by recalling some basic definitions.

Definition 2.1. Let R be a ring and M an R-module. Then:

(1) The weak dimension of M, denoted by $\mathrm{w.dim}(M)$, measures how far M is from being flat. It is defined as follows: Let n be a positive integer. We have $\mathrm{w.dim}(M) \leq n$ if there is a flat resolution

$$0 \to E_n \to E_{n-1} \to \ldots \to E_1 \to E_0 \to M \to 0.$$

If no such resolution exists, $\mathrm{w.dim}(M) = \infty$; and if n is the least such integer, $\mathrm{w.dim}(M) = n$.

(2) The weak global dimension of R, denoted by $\mathrm{w.gl.dim}(R)$, is the supremum of $\mathrm{w.dim}(M)$, where M ranges over all (finitely generated) R-modules.

Definition 2.2. Let R be a ring. Then:

(1) R is said to be a chained ring (or valuation ring) if its lattice of ideals is totally ordered by inclusion.

(2) R is called an arithmetical ring if if $R_{\mathfrak{m}}$ is a chained ring for each maximal ideal m of R.

Fields and $\mathbb{Z}_{(p)}$, where $\mathbb{Z}$ is the ring of integers and p is a prime number, are examples of chained rings. Also, $\mathbb{Z}/n^2\mathbb{Z}$ is an arithmetical ring for any positive integer n. For more examples, see [3]. For a ring R, let $\mathrm{Z}(R)$ denote the set of all zero divisors of R.

Next we give the main theorem of this section.

Theorem 2.3. *Let R be an arithmetical ring. Then* $\mathrm{w.gl.dim}(R) =$ *0, 1, or* ∞.

To prove this theorem we make the following reductions:

(1) We may assume that R is a chained ring since $\mathrm{w.gl.dim}(R)$ is the supremum of $\mathrm{w.gl.dim}(R_{\mathfrak{m}})$ for all maximal ideal $\mathfrak{m}$ of R [13, Theorem 1.3.14 (1)].

(2) We may assume that R is a chained ring with zero divisors. Then we prove that $\mathrm{w.gl.dim}(R) = \infty$ since, if R is a valuation domain, then $\mathrm{w.gl.dim}(R) \leq 1$ by [13, Corollary 4.2.6].

(3) Finally, we may assume that $(R,\mathfrak{m})$ is a chained ring with zero divisors such that $\mathrm{Z}(R)=\mathfrak{m}$, since $\mathrm{Z}(R)$ is a prime ideal, $\mathrm{Z}(R_{\mathrm{Z}(R)})=\mathrm{Z}(R)R_{\mathrm{Z}(R)}$, and $\mathrm{w.gl.dim}(R_{\mathrm{Z}(R)})\leq\mathrm{w.gl.dim}(R)$.

So our task is reduced to prove the following theorem.

Theorem 2.4 ([22, Theorem]). *Let $(R,\mathfrak{m})$ be a chained ring with zero divisors such that $\mathrm{Z}(R)=\mathfrak{m}$. Then $\mathrm{w.gl.dim}(R)=\infty$.*

To prove this theorem we first prove the following lemmas. Throughout, let $(R,\mathfrak{m})$ be a chained ring with $\mathrm{Z}(R)=\mathfrak{m}$, M an R-module, $I=\{x\in R\mid x^2=0\}$, and for $x\in M$, $(0:x)=\{y\in R\mid yx=0\}$. One can easily check that I is a nonzero ideal since R is a chained ring with zero divisors.

Lemma 2.5 ([22, Lemma 1]). *$I^2=0$, and for all $x\notin R$, $x\notin I\Rightarrow(0:x)\subseteq I$.*

Proof. To prove that $I^2=0$, it suffices to prove that $ab=0$ for all $a,b\in I$. So let $a,b\in I$. Then either $a\in bR$ or $b\in aR$, so that $ab\in a^2R=0$ or $ab\in b^2R=0$.

Now let $x\in R\setminus I$ and $y\in(0:x)$. Then either $x\in yR$ or $y\in xR$. But $x\in yR$ implies that $x^2\in xyR=0$, absurd. Therefore $y\in xR$, so that $y^2\in xyR=0$. Hence $y\in I$.

Lemma 2.6 ([22, Lemma 2]). *Let $0\neq x\in\mathrm{Z}(R)$ such that $(0:x)=yR$. Then $\mathrm{w.gl.dim}(R)=\infty$.*

Proof. We first prove that $(0:y)=xR$. The inclusion $(0:y)\supseteq xR$ is trivial since $xy=0$. Now to prove the other inclusion let $z\in(0:y)$. Then either $z=xr$ for some $r\in R$ and in this case we are done, or $x=zj$ for some $j\in R$. We may assume $j\in\mathfrak{m}$. Otherwise, j is a unit and then we return to the first case. Since $x\neq 0$, $j\notin(0:z)$, so $jR\nsubseteq(0:z)$ which implies $(0:z)\subseteq jR$, and hence $y=jk$ for some $k\in\mathfrak{m}$. But then $0=zy=zjk=xk$, so $k\in(0:x)=yR$, and hence $k=yr$ for some $r\in R$. Hence $y=kj=yrj$, and as $j\in\mathfrak{m}$ we have the equality $y=y(1-rj)(1-rj)^{-1}=0$, which contradicts the fact that x is a zero divisor. Hence $z\in xR$, and therefore $(0:y)=xR$.

Now let m_x (resp., m_y) denote the multiplication by x (resp., y). Since $(0:x)=yR$ and $(0:y)=xR$ we have the following infinite flat resolution of xR with syzygies xR and yR:

$$\ldots\longrightarrow R\xrightarrow{m_y}R\xrightarrow{m_x}R\xrightarrow{m_y}\ldots\xrightarrow{m_y}R\xrightarrow{m_x}xR\longrightarrow 0$$

We claim that xR and yR are not flat. Indeed, recall that a projective module over a local ring is free [23]. So no projective module is annihilated by x or y. Since xR is annihilated by y and yR is annihilated by x, both xR and yR are not projective. Further, xR and yR are finitely presented in view of the exact sequence $0\to yR\to R\to xR\to 0$. It follows that xR and yR are not flat (since a finitely presented flat module is projective [23, Theorem 3.61]).

Corollary 2.7 ([22, Corollary]). *If $I=\mathfrak{m}$, then I is cyclic and R has infinite weak global dimension.*

Proof. Assume that $I=\mathfrak{m}$. Then $\mathfrak{m}^2=0$. Now let $0\neq a\in\mathfrak{m}$. We claim that $\mathfrak{m}=aR$. Indeed, let $b\in\mathfrak{m}$. Since R is a chained ring, either $b=ra$ for some $r\in R$ and in this case we are done, or $a=rb$ for some $r\in R$. In the later case, either r is a unit and then $b=r^{-1}a\in aR$, or $r\in\mathfrak{m}$ which implies $a=rb=0$, which contradicts the assumption $a\neq 0$. Thus $\mathfrak{m}=aR$, as claimed. Moreover, we have $(0:a)=aR$. Indeed, $(0:a)\supseteq aR$ since $a\in I$; if $x\in(0:a)$, then $x\in \mathrm{Z}(R)=\mathfrak{m}=aR$. Hence $(0:a)=aR$. It follows that R satisfies the conditions of Lemma 2.6 and hence the weak global dimension of R is ∞.

Throughout, an element x of an R- module M is said to be regular if $(0:x)=0$.

Lemma 2.8 ([22, Lemma 3]). *Let F be a free module and $x\in F$. Then x is contained in zR for some regular element z of F.*

Proof. Let $\{y_\alpha\}$ be a basis for F and let $x:=\sum_{i=1}^{n}y_ir_i\in F$, where $r_i\in R$. Since R is a chained ring, there is $j\in\{1,2,...,n\}$ such that $\sum_{i=1}^{n}r_iR\subseteq r_jR$. So that for each $i\in\{1,2,...,n\}$, $r_i=r_js_i$ for some $s_i\in R$ with $s_j=1$. Hence $x=r_j(\sum_{i=1}^{n}(y_is_i))$. We claim that $z:=\sum_{i=1}^{n}y_is_i$ is regular. Suppose not and let $t\in R$ such that $t(\sum_{i=1}^{n}y_is_i)=0$. Then $ts_i=0$ for all $i\in\{1,2,...,n\}$. In particular $t=ts_j=0$, absurd. Therefore z is regular and $x=r_jz$, as desired.

Note, for convenience, that in the proof of Theorem 2.4 (below) we will prove the existence of a module M satisfying the conditions (1) and (2) of the next lemma; which will allow us to construct -via iteration- an infinite flat resolution of M.

Lemma 2.9 ([22, Lemma 4]). *Assume that $(0:r)$ is infinitely generated for all $0\neq r\in\mathfrak{m}$. Let M be an R-submodule of a free module N such that:*

(1) $M=M_1\bigcup M_2\bigcup M_3$, where $M_1=\bigcup\limits_{\substack{x\in M\\ x\ regular}}xR$, $M_2=\bigcup\limits_{i=0}^{\infty}yu_iR$, with y regular in N, $u_iR\subsetneqq u_{i+1}R$, and yu_i is not in M_1, and $M_3=\sum v_jR$.

(2) $yu_0R\cap xR$ is infinitely generated for some regular $x\in M$.

