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On the Class Group of $A + XB[X]$ Domains

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INTRODUCTION

All the rings considered in this paper are integral domains, and all modules and ring homomorphisms are unital. In this paper, we deal with the class group (see definition below) of $A + XB[X]$ domains. That is, let $A \subset B$ be an extension of integral domains. Then $A + XB[X]$ is a subring of the polynomial ring $B[X]$. This construction has been studied by many authors and has proven to be useful in constructing interesting examples and counterexamples, see for instance [2], [3], [5], [9], and [10].

If D is an integral domain, two well-known results on polynomial rings are that $Pic(D[X]) = Pic(D)$ if and only if D is seminormal, and $Cl(D[X]) = Cl(D)$ if and only if D is integrally closed (cf. [14] and [12], respectively). In [2], it is shown that $Pic(A + XB[X]) = Pic(A)$ if and only if B is seminormal. The purpose of this work is to study the question of when $Cl(A + XB[X]) = Cl(A)$, paying particular attention to the case where B is integrally closed. Namely, Theorem 4.4 establishes

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that if B is integrally closed and a flat overring of A, then $Cl(A+XB[X])$ is canonically isomorphic to $Cl(A)$. We also show that if B is integrally closed, then this isomorphism holds in the cases $qf(A) \subset B$ or $B = A[\mathbf{Y}]$, where $\mathbf{Y}$ is a set of indeterminates. Theorem 4.10 allows us to construct explicit examples showing that this canonical isomorphism does not hold in general even if B is integrally closed.

In this paper, $A \subset B$ is an extension of integral domains and K is the quotient field of B. Let D be an integral domain with quotient field $qf(D) = k$. By an ideal of D we mean an integral ideal of D. Given a nonzero fractional ideal I of D, we define $I^{-1} = \{x \in k \mid xI \subset D\}$ and $I_v = (I^{-1})^{-1}$. We say that I is divisorial or a v–ideal if $I_v = I$; while I is v–finite if $I = J_v$ for some finitely generated fractional ideal J of D. For I a nonzero fractional ideal of D, we define $I_t = \cup\{J_v \mid J \subset I \text{ finitely generated}\}$. Then I is a t–ideal if $I_t = I$. The mappings $I \mapsto I_v$ and $I \mapsto I_t$ are particular star-operations on fractional ideals of D, see [13, Sections 32 and 34] for a general theory. As in [7] and [8], we define the class group of D, $Cl(D)$, to be the group of t–invertible (fractional) t–ideals of D modulo the subgroup of principal ideals of D. If D is a Krull domain, then $Cl(D)$ is the usual divisor class group of D, see [11]. In this case, $Cl(D) = 0$ if and only if D is factorial.

This paper consists of four sections in addition to the introduction. In Sections 1 and 2, we state basic results on divisorial ideals and t–invertibility in $A + XB[X]$ domains. Section 3 establishes necessary and sufficient conditions for a v–invertible v–ideal or a t–invertible t–ideal of $A+XB[X]$ to be extended from A (see definition below). In Section 4, we give the proofs of the theorems mentioned above.

1. DIVISORIAL IDEALS IN $A+XB[X]$

Let $R = A+XB[X]$. In what follows, we consider the natural grading on R, that is, $R = \bigoplus_{n\geq 0} R_n$, where $R_0 = A$ and $R_n = X^nB$ for $n \geq 1$. An element (resp., an ideal) of R is said to be homogeneous if it is homogeneous with respect to this grading. If f is a polynomial over an integral domain A, we denote by A_f the content of f.

LEMMA 1.1 Let $R = A+XB[X]$ with B integrally closed. Let I be a homogeneous divisorial ideal of R, J the ideal of B generated by the coefficients of all polynomials of I, n the least integer k such that $aX^k \in I$ for some nonzero $a \in B$, and $W \subset J$ the A-module generated by all $a \in B$ such that $aX^n \in I$. Then J is a divisorial ideal of B and $I = X^nW + X^{n+1}J[X]$.