Let F be a free R-module with basis $\{y_x\mid x \text{ regular}\in M\}\cup\{z_i\mid i\in\omega\}\cup\{w_j\}$, and let $v:F\longrightarrow N$ be the map defined by: $v(y_x)=x$, $v(z_i)=yu_i$, and $v(w_j)=v_j$. Then $K=Ker(v)$ has properties $(1),(2)$, and M is not flat.

Proof. First the map v exists by [19, Theorem 4.1]. (1) By (2), there exist $r,s\in R$ such that $yu_0r=xs\neq 0$. Here $r\in\mathfrak{m}$; otherwise, $yu_0=xsr^{-1}\in M_1$, contradiction. Since $\mathrm{Z}(R)=\mathfrak{m}$, the expression for any regular element in terms of a basis for N has one coefficient a unit. Indeed, let $(n_\alpha)_{\alpha\in\Delta}$ be a basis for N and z a regular element in N with $z=\sum_{i=0}^{i=k}c_in_i$ where $c_i\in R$. As R is a chained ring, there exists $j\in\{0,...,k\}$

such that for all $i \in \{0,...,k\}$, there exists $d_i \in R$ with $c_i = c_j d_i$ and $d_j = 1$. We claim that c_j is a unit. Suppose not. Then $c_j \in Z(R)$. So there is a nonzero $d \in R$ with $dc_j = 0$, and hence $dz = dc_j \sum_{i=0}^{i=k} d_i n_i = 0$. This is absurd since z is regular.

Now, let $x = \sum_{\substack{i \in I \\ I \text{ finite}}} a_i n_i$ and $y = \sum_{\substack{i \in I \\ I \text{ finite}}} b_i n_i$. Then $b_i u_0 r = a_i s$ for all $i \in I$. Let $i_0 \in I$ such that a_{i_0} is a unit. So $s = u_0 r t$, where $t = b_{i_0} a_{i_0}^{-1} \in R$. Note that $b_{i_0} \neq 0$ since $xs \neq 0$. Clearly, $z_0 - y_x u_0 t$ is regular in F (since z_0, y_x are part of the basis of F), is not in K (otherwise, $v(z_0 - y_x u_0 t) = 0$ yields $yu_0 = xu_0 t$, which contradicts (1)), and $(z_0 - y_x u_0 t) r \in K$. We claim that $(z_0 - y_x u_0 t) r$ is not in $K_1 := \bigcup_{\substack{x' \in K \\ x' \text{ regular}}} x'R$. Suppose not and assume that $r(z_0 - u_0 t y_x) = r'x'$ with $r' \in R$ and x' regular in K. Then $r' \neq 0$ since $r \neq 0$ and as $x' \in K \subseteq F$, there are $a, b, a_i \in R$ such that $x' = az_0 - by_x + x''$, where $x'' = \sum_{\substack{y_x \neq f_i \\ z_0 \neq f_i}} a_i f_i$. Thus $r = r'a$, $ru_0 t = r'b$, and $r'x'' = 0$. Since x' is regular in F and $r'x'' = 0$, a or b is unit. We claim that a is always a unit. Indeed, if b is a unit, then $r(1 - ab^{-1}u_0 t) = 0$, so if $a \in \mathfrak{m}$, then $(1 - ab^{-1}u_0 t)$ is a unit which implies $r = 0$, absurd. So $a^{-1}x' = z_0 - a^{-1}by_x + a^{-1}x''$, $r' = a^{-1}r$, and $ru_0 t = ra^{-1}b$ which implies $z_0 - u_0 t y_x + (u_0 t - a^{-1}b)y_x + a^{-1}x'' = a^{-1}x' \in K$. By Lemma 2.8 $(u_0 t - a^{-1}b)y_x + a^{-1}x'' = pq$, fore some q regular in F and $p \in R$. But clearly since $r = r'a$, $ru_0 t = r'b$, and $r'x'' = 0$, then $rpq = 0$. Hence $rp = 0$. It follows that $(z_0 - y_x u_0 t + qp) \in K$, where q is regular in F and $p \in (0 : r)$. Thus by applying v we obtain $yu_0 - xu_0 t + pv(q) = 0$. But R is a chained ring, so p and $u_0 t$ are comparable and since $u_0 t r \neq 0$, $p = u_0 t h$ for some $h \in R$. Hence $yu_0 = (x - hv(q))u_0 t$, we show that $(x - hv(q))$ is regular in M which contradicts property (1). First clearly $(x - hv(q)) \in M$ since $x, v(q) \in M$. Now suppose that $a(x - hv(q)) = 0$ for some $a \in \mathfrak{m}$. Either $u_0 t = a'a$ for some $a' \in R$, this yields $yu_0 = (x - hv(q))aa' = 0$ also impossible, or $a = u_0 t m$ for some $m \in R$, and this yields $mu_0 y = (x - hv(q))a = 0$, so $mu_0 = 0$ as y is regular, and hence $a = mu_0 t = 0$. We conclude that $(x - hv(q))$ is regular in M and hence $yu_0 \in M_1$, the desired contradiction.

Last, let $yu_0 R \cap xR = \langle x_0, x_1, ..., x_n, ... \rangle$, where

$$\langle x_0, x_1, ..., x_i \rangle \subsetneqq \langle x_0, x_1, ..., x_i, x_{i+1} \rangle.$$

For any integer $i \geq 0$, let $x_i = yu_0 r_i$ for some $r_i \in R$. It is clear that $r_0 R \subsetneqq r_1 R \subsetneqq ... \subsetneqq r_i R \subsetneqq r_{i+1} R \subsetneqq ...$. Now, let $y' := z_0 - y_x u_0 t$, $u'_i := r_i$ for each $i \in \mathbb{N}$. Then $K = K_1 \bigcup K_2 \bigcup K_3$, where $K_1 := \bigcup_{\substack{x' \in K \\ x' \text{ regular}}} x'R$, $K_2 := \bigcup_{i=0}^{\infty} y' u'_i R$ with y' regular in F and $u'_i R \subsetneqq u'_{i+1} R$, and $K_3 := K \setminus (K_1 \bigcup K_2)$. Thus K satisfy Property (1).

(2) Since $u_0 R \subsetneqq u_1 R$, $u_0 = u_1 m'$ for some $m' \in \mathfrak{m}$. Hence $x' := z_0 - z_1 m'$ is regular in K since $v(x') = v(z_0 - z_1 m') = yu_0 - yu_1 m' = 0$ and z_0, z_1 are basis elements. We claim that $(z_0 - z_1 m')R \cap (z_0 - y_x u_0 t) r_0 R = z_0 (0 : m')$. Indeed, since z_0, z_1, y_x

are basis elements, then $(z_0 - z_1 m')R \cap (z_0 - y_x u_0 t)r_0 \subseteq z_0 R$. Also $(z_0 - z_1 m')R \cap z_0 R = z_0(0 : m')$. For, let $l \in (z_0 - z_1 m')R \cap z_0 R$. Then $l = (z_0 - z_1 m')a = z_0 a'$ for some $a, a' \in R$. Hence $a = a'$ and $am' = 0$, whence $l = az_0$ with $am' = 0$. So $l \in z_0(0 : m')$. The reverse inclusion is straightforward. Consequently, $(z_0 - z_1 m')R \cap (z_0 - y_x u_0 t)r_0 R \subseteq z_0(0 : m')$. To prove the reverse inclusion, let $k \in (0 : m')$. Then either $k = r_0 k'$ or $r_0 = kk'$, for some $k' \in R$. The second case is impossible since $r_0 u_0 \neq 0$. Hence $z_0 k = (z_0 - y_x u_0 t)r_0 k' \in (z_0 - y_x u_0 t)r_0 R$. Further, $z_0 k \in (z_0 - z_1 m')R$. Therefore our claim is true. But z_0 is regular, so $z_0(0 : m') \cong (0 : m')$ which is infinitely generated by hypothesis. Therefore $y' u_0' R \cap x' R$ is infinitely generated, as desired.

Finally, M is not flat. Suppose not, then by [23, Theorem 3.57], there is an R-map $\theta : F \longrightarrow K$ such that $\theta\left((z_0 - y_x u_0 t)r_0\right) = (z_0 - y_x u_0 t)r_0$. Assume that $\theta(z_0) = az_0 + by_x + Z_1$ for some $a, b \in R$ and $\theta(y_x) = a' z_0 + b' y_x + Z_2$ for some $a', b' \in R$. Then $r_0 a - r_0 u_0 t a' = r_0$, $r_0 b - r_0 u_0 t b' = -r_0 u_0 t$, and $r_0 Z_1 - r_0 u_0 t Z_2 = 0$. Hence $r_0(1 - a + u_0 t a') = 0$ and since $r_0 \neq 0$, a or a' is a unit. Suppose that a is a unit and without loss of generality we can assume that $a = 1$. Thus we have the equation $z_0 - u_0 t y_x - u_0 t a' z_0 + (u_0 t - u_0 t b' + b)y_x + Z_1 - u_0 t Z_2 = \theta(z_0) - u_0 t \theta(Z_2) \in K$. By Lemma 2.8, $-u_0 t a' z_0 + (u_0 t - u_0 t b' + b)y_x + Z_1 - u_0 t Z_2 = pq$, where q is regular in F and, clearly, $r_0 p = 0$ since $r_0 u_0 t a' = 0$. Thus $z_0 - u_0 t y_x + pq \in K$, which is absurd (as seen before in the second paragraph of the proof of Lemma 2.9).

Now we are able to prove Theorem 2.4.

Proof of Theorem 2.4. If $(0 : r)$ is cyclic for some $r \in \mathfrak{m}$, then R has infinite weak global dimension by Lemma 2.6. Next suppose that $(0 : r)$ is not cyclic, for all $0 \neq r \in \mathfrak{m}$. Which is equivalent to assume that $(0 : r)$ is infinitely generated for all $0 \neq r \in \mathfrak{m}$, since R is a chained ring.

Let $0 \neq a \in I$ and $b \in \mathfrak{m} \setminus I$. Note that b exists since $I \neq \mathfrak{m}$ by the proof of Corollary 2.7. Let N be a free R-module on two generators y, y' and let $M := (y - y'b)R + y(0 : a)$. Then:

(A) $M_1 := \bigcup\limits_{\substack{x \in M \\ x \text{ regular}}} xR = \{(yt - y'b)r \mid 1 - t \in (0 : a), r \in R)$. To show this equality, let c be a regular element in M. Then $c = (r_1 + r_2)y - r_1 b y'$ for some $r_1 \in R, r_2 \in (0 : a)$. We claim that r_1 is a unit. Suppose not. So either $r_1 \in (r_2)$ hence $ac = 0$, or $r_2 = nr_1$ for some $n \in R$ and since $r_1 \in \mathfrak{m} = \mathrm{Z}(R)$, there is $r_1' \neq 0$ such that $r_1 r_1' = 0$, so $r_1' c = 0$. In both cases there is a contradiction with the fact that c is regular. Thus, r_1 is a unit. It follows that $c = (1 + r_1^{-1} r_2)y r_1 - by' r_1 \in \{(yt - y'b)r \mid 1 - t \in (0 : a), r \in R\}$. Now let $c = yt - y'b$, where $(1 - t) \in (0 : a)$. Then c is regular. Indeed, if $rc = 0$ for some $r \in R$, then $rt = 0$. Moreover, either $r = na$ for some $n \in R$, and in this case $r(1 - t) = na(1 - t) = 0$, so $r = rt = 0$ as desired, or $a = nr$ for some $n \in R$, so $a = at = nrt = 0$, absurd.

(B) There exists a countable chain of ideals $u_0 R \subsetneqq u_1 R \subsetneqq \ldots$ where $u_i \in (0 : a) \setminus (0 : b)$. Since $0 \neq a \in I$ and $b \in \mathfrak{m} \setminus I$, $(a) \subseteq (b)$. Thus $(0 : b) \subseteq (0 : a)$. Moreover $(0 : b) \subsetneqq (0 : a)$; otherwise, $a \in (0 : a) = (0 : b)$, and hence $ab = 0$. Hence $b \in (0 : a) = (0 : b) \subseteq I$ by Lemma 2.5, absurd. Now let $u_0 \in (0 : a) \setminus (0 : b)$. Since $(0 : a)$ is infinitely generated, there are $u_1, u_2, ...$ such that $(u_0) \subsetneqq (u_0, u_1) \subsetneqq ... \subseteq (0 : a)$. So $u_0R \subsetneqq u_1R \subsetneqq ...$ and necessarily $u_i \notin (0 : b)$ for all $i \geq 1$ since $u_0 \notin (0 : b)$.

Note that $yu_i \in M$(since $u_i \in (0 : a)$). Also $yu_i \notin M_1$; otherwise, if $yu_i = ytr - y'br$ with $1 - t \in (0 : a)$ and $r \in R$, then $u_i = tr$ and $br = 0$. Hence $bu_i = btr = 0$ and thus $u_i \in (0 : b)$, contradiction. Also note that y is regular in N (part of the basis) and $y \notin M$; if $y = (y - y'b)r_1 + r_2y$ with $r_1 \in R$ and $r_2 \in (0 : a)$, then $r_1b = 0$ and $r_1 + r_2 = 1$. So $r_1 \in \mathfrak{m}$, $ar_1 = a$, and hence $a = 0$, absurd.

(A) and (B) imply that (1) of Lemma 2.9 holds.

Let us show that $yu_0R \cap (y - y'b)R = y(0 : b)$. Indeed, if $c = yu_0r = (y - y'b)r'$ where $r, r' \in R$, then $u_0r = r'$ and $r'b = 0$. Hence $c \in y(0 : b)$. If $c = ry$ where $rb = 0$, then $r = u_0t$ for some $t \in R$ as $u_0 \in (0 : a) \setminus (0 : b)$. Thus $c = r(y - y'b)$. Now $y(0 : b) \cong (0 : b)$ is infinitely generated. Therefore (2) of Lemma 2.9 holds.

Since K satisfies the properties of M we can consider it as a new module M, and then there is a free module F_1 and a map $v_1 : F_1 \longrightarrow F$ such that $K_1 = Ker(v_1)$ satisfies the same conditions of K and K_1 is not flat. We can repeat this iteration above to get the infinite flat resolution of M:

$$... \to F_n \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0.$$

with none of the syzygies $K, K_1, K_2, ...$ is flat. Therefore R has an infinite weak global dimension. □

3 Weak global dimension of Gaussian rings

In 2005, Glaz proved that if R is a Gaussian coherent ring, then $\mathrm{w.gl.dim}(R) = 0$, 1, or ∞ [14]. In this section, we will see that the same conclusion holds for the larger class of Prüfer coherent rings and fore some contexts of Gaussian rings. We start by recalling the definitions of Gaussian, Prüfer, and coherent rings.

Definition 3.1. Let R be a ring. Then:

(1) R is called a Gaussian ring if for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$, where $c(f)$, the content of f, is the ideal of R generated by the coefficients of f.
(2) R is called a Prüfer ring if every nonzero finitely generated regular ideal is invertible (or, equivalently, projective)
(3) R is called a coherent ring if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R.

Recall that Arithmetical ring ⇒ Gaussian ring ⇒ Prüfer ring. To see the proofs of the above implications and that they cannot be reversed, in general, we refer the reader to [5, 14, 15] and Section 5 of this paper.

Noetherian rings, valuation domains, and $K[x_1, x_2, ...]$ where K is a field are examples of coherent rings. For more examples, see [13].

Let $Q(R)$ denote the total ring of fractions of R and $\mathrm{Nil}(R)$ its nilradical. The following proposition is the first main result of this section.

Proposition 3.2 ([5, proposition 6.1]). *Let R be a coherent Prüfer ring. Then the weak global dimension of R is equal to 0, 1, or ∞.*

The proof of this proposition relies on the following lemmas. Recall that a ring R is called regular if every finitely generated ideal of R has a finite projective dimension; and von Neumann regular if every R-module is flat.

Lemma 3.3 ([13, Corollary 6.2.4]). *Let R be a coherent regular ring. Then $Q(R)$ is a von Neumann regular ring.* □

Lemma 3.4 ([14, Lemma 2.1]). *Let R be a local Gaussian ring and $I = (a_1, ..., a_n)$ be a finitely generated ideal of R. Then $I^2 = (a_i^2)$, for some $i \in \{1, 2, ..., n\}$.*

Proof. We first assume that $I = (a, b)$. Let $f(x) := ax + b$, $g(x) := ax - b$, and $h(x) := bx + a$. Since R is Gaussian, $c(fg) = c(f)c(g)$, so that $(a,b)^2 = (a^2, b^2)$, also $c(fh) = c(f)c(h)$ which implies that $(a,b)^2 = (ab, a^2 + b^2)$. Hence $(a^2, b^2) = (ab, a^2 + b^2)$, whence $a^2 = rab + s(a^2 + b^2)$, for some r and s in R. That is, $(1-s)a^2 + rab + sb^2 = 0$. Since R is a local ring, either s or $1-s$ is a unit in R. If s is a unit in R, then $b^2 + rs^{-1}ab + (s^{-1} - 1)a^2 = 0$. Next we show that $ab \in (a^2)$. Let $k(x) := (b + \alpha a)x - a$, where $\alpha := rs^{-1}$. Then $c(hk) = c(h)c(k)$ implies that $(b(b+\alpha a), \alpha a^2, -a^2) = (a,b)((b+\alpha a), a)$. But clearly $(b(b+\alpha a), \alpha a^2, -a^2) = ((s^{-1}-1)a^2, \alpha a^2, -a^2) = (a^2)$. Thus $(a^2) = (a,b)((b+\alpha a), a)$. In particular, $ab \in (a^2)$ and so does b^2. If $1-s$ is unit, similar arguments imply that ab, and hence $a^2 \in (b^2)$. Thus for any two elements a and b, $ab \in (b^2)$ or (a^2). It follows that $I^2 = (a_1, ..., a_n)^2 = (a_1^2, ..., a_n^2)$. An induction on n leads to the conclusion.

Recall that a ring R is called reduced if it has no non-zero nilpotent elements.