Proof. Since I is homogeneous, it is easy to see that $I \subset X^nW+X^{n+1}J[X]$. Conversely, since I is divisorial and $X^nW \subset I$, it suffices to show that $X^{n+1}J_v[X] \subset \frac{f}{g}R$ for each $f,g \in R$ such that $I \subset \frac{f}{g}R$. Choose a nonzero $a \in W$. Let $f,g \in R$ such that $I \subset \frac{f}{g}R$. Since $aX^n \in I$, we have $\frac{f}{g} = \frac{aX^n}{r}$ for some $r \in R$. Hence we can assume $f = aX^n$. Let $0 \neq h \in I$. Then $h \in \frac{aX^n}{g}R$, and hence $gh \in aX^nR \subset aB[X]$ and $A_{gh} \subset aB$. Since B is integrally closed, by [15, Lemme 1, Sect.2], $(A_gA_h)_v = (A_{gh})_v$. Hence $A_gA_h \subset aB$, so that $gA_h[X] \subset aB[X]$. Then $gJ[X] \subset aB[X]$, and by taking the v-closure, we get $gJ_v[X] \subset aB[X]$ [12, Lemma 1.6]. Since $XB[X] \subset R$, then $X^{n+1}J_v[X] \subset \frac{aX^n}{g}R$; whence $X^{n+1}J_v[X] \subset I$, as desired.

LEMMA 1.2 Let $R = A+XB[X]$ with B integrally closed. Then for each divisorial ideal I of R, there exist $u \in K[X, X^{-1}]$ and J a homogeneous divisorial ideal of R such that $I = uJ$.

Proof. Let S be the (multiplicatively closed) set of nonzero homogeneous elements of R. We have $R_S = K[X, X^{-1}]$. Let I be a divisorial ideal of R. Then $IR_S = fR_S$ for some $f \in B[X]$ such that $f(0) \neq 0$. Hence $I \subset fK[X]$. Since B is integrally closed, by [15, Section 2, Lemme 1], $fK[X] \cap B[X] = fA_f^{-1}[X]$; hence $I \subset fA_f^{-1}[X]$. Let $0 \neq b \in A_f$ and set $J = bXf^{-1}I$. Clearly, J is a divisorial ideal of R. We next show that J is homogeneous. Let $f, g \in R$ such that $J \subset \frac{f}{g}R$. Since $J \cap S \neq \emptyset$, and as in the proof of Lemma 1.1, we can assume that $f = aX^m$ for some nonzero element a of B and some integer $m \geq 0$. Now let $0 \neq h \in J$. Then $h \in \frac{aX^m}{g}R$, and hence $gh \in aX^mR \subset aB[X]$. Then $A_gA_h \subset aB$ by an argument similar to that in the proof of Lemma 1.1. On the other hand, $gh \in aX^mR$ implies that $g = X^rg_1$ and $h = X^sh_1$ with $r+s \geq m$ and $g_1(0)h_1(0) \neq 0$. We have $A_g = A_{g_1}$ and $A_h = A_{h_1}$, so $A_{g_1}A_{h_1} \subset aB$, and thus $g_1A_{h_1}[X] \subset aB[X]$. Therefore $X^{s+1}A_{h_1}[X] \subset J$. It follows that the homogeneous components of h are in J, which proves that J is homogeneous.

REMARK 1.3 Lemma 1.2 can be generalized to N-graded domains by using other techniques. Let $R = \bigoplus_{n\geq 0} R_n$ be an N-graded integral domain and S be the (multiplicatively closed) set of nonzero homogeneous elements of R. Then R_S is a Z-graded domain. In [1], the authors define a graded domain R to be almost normal if it is integrally closed with respect to nonzero homogeneous elements of R_S of nonzero degree. They showed [1, Corollary 3.8] that the following statements are equivalent:

(i) R is almost normal.

(ii) For each v–ideal I of R, $I = uJ$ for some $u \in R_S$ and some homogeneous v–ideal J of R.

If $R = A + XB[X]$ is graded in the natural way, it is not difficult to show that R is almost normal if and only if B is integrally closed.

By Lemma 1.1 and Remark 1.3, we have the following theorem.

THEOREM 1.4 Let $R = A + XB[X]$. The following statements are equivalent.
(1) B is integrally closed.
(2) For each v-ideal I of R, $I = u(W + XJ[X])$ for some $u \in K[X, X^{-1}]$, J a v-ideal of B, and $W \subset J$ a nonzero A-module.