Lemma 3.5 ([14, Theorem 2.2]). *Let R be a ring. Then* $\mathrm{w.gl.dim}(R) \leq 1$ *if and only if R is a Gaussian reduced ring.*

Proof. Assume that $\mathrm{w.gl.dim}(R) \leq 1$. By [13, Corollary 4.2.6], R_p is a valuation domain for every prime ideal p of R. As valuation domains are Gaussian, R is locally Gaussian, and therefore Gaussian. Further, R is reduced. For, let $x \in R$ such that x is nilpotent. We claim that $x = 0$. Suppose not and let $n \geq 2$ be an integer such that $x^n = 0$. Then there exists a prime ideal q in R such that $x \neq 0$ in R_q [2, Proposition 3.8]. It follows that $x^n = 0$ in R_q, a contradiction since R_q is a domain.

Conversely, since R is Gaussian reduced, R_p is a local, reduced, Gaussian ring for any prime ideal p of R. We claim that R_p is a domain. Indeed, let a and b in R_p such that $ab = 0$. By Lemma 3.4, $(a,b)^2$=$(b)^2$ or (a^2). Say $(a,b)^2 = (b^2)$. Then $a^2 = tb^2$ for some $t \in R_p$. Thus $a^3 = tb(ab) = 0$. Since R_p is reduced, $a = 0$, and R_p is a domain. Therefore R_p is a valuation domain for all prime ideals p of R. So $\mathrm{w.gl.dim}(R) \leq 1$ by [13, Corollary 4.2.6].

Lemma 3.6 ([5, Theorem 3.3]). *Let R be a Prüfer ring. Then R is Gaussian if and only if $Q(R)$ is Gaussian.* □

Lemma 3.7 ([5, Theorem 3.12(ii)]). *Let R be a ring. Then* $\mathrm{w.gl.dim}(R) \leq 1$ *if and only if R is a Prüfer ring and* $\mathrm{w.gl.dim}(Q(R)) \leq 1$.

Proof. If $\mathrm{w.gl.dim}(R) \leq 1$, R is Prüfer and, by localization, $\mathrm{w.gl.dim}(Q(R)) \leq 1$. Conversely, assume that R is a Prüfer ring such that $\mathrm{w.gl.dim}(Q(R)) \leq 1$. By Lemma 3.5, $Q(R)$ is a Gaussian reduced ring. So R is reduced and, by Lemma 3.6, R is Gaussian. By Lemma 3.5, $\mathrm{w.gl.dim}(R) \leq 1$.

Proof of Proposition 3.2. Assume that $\mathrm{w.gl.dim}(R) = n < \infty$ and let I be any finitely generated ideal of R. Then I has a finite weak dimension. Since R is a coherent ring, I is finitely presented. Hence the weak dimension of I equals its projective dimension by [13, Corollary 2.5.5]. Whence, as I is an arbitrary finitely generated ideal of R, R is a regular ring. So, by [13, Corollary 6.2.4], $Q(R)$ is von Neumann regular. By Lemma 3.7, $\mathrm{w.gl.dim}(R) \leq 1$. □

The following is an example of a coherent Prüfer ring with infinite weak global dimension.

Example 3.8. Let $R = \mathbb{R} \ltimes \mathbb{C}$. Then R is coherent by [20, Theorem 2.6], Prüfer by Theorem 4.2, and $\mathrm{w.gl.dim}(R) = \infty$ by Lemma 4.1.

In order to study the weak global dimension of an arbitrary Gaussian ring, we make the following reductions:

(1) We may assume that R is a local Gaussian ring since $\mathrm{w.gl.dim}(R)$ is the supremum of $\mathrm{w.gl.dim}(R_m)$ for all maximal ideal m of R [13, Theorem 1.3.14 (1)].

(2) We may assume that R is a non-reduced local Gaussian ring since every reduced Gaussian ring has weak global dimension at most 1 by Lemma 3.5.

(3) Finally, we may assume that $(R, \mathfrak{m})$ is a local Gaussian ring with the maximal ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathrm{Nil}(R)$. For, the prime ideals of a local Gaussian ring R are linearly ordered, so that $\mathrm{Nil}(R)$ is a prime ideal, and $\mathrm{w.gl.dim}(R) \geq \mathrm{w.gl.dim}(R_{\mathrm{Nil}(R)})$.

Next we announce the second main result of this section.

Theorem 3.9 ([5, Theorem 6.4]). *Let R be a Gaussian ring with a maximal ideal* $\mathfrak{m}$ *such that* $\mathrm{Nil}(R_{\mathfrak{m}})$ *is a nonzero nilpotent ideal. Then* $\mathrm{w.gl.dim}(R) = \infty$.

The proof of this theorem involves the following results:

Lemma 3.10. *Consider the following exact sequence of R-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

where M is flat. Then either the three modules are flat or $\mathrm{w.dim}(M'') = \mathrm{w.dim}(M') + 1$.

Proof. This is a classic result. We offer here a proof for the sake of completeness. Suppose that M'' is flat. Then by the long exact sequence theorem [23, Theorem 8.3] we get the exact sequence

$$0 = Tor_2(M'', N) \longrightarrow Tor_1(M', N) \longrightarrow Tor_1(M, N) = 0$$

for any R-module N. Hence $Tor_1(M',N) = 0$ which implies that M' is flat.

Next, assume that M'' is not flat. In this case, we claim that

$$\mathrm{w.dim}(M'') = \mathrm{w.dim}(M') + 1.$$

Indeed, let $\mathrm{w.dim}(M') = n$. Then we have the exact sequence

$$0 = Tor_{n+2}(M,N) \longrightarrow Tor_{n+2}(M'',N) \longrightarrow Tor_{n+1}(M',N) = 0$$

for any R-module N. Hence $Tor_{n+2}(M'',N) = 0$ for any R-module N which implies

$$\mathrm{w.dim}(M'') \leq n+1 = \mathrm{w.dim}(M') + 1$$

Now let $\mathrm{w.dim}(M'') = m$. Then we have the exact sequence

$$0 = Tor_{m+1}(M'',N) \longrightarrow Tor_m(M',N) \longrightarrow Tor_m(M,N) = 0$$

for any R-module N. Hence $Tor_m(M',N) = 0$ for any R-module N which implies that

$$\mathrm{w.dim}(M'') = m \geq \mathrm{w.dim}(M') + 1$$

Consequently, $\mathrm{w.dim}(M'') = \mathrm{w.dim}(M') + 1$.

Recall that an exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is pure if it remains exact when tensoring it with any R-module. In this case, we say that M' is a pure submodule of M [23].

Lemma 3.11 ([5, Lemma 6.2]). *Let $(R,\mathfrak{m})$ be a local ring which is not a field. Then* $\mathrm{w.dim}(R/\mathfrak{m}) = \mathrm{w.dim}(\mathfrak{m}) + 1$.

Proof. Consider the short exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0.$$

Assume that $R/\mathfrak{m}$ is flat. By [13, Theorem 1.2.15 (1,2,3)], $\mathfrak{m}$ is pure and $(aR)\mathfrak{m} = aR \cap \mathfrak{m} = aR$ for all $a \in \mathfrak{m}$. Hence $a\mathfrak{m} = aR$, for all $a \in \mathfrak{m}$, and so by Nakayama's Lemma, $a = 0$, absurd. By Lemma 3.10, $\mathrm{w.dim}(R/\mathfrak{m}) = \mathrm{w.dim}_R(\mathfrak{m}) + 1$.

Proposition 3.12 ([5, Proposition 6.3]). *Let $(R,\mathfrak{m})$ be a local ring with nonzero nilpotent maximal ideal. Then* $\mathrm{w.dim}(\mathfrak{m}) = \infty$.

Proof. Let n be the minimum integer such that $\mathfrak{m}^n = 0$. We claim that for all $1 \leq k < n$, $\mathrm{w.dim}(\mathfrak{m}^{n-k}) = \mathrm{w.dim}(\mathfrak{m}) + 1$. Indeed, let $k = 1$. Then $\mathfrak{m}^{n-1}\mathfrak{m} = 0$, so $\mathfrak{m}^{n-1}$ is an $(R/\mathfrak{m})$-vector space, hence $0 \neq \mathfrak{m}^{n-1} \cong \bigoplus R/\mathfrak{m}$, implies that $\mathrm{w.dim}_R(\mathfrak{m}^{n-1}) = \mathrm{w.dim}(R/\mathfrak{m}) = \mathrm{w.dim}(\mathfrak{m}) + 1$ by Lemma 3.11 . Now let h be the maximum integer in $\{1,...,n-1\}$ such that $\mathrm{w.dim}(\mathfrak{m}^{n-k}) = \mathrm{w.dim}(\mathfrak{m}) + 1$ for all $k \leq h$. Assume by way of contradiction that $h < n-1$. Then we have the exact sequence:

$$0 \to \mathfrak{m}^{n-h} \to \mathfrak{m}^{n-(h+1)} \to \mathfrak{m}^{n-(h+1)} / \mathfrak{m}^{n-h} \to 0 \qquad (*)$$

where $\mathfrak{m}^{n-(h+1)} / \mathfrak{m}^{n-h}$ is a nonzero $(R/\mathfrak{m})$-vector space. So by Lemma 3.11, we have $\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)} / \mathfrak{m}^{n-h}) = \mathrm{w.dim}(\mathfrak{m}) + 1$. By hypothesis, $\mathrm{w.dim}(\mathfrak{m}^{n-h}) = \mathrm{w.dim}(\mathfrak{m}) + 1$. Let us show that $\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)}) = \mathrm{w.dim}(\mathfrak{m}) + 1$. Indeed, if $l := \mathrm{w.dim}(\mathfrak{m}) + 1$, then by applying the long exact sequence theorem to $(*)$, we get

$$0 = Tor_{l+1}(\mathfrak{m}^{n-h}, N) \longrightarrow Tor_{l+1}(\mathfrak{m}^{n-(h+1)}, N) \longrightarrow Tor_{l+1}(\tfrac{\mathfrak{m}^{n-(h+1)}}{\mathfrak{m}^{n-h}}, N) = 0$$

for any R-module N. Hence $Tor_{l+1}(\mathfrak{m}^{n-(h+1)}, N) = 0$ for any R-module N which implies

$$\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)}) \le l = \mathrm{w.dim}(\mathfrak{m}) + 1$$

Further, if $\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)}) \lneqq l$, then we have

$$0 = Tor_{l+1}(\tfrac{\mathfrak{m}^{n-(h+1)}}{\mathfrak{m}^{n-h}}, N) \longrightarrow Tor_{l}(\mathfrak{m}^{n-h}, N) \longrightarrow Tor_{l}(\mathfrak{m}^{n-(h+1)}, N) = 0$$

for any R-module N. Hence $Tor_l(\mathfrak{m}^{n-h}, N) = 0$ for any R-module N which implies that $\mathrm{w.dim}(\mathfrak{m}^{n-h})) \le l - 1$, absurd. Hence $\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)}) = \mathrm{w.dim}(\mathfrak{m}) + 1$, the desired contradiction. Therefore the claim is true and, in particular, for $k = n-1$, we have $\mathrm{w.dim}(\mathfrak{m}) = \mathrm{w.dim}(\mathfrak{m}) + 1$, which yields $\mathrm{w.dim}(\mathfrak{m}) = \infty$.

Proof of Theorem 3.9. Suppose that R is Gaussian and $\mathfrak{m}$ is a maximal ideal in R such that $\mathrm{Nil}(R_{\mathfrak{m}})$ is a nonzero nilpotent ideal. Then $R_{\mathfrak{m}}$ is also Gaussian and $\mathrm{Nil}(R_{\mathfrak{m}})$ is a prime ideal in R. Moreover $\mathrm{Nil}(R_{\mathfrak{m}}) = pR_{\mathfrak{m}} \neq 0$ for some prime ideal p in R. Now, the maximal ideal pR_p of R_p is nonzero since $0 \neq pR_{\mathfrak{m}} \subseteq pR_p$. Also by assumption, there is a positive integer n such that $(pR_{\mathfrak{m}})^n = 0$, whence $p^n = 0$. So $(pR_p)^n = 0$ and hence pR_p is nilpotent. Therefore R_p is a local ring with nonzero nilpotent maximal ideal. By Proposition 3.12, $\mathrm{w.gl.dim}(R_p) = \infty$. Since $\mathrm{w.gl.dim}(R) \ge \mathrm{w.gl.dim}(R_S)$ for any localization R_S of R, we get $\mathrm{w.gl.dim}(R) = \infty$. $\square$

In the previous section, we saw that the weak global dimension of an arithmetical ring is 0, 1, or ∞. In this section, we saw that the same result holds if R is Prüfer coherent or R is a Gaussian ring with a maximal ideal $\mathfrak{m}$ such that $\mathrm{Nil}(R_{\mathfrak{m}})$ is a nonzero nilpotent ideal.

The question of whether this result is true for an arbitrary Gaussian ring was the object of Bazzoni-Glaz conjecture which sustained that the weak global dimension of a Gaussian ring is 0, 1, or ∞. In a first preprint [9], Donadze and Thomas claim to prove this conjecture in all cases except when the ring R is a non-reduced local Gaussian ring with nilradical N satisfying $N^2 = 0$. Then in a second preprint [10], they claim to prove the conjecture for all cases.

4 Gaussian rings via trivial ring extensions

In this section, we will use trivial ring extensions to construct new examples of non-arithmetical Gaussian rings , non-Gaussian Prüfer rings, and illustrative examples for Theorem 2.4 and Theorem 3.9. Let A be a ring and M an R-module. The trivial ring extension of A by M (also called the idealization of M over A) is the ring $R := A \ltimes M$ whose underlying group is $A \times M$ with multiplication given by

$$(a,x)(a',x') = (aa', ax' + a'x).$$

Recall that if I is an ideal of A and M' is a submodule of M such that $IM \subseteq M'$, then $J := I \ltimes M'$ is an ideal of R; ideals of R need not be of this form [20, Example 2.5]. However, the form of the prime (resp., maximal) ideals of R is $p \ltimes M$, where p is a prime (resp., maximal) ideal of A [17, Theorem 25.1(3)]. Suitable background on trivial extensions is [13, 17, 20].

The following lemma is useful for the construction of rings with infinite weak global dimension.

Lemma 4.1 ([3, Lemma 2.3]). *Let K be a field, E a nonzero K-vector space, and $R := K \ltimes E$. Then* $\mathrm{w.gl.dim}(R) = \infty$.

Proof. First note that $R^{(I)} \cong A^{(I)} \ltimes E^{(I)}$. So let us identify $R^{(I)}$ with $A^{(I)} \ltimes E^{(I)}$ as R-modules. Now let $\{f_i\}_{i\in I}$ be a basis of E and $J := 0 \ltimes E$. Consider the R-map $u : R^{(I)} \longrightarrow J$ defined by $u((a_i, e_i)_{i\in I}) = (0, \sum_{i\in I} a_i f_i)$. Then we have the following short exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ker}(u) \longrightarrow R^{(I)} \xrightarrow{u} J \longrightarrow 0$$

But $\mathrm{Ker}(u) = 0 \ltimes E^{(I)}$. Indeed, clearly $0 \ltimes E^{(I)} \subseteq \mathrm{Ker}(u)$. Now suppose $u((a_i, e_i)) = (0,0)$. Then $\sum_{i\in I} a_i f_i = 0$, hence $a_i = 0$ for each i as $\{f_i\}_{i\in I}$ is a basis for E and we have the equality. Therefore the above exact sequence becomes

$$0 \longrightarrow 0 \ltimes E^{(I)} \longrightarrow R^{(I)} \xrightarrow{u} J \longrightarrow 0 \qquad (*)$$

We claim that J is not flat. Suppose not. Then $0 \ltimes E^{(I)} \bigcap JR^{(I)} = (0 \ltimes E^{(I)})J$ by [23, Theorem 3.55]. But $(0 \ltimes E^{(I)})J = 0$. We use the above identification to obtain $0 = 0 \ltimes E^{(I)} \bigcap JR^{(I)} = (J)^{(I)} \bigcap J^{(I)} = J^{(I)} = 0 \ltimes E^{(I)}$, absurd (since $E \neq 0$).

Now, by Lemma 3.10, $\mathrm{w.dim}(J) = \mathrm{w.dim}(J^{(I)}) + 1 = \mathrm{w.dim}(J) + 1$. It follows that $\mathrm{w.gl.dim}(R) = \mathrm{w.dim}(J) = \infty$.

Next, we announce the main result of this section.

Theorem 4.2 ([3, Theorem 3.1]). *Let $(A, \mathfrak{m})$ be a local ring, E a nonzero $\frac{A}{\mathfrak{m}}$-vector space, and $R := A \ltimes E$ the trivial ring extension of A by E. Then:*

(1) *R is a total ring of quotients and hence a Prüfer ring.*
(2) *R is Gaussian if and only if A is Gaussian.*
(3) *R is arithmetical if and only if $A := K$ is a field and* $\dim_K(E) = 1$.

(4) $\mathrm{w.gl.dim}(R) \gneqq 1$. *If* $\mathfrak{m}$ *admits a minimal generating set, then* $\mathrm{w.gl.dim}(R)$ *is infinite.*

Proof. (1) Let $(a,e) \in R$. Then either $a \in \mathfrak{m}$ in which case we get $(a,e)(0,e) = (0,ae) = (0,0)$; or $a \notin \mathfrak{m}$ which implies a is a unit and hence $(a,e)(a^{-1},-a^{-2}e) = (1,0)$, the unity of R. Therefore R is a total ring of quotients and hence a Prüfer ring.

(2) Suppose that R is Gaussian. Then, since $A \cong \frac{R}{0\ltimes E}$ and the Gaussian property is stable under factor rings, A is Gaussian.

Conversely, assume that A is Gaussian and let $F := \sum(a_i,e_i)X^i$ be a polynomial in $R[X]$. If $a_i \notin \mathfrak{m}$ for some i, then (a_i,e_i) is invertible since we have $(a_i,e_i)(a_i^{-1},-a^{-2}e_i) = (1,0)$. We claim that F is Gaussian. Indeed, for any $G \in R[X]$, we have $c(F)c(G) = Rc(G) = c(G) \subseteq c(FG)$. The reverse inclusion always holds. If $a_i \in \mathfrak{m}$ for each i, let $G := \sum(a'_j,e'_j)X^j \in R[X]$. We may assume, without loss of generality, that $a'_j \in \mathfrak{m}$ for each j (otherwise, we return to the first case) and let $f := \sum a_iX^i$ and $g := \sum a'_jX^j$ in $A[X]$. Then $c(FG) = c(fg) \ltimes c(fg)E$. But since E is an $\frac{A}{\mathfrak{m}}$-vector space, $\mathfrak{m}E = 0$ yields $c(FG) = c(fg) \ltimes 0 = c(f)c(g) \ltimes 0 = c(F)c(G)$, since A is Gaussian. Therefore R is Gaussian, as desired.