2. v-INVERTIBLE IDEALS AND t-INVERTIBLE IDEALS IN $A + XB[X]$

LEMMA 2.1 Let $R = A + XB[X]$. Let F_1 (resp., F_2) be a nonzero fractional ideal of A (resp., B) such that $F_1 \subset F_2$. Then $F_1 + XF_2[X]$ is a fractional ideal of R, and we have $(F_1 + XF_2[X])^{-1} = F_1^{-1} \cap F_2^{-1} + XF_2^{-1}[X]$.

Proof. It is obvious that $I = F_1 + XF_2[X]$ is a fractional ideal of R. Now since $F_1 \subset I$, if $u \in I^{-1}$, then $u \in K[X]$. Thus if $u \in K[X]$, then $u \in I^{-1}$ if and only if $u(0)F_1 \subset A$ and $uF_2[X] \subset B[X]$. Hence $u \in I^{-1}$ if and only if $u \in F_1^{-1} \cap F_2^{-1} + XF_2^{-1}[X]$.

LEMMA 2.2 Let $R = A + XB[X]$. Then $XB[X]$ and $B[X]$ are divisorial ideals of R.

Proof. Let $C(A, B) = \{x \in A \mid xB \subset A\}$. It is easy to see that $[R : B[X]] = C(A, B) + XB[X]$. If $C(A, B) = 0$, then $(B[X])^{-1} = XB[X]$; hence $(B[X])_v = B[X]$. If $C(A, B) \neq 0$, by Lemma 2.1, $(B[X])_v = (C(A, B)^{-1} \cap B) + XB[X] = B[X]$. Hence $B[X]$ and $XB[X]$ are divisorial ideals of R.

THEOREM 2.3 Let $R = A + XB[X]$ with B integrally closed. If I is a fractional v-invertible v-ideal, then $I = u(J_1 + XJ_2[X])$ for some $u \in qf(R)$, J_2 a v-invertible v-ideal of B, and $J_1 \subset J_2$ a nonzero ideal of A.

Proof. By Theorem 1.4, we can assume that $I = W + XJ[X]$ for some v-ideal J of B and $W \subset J$ a nonzero A-module. First, we show that there exists nonzero $c \in K$ such that $cW \subset A$ and $cJ \subset B$. Let $a \in W$ be a nonzero element. Then one can easily show that $aI^{-1} \subset R$ satisfies the hypothesis of Lemma 1.1. Thus there exist an integer m, J' a divisorial ideal of B, and $W' \subset J'$ a nonzero A-module such that

$$aI^{-1} = X^m W' + X^{m+1} J'[X].$$

Since I is v-invertible,

$$aR = X^m((W + XJ[X])(W' + XJ'[X]))_v.$$

On the other hand, we have $(W + XJ[X])(W' + XJ'[X]) \subset B[X]$. Further, since $B[X]$ is divisorial, $aR \subset X^m B[X]$. Hence $m = 0$ and

$$aR = ((W + XJ[X])(W' + XJ'[X]))_v.$$

Thus $a^{-1}WW' \subset A$ and $a^{-1}JJ' \subset B$. Let $c \in a^{-1}W'$ be a nonzero element. Then $J_1 = cW \subset A$ and $J_2 = cJ \subset B$. Hence there exist $J_2 \subset B$ a divisorial ideal of B and $J_1 \subset J_2$ a nonzero ideal of A such that $I = u(J_1 + XJ_2[X])$ for some $u \in qf(R)$. It remains to show that J_2 is v-invertible. By Lemma 2.1, we have

$$I^{-1} = u^{-1}(J_1^{-1} \cap J_2^{-1} + XJ_2^{-1}[X]).$$

Hence, $II^{-1} \subset J_1(J_1^{-1} \cap J_2^{-1}) + XJ_2J_2^{-1}[X] \subset R$, and since I is v-invertible, we have

$$(J_1(J_1^{-1} \cap J_2^{-1}) + XJ_2J_2^{-1}[X])^{-1} = R.$$

By applying Lemma 2.1, we conclude that $(J_2J_2^{-1})^{-1} = B$. Hence J_2 is a v-invertible v-ideal of B.

COROLLARY 2.4 Let $R = A + XB[X]$ with B integrally closed. If I is a fractional t-invertible t-ideal, then $I = u(J_1 + XJ_2[X])$ for some $u \in qf(R)$, J_2 a t-invertible t-ideal of B and $J_1 \subset J_2$ a nonzero ideal of A.