(3) Suppose that R is arithmetical. First we claim that A is a field. On the contrary, assume that A is not a field. Then $\mathfrak{m} \neq 0$, so there is $a \neq 0 \in \mathfrak{m}$. Let $e \neq 0 \in E$. Since R is a local arithmetical ring (i.e., chained ring), either $(a,0) = (a',e')(0,e) = (0,a'e)$ for some $(a',e') \in R$ which contradicts $a \neq 0$; or $(0,e) = (a'',e'')(a,0) = (a'a,0)$ for some $(a'',e'') \in R$ which contradicts $e \neq 0$. Hence A is a field. Next, we show that $\dim_K(E) = 1$. Let e,e' be two nonzero vectors in E. We claim that they are linearly dependent. Indeed, since R is a local arithmetical ring, either $(0,e) = (a,e'')(0,e') = (0,ae')$ for some $(a,e'') \in R$, hence $e = ae'$; or similarly if $(0,e') \in (0,e)R$. Consequently, $\dim_K(E) = 1$.

Conversely, let J be a nonzero ideal in $K \ltimes K$ and let (a,b) be a nonzero element of J. So $(0,a^{-1})(a,b) = (0,1) \in J$. Hence $0 \ltimes K \subseteq J$. But $0 \ltimes K$ is maximal since 0 is the maximal ideal in K. So the ideals of $K \ltimes K$ are $(0,0)K \ltimes K$, $0 \ltimes K = R(0,1)$, and $K \ltimes K$. Therefore $K \ltimes K$ is a principal ring and hence arithmetical.

(4) First $\mathrm{w.gl.dim}(R) \gneqq 1$. Let $J := 0 \ltimes E$ and $\{f_i\}_{i\in I}$ be a basis of the $\frac{A}{\mathfrak{m}}$-vector space E. Consider the map $u : R^{(I)} \longrightarrow J$ defined by $u((a_i,e_i)_{i\in I}) = (0, \sum\limits_{i\in I} a_if_i)$. Here we are using the same identification that has been used in Lemma 4.1. Then clearly $\mathrm{Ker}(u) = (\mathfrak{m}\ltimes E)^{(I)}$. Hence we have the short exact sequence of R-modules

$$0 \longrightarrow (\mathfrak{m}\ltimes E)^{(I)} \longrightarrow R^{(I)} \xrightarrow{u} J \longrightarrow 0 \qquad (1)$$

We claim that J is not flat. Otherwise, by [23, Theorem3.55], we have

$$J^{(I)} = (\mathfrak{m}\ltimes E)^{(I)} \cap JR^{(I)} = J(\mathfrak{m}\ltimes E^{(I)}) = 0.$$

Hence, by [23, Theorem 2.44], $\mathrm{w.gl.dim}(R) \gneqq 1$.

Next, assume that $\mathfrak{m}$ admits a minimal generating set. Then $\mathfrak{m}\ltimes E$ admits a minimal generating set (since E is a vector space). Now let $(b_i,g_i)_{i\in L}$ be a minimal generating set of $\mathfrak{m}\ltimes E$. Consider the R-map $v : R^{(L)} \longrightarrow \mathfrak{m}\ltimes E$ defined by $v((a_i,e_i)_{i\in L}) = \sum\limits_{i\in L}(a_i,e_i)(b_i,g_i)$. Then we have the exact sequence

$$0 \longrightarrow \mathrm{Ker}(v) \longrightarrow R^{(L)} \xrightarrow{v} \mathfrak{m} \ltimes E \longrightarrow 0 \qquad (2)$$

We claim that $\mathrm{Ker}(v) \subseteq (\mathfrak{m} \ltimes E)^{(L)}$. On the contrary, suppose that there is $x = ((a_i, e_i)_{i\in L}) \in \mathrm{Ker}(v)$ and $x \notin (\mathfrak{m} \ltimes E)^{(L)}$. Then $\sum_{i\in L}(a_i, e_i)(b_i, g_i) = 0$ and as $x \notin (\mathfrak{m} \ltimes E)^{(L)}$, there is (a_j, e_j) with $a_j \notin \mathfrak{m}$. So that (a_j, e_j) is a unit, which contradicts the minimality of $(b_i, g_i)_{i\in L}$. It follows that

$$\mathrm{Ker}(v) = V \ltimes E^{(L)} = (V \ltimes 0) \bigoplus (0 \ltimes E^{(L)}) = (V \ltimes 0) \bigoplus J^{(L)}$$

where $V := \{(a_i)_{i\in L} \in \mathfrak{m}^i \mid \sum_{i\in L} a_i b_i = 0\}$. Indeed, if $x \in \mathrm{Ker}(v)$, then $x = (a_i, b_i)_{i\in L}$ where $a_i \in \mathfrak{m}$, $b_i \in E$, with $\sum_{i\in L} a_i b_i = 0$, hence $\mathrm{Ker}(v) \subseteq V \ltimes E^{(L)}$. The other inclusion is trivial. Now, by Lemma 3.10 applied to (1), we get

$$\mathrm{w.dim}(J) = \mathrm{w.dim}((\mathfrak{m} \ltimes E)^I) + 1 = \mathrm{w.dim}(\mathfrak{m} \ltimes E) + 1.$$

On the other hand, from (2) we obtain

$$\mathrm{w.dim}(J) \leq \mathrm{w.dim}(V \ltimes 0 \oplus J^L) = \mathrm{w.dim}(\mathrm{Ker}(v)) \leq \mathrm{w.dim}(\mathfrak{m} \ltimes E).$$

It follows that

$$\mathrm{w.dim}(J) \leq \mathrm{w.dim}(J) - 1.$$

Consequently, $\mathrm{w.gl.dim}(R) = \mathrm{w.dim}(J) = \infty$.

Next, we give examples of non-arithmetical Gaussian rings.

Example 4.3.(1) Let p be a prime number. Then $(\mathbb{Z}_{(p)}, p\mathbb{Z}_{(p)})$ is a non-trivial valuation domain. Hence $\mathbb{Z}_{(p)} \ltimes \frac{\mathbb{Z}}{p\mathbb{Z}}$ is a non-arithmetical Gaussian total ring of quotients by Theorem 4.2.

(2) Since $\dim_{\mathbb{R}}(\mathbb{C}) = 2 \gneqq 1$, $\mathbb{R} \ltimes \mathbb{C}$ is a non arithmetical Gaussian total ring of quotient. In general, if K is a field and E is a K-vector space with $\dim_K(E) \gneqq 1$, then $R := K \ltimes E$ is a non-arithmetical Gaussian total ring of quotients by Theorem 4.2.

Next, we provide examples of non-Gaussian total rings of quotients and hence non-Gaussian Prüfer rings.

Example 4.4. Let $(A, \mathfrak{m})$ be a non-valuation local domain. By Theorem 4.2, $R := A \ltimes \frac{A}{\mathfrak{m}}$ is a non-Gaussian total ring of quotients, hence a non-Gaussian Prüfer ring.

The following is an illustrative example for Theorem 2.4.

Example 4.5. Let $R := \mathbb{R} \ltimes \mathbb{R}$. Then R is a local ring with maximal ideal $0 \ltimes \mathbb{R}$ and $\mathrm{Z}(R) = 0 \ltimes \mathbb{R}$. Further, R is arithmetical by Theorem 4.2. By Osofsky's Theorem (Theorem 2.4) or by Lemma 4.1, $\mathrm{w.gl.dim}(R) = \infty$.

Now we give an example of a non-coherent local Gaussian ring with nilpotent maximal ideal and infinite weak global dimension (i.e., an illustrative example for Theorem 3.9).

Example 4.6. Let K be a field and X an indeterminate over K and let $R := K \ltimes K[X]$. Then:

(1) R is a non-arithmetical Gaussian ring since K is Gaussian and $\dim_K(K[X]) = \infty$ by Theorem 4.2.
(2) R is not a coherent ring since $\dim_K(K[X]) = \infty$ by [20, Theorem 2.6].
(3) R is local with maximal ideal $\mathfrak{m} = 0 \ltimes K[X]$ by [17, Theorem 25.1(3)]. Also $\mathfrak{m}$ is nilpotent since $\mathfrak{m}^2 = 0$. Therefore, by Theorem 3.9, $\mathrm{w.gl.dim}(R) = \infty$.

5 Weak global dimension of fqp-rings

Recently, Abuhlail, Jarrar, and Kabbaj studied commutative rings in which every finitely generated ideal is quasi-projective (fqp-rings). They investigated the correlation of fqp-rings with well-known Prüfer conditions; namely, they proved that fqp-rings stand strictly between the two classes of arithmetical rings and Gaussian rings [1, Theorem 3.2]. Also they generalized Osofsky's Theorem on the weak global dimension of arithmetical rings (and partially resolved Bazzoni-Glaz's related conjecture on Gaussian rings) by proving that the weak global dimension of an fqp-ring is 0, 1, or ∞ [1, Theorem 3.11]. In this section, we will give the proofs of the above mentioned results. Here too, the needed examples in this section will be constructed by using trivial ring extensions. We start by recalling some definitions.