Proof. It remains to show that J_2 and J_2^{-1}, from Theorem 2.3, are v-finite. Since I is a t-invertible t-ideal, then $J_1 + XJ_2[X] = (f_1, \ldots, f_n)_v$ for some $f_1, \ldots, f_n \in R$. Thus there exists $F_1 \subset J_1$ (resp., $F_2 \subset J_2$) a finitely generated ideal of A (resp., B) such that $F_1 \subset F_2$ and $f_1, \ldots, f_n \in F_1 + XF_2[X]$. Hence $J_1 + XJ_2[X] = (F_1 + XF_2[X])_v$. Applying Lemma 2.1 yields $J_2^{-1} = F_2^{-1}$; hence J_2 is v-finite. Similarly, one shows that J_2^{-1} is also v-finite by using the fact that I^{-1} is v-finite.

3. t-INVERTIBLE IDEALS OF $A + XB[X]$ EXTENDED FROM A

A fractional ideal I of $R = A + XB[X]$ is said to be extended from A if $I = uJR$ for some $u \in qf(R)$ and some ideal J of A.

LEMMA 3.1 Let $R = A + XB[X]$ with B integrally closed and I be a fractional divisorial ideal of R. Then the following statements are equivalent.

(1) There exist $u \in qf(R)$ and W a nonzero A-module ($\subset B$) such that $I = uWR$.
(2) $IB[X]$ is a divisorial ideal of $B[X]$.

Proof. (1) $\Rightarrow$ (2). We can assume that $I = XWR$; hence $I = XW + X^2WB[X]$. By Lemma 1.1, WB is a divisorial ideal of B; so $IB[X] = XWB[X]$ is divisorial in $B[X]$.
(2) $\Rightarrow$ (1) By Theorem 1.4, $I = u(W + XJ[X])$, where $u \in K[X, X^{-1}]$, J is a divisorial ideal of B and $W \subset J$ is a nonzero A-module. Let $I_1 = u^{-1}IB[X]$. Then we have $I_1 = WB + XJ[X]$. Applying Lemma 2.1 to I_1 in the case $A = B$, we get $(I_1)_v = J[X]$. Further, since I_1 is divisorial in $B[X]$, then $J = WB$. Therefore $I = uWR$.

REMARK 3.2 If B is integrally closed, then divisorial ideals of R are not always of the form uWR, where $u \in qf(R)$ and $W \subset B$ is a nonzero A-module. For, let $A = Z$ and $B = Z[i]$. Let's consider the ideal $I = 2Z + (1+i)XZ[i][X]$ of $R = Z + XZ[i][X]$. By applying Lemma 2.1, one can easily show that I is a divisorial ideal. Notice that I is also a t–invertible t–ideal (see Remark 4.15 and Example 4.16). Now assume $I = uWR$. Then $u \in Q(i)$, so $uW = 2Z$ and $uWZ[i] = (1+i)Z[i]$. Hence $2Z[i] = (1+i)Z[i]$, a contradiction.

LEMMA 3.3 Let $R = A + XB[X]$. Let I be a divisorial ideal of R of the form $I = J_1 + XJ_2[X]$, where J_2 is an ideal of B and $J_1 \subset J_2$ is a nonzero ideal of A. Then the following statements are equivalent.
(1) I is extended from A.
(2) $J_2 = J_1B$.
(3) $IB[X]$ is divisorial in $B[X]$.

Proof. (1) $\Rightarrow$ (2) We assume that $I = uJR$ for some $u \in qf(R)$ and some ideal J of A. Since $I \cap A \neq 0$, $u \in qf(A)$. It follows that $J_1 = uJ$ and $J_2 = uJB$. Hence $J_2 = J_1B$.
(2) $\Rightarrow$ (1) Clear.
(2) $\Leftrightarrow$ (3) Notice that by Lemma 2.1, J_2 is necessarily a divisorial ideal of B. We have $IB[X] = J_1B + XJ_2[X]$, and applying Lemma 2.1 in the case where $A = B$ yields $(IB[X])_v = J_2[X]$. Therefore $IB[X]$ is divisorial if and only if $J_2 = J_1B$.

THEOREM 3.4 Let $R = A + XB[X]$ with B integrally closed. Let I be a fractional v–invertible v–ideal of R. Then the following statements are equivalent.
(1) I is extended from A.
(2) $IB[X]$ is a divisorial ideal of $B[X]$.

Proof. It follows from Theorem 2.3 and Lemma 3.3.

REMARK 3.5 The implication (2) $\Rightarrow$ (1) in Theorem 3.4 is not true in general if I is a v–ideal which is not v–invertible. To see this, let A and B be such that $C(A, B) = 0$ (see Lemma 2.2), and consider the fractional ideal $I = B[X]$ of R. By Lemma 2.2, I is a divisorial ideal of R, but it is not v–invertible in R since $I^{-1} = XB[X]$ and $(II^{-1})_v = XB[X]$. Note that $IB[X] = B[X]$ is divisorial in $B[X]$. If $I = uJR$ for some $u \in qf(R)$ and some ideal J of A, then $u \in K$. Hence $B = uJ$, and thus $u^{-1}B = J \subset A$. Hence $u^{-1} \in C(A, B)$, a contradiction. Note that in this case, the implication (2) $\Rightarrow$ (1) in Lemma 3.1 is true, namely $W = B$.

LEMMA 3.6 Let $R = A + XB[X]$. Then R is a flat A-module if and only if B is a flat A-module.

Proof. Just note that $R = A \oplus \bigoplus_{n\geq 1} X^nB$.

LEMMA 3.7 Let $S \subset T$ be an extension of integral domains such that T is a flat S-module. If I is a finitely generated ideal of S, then $(IT)^{-1} = I^{-1}T$

Proof. See for instance [6, Alg. Comm., Chap.1].

LEMMA 3.8 Let $R = A + XB[X]$ and J be an ideal of A.
(1) If $(JR)_v = R$, then $J_v = A$.
(2) If $(JR)_t = R$, then $J_t = A$.

Proof. (1) Assume $(JR)_v = R$. Let $u \in qf(A)$ such that $J \subset uA$. Then $JR \subset uR$, and hence $R = (JR)_v \subset uR$. Thus $1 \in uA$ and $J_v = A$. (2) is a consequence of (1) since $(JR)_t = \cup\{(FR)_v \mid F \subset J \text{ finitely generated }\}$.

PROPOSITION 3.9 Let $R = A + XB[X]$ such that B is a flat A-module. Let J be an ideal of A. Then the following statements are equivalent.
(1) J is a t–invertible t–ideal of A.
(2) JR is a t–invertible t–ideal of R.

Proof. If B is a flat A-module, then R is a flat A-module by Lemma 3.6, and by [4, Prop. 2.2], we have (1) $\Rightarrow$ (2).
(2) $\Rightarrow$ (1) Assume that $I = JR$ is a t–invertible t–ideal of R. Then $J = I \cap A$ is a t–ideal. To see this, let $F \subset J$ be a finitely generated ideal of A. By using the formula $(F_vR)_v = (FR)_v$ which is a consequence of Lemma 3.7 (see [4, Prop. 2.2], we conclude that $F_v \subset J$. On the other hand, since JR is t–invertible, there exists $J_1 \subset J$ a finitely generated ideal of A such that $JR = (J_1R)_v$. Thus $(JJ_1^{-1}R)_t = ((JR)(JR)^{-1})_t = R$, and by Lemma 3.8, $(JJ_1^{-1})_t = A$. Hence J is a t–invertible t–ideal of A.

THEOREM 3.10 Let $R = A + XB[X]$ such that B is integrally closed and a flat A-module. Let I be a fractional t–invertible t–ideal of R. Then the following statements are equivalent.

(1) $I = uJR$ for some $u \in qf(R)$ and some t-invertible t-ideal J of A.
(2) $IB[X]$ is a divisorial ideal of $B[X]$

Proof. (1) $\Rightarrow$ (2) is a particular case of (1) $\Rightarrow$ (2) in Lemma 3.1. Since t-invertible t-ideals are v-invertible v-ideals, (2) $\Rightarrow$ (1) is a consequence of (2) $\Rightarrow$ (1) of Theorem 3.4 and Proposition 3.9.