Definition 5.1.(1) Let M be an R-module. An R-module M' is M-projective if the map $\psi : \mathrm{Hom}_R(M', M) \longrightarrow \mathrm{Hom}_R(M', \frac{M}{N})$ is surjective for every submodule N of M.
(2) M' is quasi-projective if it is M'-projective.

Definition 5.2. A commutative ring R is said to be an fqp-ring if every finitely generated ideal of R is quasi-projective.

The following theorem establishes the relation between the class of fqp-rings and the two classes of arithmetical and Gaussian rings.

Theorem 5.3 ([1, Theorem 3.2]). *For a ring R, we have*

$$R\ arithmetical \ \Rightarrow\ R\ fqp-ring \ \Rightarrow\ R\ Gaussian$$

where the implications are irreversible in general.

The proof of this theorem needs the following results.

Lemma 5.4 ([1, Lemma 2.2]). *Let R be a ring and let M be a finitely generated R-module. Then M is quasi-projective if and only if M is projective over* $\frac{R}{\mathrm{Ann}(M)}$. $\square$

Lemma 5.5 ([12, Corollary 1.2]). *Let $M_{i1\leq i\leq n}$ be a family of R-modules. Then: $\bigoplus_{i=1}^{n} M_i$ is quasi-projective if and only if M_i is M_j-projective $\forall\ i, j \in \{1,\ 2,\ \ldots,\ \}$.* $\square$

Lemma 5.6 ([1, Lemma 3.6]). *Let R be an fqp-ring. Then $S^{-1}R$ is an fqp-ring, for any multiplicative closed subsets of R.*

Proof. Let J be a finitely generated ideal of $S^{-1}R$. Then $J = S^{-1}I$ for some finitely generated ideal I of R. Since R is an fqp-ring, I is quasi-projective and hence, by Lemma 5.4, I is projective over $\frac{R}{\mathrm{Ann}(I)}$. By [23, Theorem 3.76], $J := S^{-1}I$ is projective over $\frac{S^{-1}R}{S^{-1}\mathrm{Ann}(I)}$. But $S^{-1}\mathrm{Ann}(I) = \mathrm{Ann}(S^{-1}I) = \mathrm{Ann}(J)$ by [2, Proposition 3.14]. Therefore $J := S^{-1}I$ is projective over $\frac{S^{-1}R}{\mathrm{Ann}(S^{-1}I)}$. Again by Lemma 5.4, J is quasi-projective. It follows that $S^{-1}R$ is an fqp-ring.

Lemma 5.7 ([1, Lemma 3.8]). *Let R be a local ring and $a,\ b$ two nonzero elements of R such that (a) and (b) are incomparable. If $(a,\ b)$ is quasi-projective, then $(a)\cap(b) = 0$, $a^2 = b^2 = ab = 0$, and $\mathrm{Ann}(a) = \mathrm{Ann}(b)$.*

Proof. Let $I := (a,\ b)$ be quasi-projective. Then by [26, Lemma 2], there exist $f_1,\ f_2 \in \mathrm{End}_R(I)$ such that $f_1(I) \subseteq (a)$, $f_2(I) \subseteq (b)$, and $f_1 + f_2 = 1_I$. Now let $x \in (a)\cap(b)$. Then $x = r_1a = r_2b$ for some $r_1,\ r_2\ \in R$. But $x = f_1(x) + f_2(x) = f_1(r_1a) + f_2(r_2b) = r_1f_1(a) + r_2f_2(b) = r_1a'a + r_2b'b = a'x + b'x$ where $a',\ b'\ \in R$. We claim that a' is a unit. Suppose not. Since R is local, $1-a'$ is a unit. But $a = f_1(a) + f_2(a) = a'a + f_2(a)$. Hence $(1-a')a = f_2(a) \subseteq (b)$ which implies that $a \in (b)$. This is absurd since (a) and (b) are incomparable. Similarly, b' is a unit. It follows that $(a' - (1-b'))$ is a unit. But $x = a'x + b'x$ yields $(a' - (1-b'))x = 0$. Therefore $x = 0$ and $(a)\cap(b) = 0$.

Next, we prove that $a^2 = b^2 = ab = 0$. Obviously, $(a)\cap(b) = 0$ implies that $ab = 0$. So it remains to prove that $a^2 = b^2 = 0$. Since $(a)\cap(b) = 0$, $I = (a)\oplus(b)$. By Lemma 5.5, (b) is (a)-projective. Let $\varphi : (a) \longrightarrow \frac{(a)}{a\,\mathrm{Ann}(b)}$ be the canonical map and $g : (b) \longrightarrow \frac{(a)}{a\,\mathrm{Ann}(b)}$ be defined by $g(rb) = r\bar{a}$. If $r_1b = r_2b$, then $(r_1 - r_2)b = 0$. Hence $r_1 - r_2 \in \mathrm{Ann}(b)$ which implies that $(r_1 - r_2)\bar{a} = 0$. So $g(r_1b) = g(r_2b)$. Consequently, g is well defined. Clearly g is an R-map. Now, since (b) is (a)-projective, there exists an R-map $f : (b) \longrightarrow (a)$ with $\varphi\circ f = g$. For b, we have $f(b) \in (a)$, hence $f(b) = ra$ for some $r \in R$. Also $(\varphi\circ f)(b) = g(b)$. Hence $f(b) - a \in a\,\mathrm{Ann}(b)$. Whence $ra - a = at$ for some $t \in \mathrm{Ann}(b)$ which implies that $(t+1)a = ra$. By multiplying the last equality by a we obtain, $(t+1)a^2 = ra^2$. But $ab = 0$ implies $0 = f(ab) = af(b) = ra^2$. Hence $(t+1)a^2 = 0$. Since $t \in \mathrm{Ann}(b)$ and R is local, $(t+1)$ is a unit. It follows that $a^2 = 0$. Likewise $b^2 = 0$.

Last, let $x \in \mathrm{Ann}(b)$. Then $f(xb) = xra = 0$. The above equality $(t+1)a = ra$ implies $(t+1-r)a = 0$. But $t+1$ is a unit and R is local. So that r is a unit ($b \neq 0$). Hence $xa = 0$. Whence $x \in \mathrm{Ann}(a)$ and $\mathrm{Ann}(b)\ \subseteq\ \mathrm{Ann}(a)$. Similarly we can show that $\mathrm{Ann}(a)\ \subseteq\ Ann(b)$. Therefore $\mathrm{Ann}(a)\ =\ \mathrm{Ann}(b)$.

Proof of Theorem 5.3. R arithmetical $\Rightarrow$ R fqp-ring.

Let R be an arithmetical ring, I a nonzero finitely generated ideal of R, and p a prime ideal of R. Then $I_p := IR_p$ is finitely generated. But R is arithmetical, hence R_p is a chained ring and I_p is a principal ideal of R_p. By [21], I_p is quasi-projective. By [28, 19.2] and [29], it suffices to prove that $(\mathrm{Hom}_R(I,\ I))_p \cong \mathrm{Hom}_{R_p}(I_p,\ I_p)$. But $\mathrm{Hom}_{R_p}(I_p,\ I_p) \cong \mathrm{Hom}_R(I,\ I_p)$ by the adjoint isomorphisms theorem [23, Theorem 2.11] (since $\mathrm{Hom}_{S^{-1}R}(S^{-1}N, S^{-1}M) \cong \mathrm{Hom}(N, S^{-1}M)$ where $S^{-1}N \cong N \bigotimes_R S^{-1}R$ and $S^{-1}M \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}M)$). So let us prove that

$$(\mathrm{Hom}_R(I,\ I))_p \cong \mathrm{Hom}_R(I,\ I_p).$$

Let

$$\phi : (\mathrm{Hom}_R(I,\ I))_p \longrightarrow \mathrm{Hom}_R(I,\ I_p)$$

be the function defined by $\frac{f}{s} \in (\mathrm{Hom}_R(I,\ I))_p$, $\phi(\frac{f}{s}) : I \longrightarrow I_p$ with $\phi(\frac{f}{s})(x) = \frac{f(x)}{s}$, for each $x \in I$. Clearly ϕ is a well-defined R-map. Now suppose that $\phi(\frac{f}{s}) = 0$. I is finitely generated, so let $I = (x_1,\ x_2, ...,\ x_n)$, where n is an integer. Then for every $i \in \{1,\ 2,\ ...,\ n\}$, $\phi(\frac{f}{s})(x_i) = \frac{f(x_i)}{s} = 0$, whence there exists $t_i \in R \setminus p$ such that $t_i f(x_i) = 0$. Let $t := t_1 t_2 ... t_n$. Clearly, $t \in R \setminus p$ and $tf(x) = 0$, for all $x \in I$. Hence $\frac{f}{s} = 0$. Consequently, ϕ is injective. Next, let $g \in \mathrm{Hom}_R(I,\ I_p)$. Since I_p is principal in R_p, $I_p = aR_p$ for some $a \in I$. But $g(a) \in I_p$. Hence $g(a) = \frac{ca}{s}$ for some $c \in R$ and $s \in R \setminus p$. Let $x \in I$. Then $\frac{x}{1} \in I_p = aR_p$. Hence $\frac{x}{1} = \frac{ra}{u}$ for some $r \in R$ and $u \in R \setminus p$. So there exists $t \in R \setminus p$ such that $tux = tra$. Now, let $f : I \longrightarrow I$ be the multiplication by c. (i.e., for $x \in I$, $f(x) = cx$). Then $f \in \mathrm{Hom}_R(I,\ I)$ and we have

$$\phi(\frac{f}{s})(x) = \frac{f(x)}{s} = \frac{cx}{s} = \frac{c}{s}\frac{x}{1} = \frac{cra}{su} = \frac{r}{u}g(a) = \frac{1}{tu}g(tra) = \frac{1}{tu}g(txu) = g(x).$$

Therefore ϕ is surjective and hence an isomorphism, as desired.

R fqp-ring $\Rightarrow$ R Gaussian

Recall that, if $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$, then R is a Gaussian ring if and only if for any two elements a, b in R, $(a,b)^2 = (a^2)$ or (b^2) and if $(a,b)^2 = (a^2)$ and $ab = 0$, then $b^2 = 0$ [5, Theorem 2.2 (d)].

Let R be an fqp-ring and let P be any prime ideal of R. Then by Lemma 5.6 R_p is a local fqp-ring. Let $a,\ b \in R_P$. We investigate two cases. The first case is $(a,\ b) = (a)$ or (b), say (b). So $(a,\ b)^2 = (b^2)$. Now assume that $ab = 0$. Since $a \in (b)$, $a = cb$ for some $c \in R$. Therefore $a^2 = cab = 0$. The second case is $I := (a,\ b)$ with $I \neq (a)$ and $I \neq (b)$. Necessarily, $a \neq 0$ and $b \neq 0$. By Lemma 5.7, $a^2 = b^2 = ab = 0$. Both cases satisfy the conditions that were mentioned at the beginning of this proof (The conditions of [5, Theorem 2.2 (d)]). Hence R_p is Gaussian. But p being an arbitrary prime ideal of R and the Gaussian notion being a local property, then R is Gaussian.

To prove that the implications are irreversible in general, we will use the following theorem to build examples for this purpose.

Theorem 5.8 ([1, Theorem 4.4]). *Let $(A,\ \mathfrak{m})$ be a local ring and E a nonzero $\frac{A}{\mathfrak{m}}$-vector space. Let $R := A \ltimes E$ be the trivial ring extension of A by E. Then R is an fqp-ring if and only if* $\mathfrak{m}^2 = 0$.

The proof of this theorem depends on the following lemmas.

Lemma 5.9 ([24, Theorem 2]). *Let R be a local fqp-ring which is not a chained ring. Then* $(\mathrm{Nil}(R))^2 = 0$.

Lemma 5.10 ([1, Lemma 4.5]). *Let R be a local fqp-ring which is not a chained ring. Then* $\mathrm{Z}(R) = \mathrm{Nil}(R)$.

Proof. We always have $\mathrm{Nil}(R) \subseteq \mathrm{Z}(R)$. Now, let $s \in \mathrm{Z}(R)$. Then there exists $t \neq 0 \in R$ such that $st = 0$. Since R is not chained, there exist nonzero elements $x,\ y\ \in R$ such that (x) and (y) are incomparable. By Lemma 5.7, $x^2 = xy = y^2 = 0$. Either (x) and (s) are incomparable and hence, by Lemma 5.7, $s^2 = 0$. Whence $s \in \mathrm{Nil}(R)$. Or (x) and (s) are comparable. In this case, either $s = rx$ for some $r \in R$ which implies that $s^2 = r^2x^2 = 0$ and hence $s \in \mathrm{Nil}(R)$. Or $x = sx'$ for some $x' \in R$. Same arguments applied to (s) and (y) yield either $s \in \mathrm{Nil}(R)$ or $y = sy'$ for some $y' \in R$. Since (x) and (y) are incomparable, (x') and (y') are incomparable. Hence, by Lemma 5.7, $(x') \cap (y') = 0$. If (x') and (t) are incomparable, then by Lemma 5.7, $\mathrm{Ann}(x') = \mathrm{Ann}(t)$. So that $s \in \mathrm{Ann}(x')$ which implies that $x = sx' = 0$, absurd. If $(t) \subseteq (x')$, then $(t) \cap (y')\ \subseteq (x') \cap (y') = 0$. So (t) and (y') are incomparable, whence similar arguments as above yield $y = 0$, absurd. Last, if $(x') \subseteq (t)$, then $x' = r't$ for some $r' \in R$. Hence $x = sx' = str' = 0$, absurd. Therefore all the possible cases lead to $s \in \mathrm{Nil}(R)$. Consequently, $\mathrm{Z}(R) = \mathrm{Nil}(R)$.

Lemma 5.11 ([1, Lemma 4.6]). *Let $(R,\ \mathfrak{m})$ be a local ring such that $\mathfrak{m}^2 = 0$. Then R is an fqp-ring.*

Proof. Let I be a nonzero proper finitely generated ideal of R. Then $I \subsetneq \mathfrak{m}$ and $\mathfrak{m}I = 0$. Hence $\mathfrak{m} \subseteq \mathrm{Ann}(I)$, whence $\mathfrak{m} = \mathrm{Ann}(I)$ $(I \neq 0)$. So that $\frac{R}{\mathrm{Ann}(I)} \cong \frac{A}{\mathfrak{m}}$ which implies that I is a free $\frac{R}{\mathrm{Ann}(I)}$-module, hence projective over $\frac{R}{\mathrm{Ann}(I)}$. By Lemma 5.4, I is quasi-projective. Consequently, R is an fqp-ring.

Proof of Theorem 5.8. Assume that R is an fqp-ring. We may suppose that A is not a field. Then R is not a chained ring since $((a,\ 0)$ and $((0,\ e))$ are incomparable where $a \neq 0\ \in \mathfrak{m}$ and $e = (1,\ 0,\ 0,\ \ldots)\ \in E$. Also R is local with maximal $\mathfrak{m} \ltimes E$. By Lemma 5.10, $\mathrm{Z}(R) = \mathrm{Nil}(R)$. But $\mathfrak{m} \ltimes E = \mathrm{Z}(R)$. For, let $(a,e)\ \in \mathfrak{m} \ltimes E$. Since E is an $\frac{A}{\mathfrak{m}}$-vector space, $(a,e)(0,e) = (0,ae) = (0,0)$. Hence $\mathfrak{m} \ltimes E \subseteq \mathrm{Z}(R)$. The other inclusion holds since $\mathrm{Z}(R)$ is an ideal. Hence $\mathfrak{m} \ltimes E = \mathrm{Nil}(R)$. By Lemma 5.9, $(\mathrm{Nil}(R))^2 = 0 = (\mathfrak{m} \ltimes E)^2$. Consequently, $\mathfrak{m}^2 = 0$.

Conversely, $\mathfrak{m}^2 = 0$ implies $(\mathfrak{m} \ltimes E)^2 = 0$ and hence by Lemma 5.11, R is an fqp-ring. □

Now we can use Theorem 5.8 to construct examples which prove that the implications in Theorem 5.3 cannot be reversed in general. The following is an example of an fqp-ring which is not an arithmetical ring

Example 5.12. $R := \frac{\mathbb{R}[X]}{(X^2)} \ltimes \mathbb{R}$ is an fqp-ring by Theorem 5.8, since R is local with a nilpotent maximal ideal $\frac{(X)}{(X^2)} \ltimes \mathbb{R}$. Also, since $\frac{\mathbb{R}[X]}{(X^2)}$ is not a field, R is not arithmetical by Theorem 4.2.

The following is an example of a Gaussian ring which is not an fqp-ring.

Example 5.13. $R := \mathbb{R}[X]_{(X)} \ltimes \mathbb{R}$ is Gaussian by Theorem 4.2. Also, by Theorem 5.8, R is not an fqp-ring.

Now the natural question is what are the values of the weak global dimension of an arbitrary fqp-ring? The answer is given by the following theorem.

Theorem 5.14 ([1, Theorem 3.11]). *Let R be an fqp-ring. Then* $\mathrm{w.gl.dim}(R) = 0$*, 1, or* ∞.

Proof. Since $\mathrm{w.gl.dim}(R) = \sup\{\mathrm{w.gl.dim}(R_p) \mid p \text{ prime ideal of } R\}$, one can assume that R is a local fqp-ring. If R is reduced, then $\mathrm{w.gl.dim}(R) \leq 1$ by Lemma 3.5. If R is not reduced, then $\mathrm{Nil}(R) \neq 0$. By Lemma 5.9, either $(\mathrm{Nil}(R))^2 = 0$, in this case, $\mathrm{w.gl.dim}(R) = \infty$ by Theorem 3.9 (since an fqp-ring is Gaussian); or R is a chained ring with zero divisors ($\mathrm{Nil}(R) \neq 0$), in this case $\mathrm{w.gl.dim}(R) = \infty$ by Theorem 2.3. Consequently, $\mathrm{w.gl.dim}(R) = 0$, 1, or ∞.

It is clear that Theorem 5.14 generalizes Osofsky's Theorem on the weak global dimension of arithmetical rings (Theorem 2.3) and partially resolves Bazzoni-Glaz Conjecture on Gaussian rings.

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