4. THE CLASS GROUP OF $A + XB[X]$

LEMMA 4.1 Let S be an integral domain and T an overring of S. Then the following statements are equivalent.
(1) T is a flat S-module.
(2) For each maximal ideal M of T, $T_M = S_{M \cap S}$.

Proof. See [11, Lemma 6.5].

LEMMA 4.2 Let $R = A + XB[X]$ such that B is a flat A-module. Then, $B[X]$ is a flat R-module if and only if B is an overring of A.

Proof. We will use Lemma 4.1. First suppose that $B[X]$ is a flat R-module and let M be a maximal ideal of $B[X]$ such that $XB[X] \subset M$; then $B[X]_M = R_{M \cap R}$. Let $x \in B$. Then $x \in R_{M \cap R}$, and hence $x = \frac{f}{g}$ for some $f, g \in R$ with $g \notin M$. Since $XB[X] \subset M$, $g(0) \neq 0$, so that $x = \frac{f(0)}{g(0)} \in qf(A)$, hence $B \subset qf(A)$.

Conversely, assume that B is an overring of A and let M be a maximal ideal of $B[X]$. We will show that $B[X]_M = R_{M \cap R}$. If $X \in M$, then $M = m + XB[X]$ for some maximal ideal m of B, and we have $M \cap R = (m \cap A) + XB[X]$. Since B is a flat A-module, by Lemma 4.1, $B_m = A_{m \cap A}$, and one can easily verify that $B[X]_M = R_{M \cap R}$. Now if $X \notin M$, let $u \in B[X]_M$. Then $u = \frac{f}{g}$ for some $f, g \in B[X]$ with $g \notin M$; thus $u = \frac{Xf}{Xg} \in R_{M \cap R}$. Hence $B[X]_M \subset R_{M \cap R}$ and $B[X]_M = R_{M \cap R}$.

LEMMA 4.3 Let $R = A + XB[X]$ such that B is a flat A-module. Then the canonical map $\varphi : Cl(A) \to Cl(R)$, $[J] \mapsto [JR]$ is well-defined and it is an injective homomorphism.

Proof. Since B is a flat A-module, by Lemma 3.6, R is a flat A-module. Hence by [4, Prop. 2.2], φ is well-defined and it is a homomorphism. φ is injective since R is a faithfully flat A-module.

THEOREM 4.4 Let $R = A + XB[X]$ such that B is integrally closed and a flat overring of A. Then $Cl(A + XB[X]) \cong Cl(A)$.

Proof. It suffices to show that the canonical homomorphism φ in Lemma 4.3 is surjective. Let I be a t-invertible t-ideal of R. Since B is a flat overring of A, by Lemma 4.2, $B[X]$ is a flat R-module, and thus $IB[X]$ is a t-invertible t-ideal of $B[X]$ [4, Prop. 2.2]. The surjectivity of φ now follows from Theorem 3.10.

COROLLARY 4.5 Let S be a multiplicatively closed subset of A. If A is integrally closed, then $Cl(A + XA_S[X]) \cong Cl(A)$.

For $A = B$, we have the following corollary [12, Theorem 3.6]

COROLLARY 4.6 If A is integrally closed, then $Cl(A[X]) \cong Cl(A)$.

THEOREM 4.7 Let $R = A + XB[X]$. If B is integrally closed and $qf(A) \subset B$, then $Cl(A + XB[X]) \cong Cl(A)$

Proof. Since $qf(A) \subset B$, B is a flat A-module, and hence by Lemma 4.3, it suffices to show that the canonical homomorphism $\varphi : Cl(A) \to Cl(R)$ is surjective. Let I be a t-invertible t-ideal of R. By Corollary 2.4, $I = u(J_1 + XJ_2[X])$, where $u \in qf(R)$, J_2 is a t-invertible t-ideal of B, and $J_1 \subset J_2$ is a nonzero ideal of A. Since $qf(A) \subset B$, $J_2 = B$, and hence $I = uJ_1R$ is extended from A. By Proposition 3.9, J_1 is a t-invertible t-ideal of A, as desired.

COROLLARY 4.8 If B is integrally closed and A is a field, then $Cl(A+XB[X]) = 0$.

THEOREM 4.9 Let $Y = \{Y_i\}_i$ be a set of indeterminates. If A is integrally closed, then $Cl(A + XA[Y][X]) \cong Cl(A)$.

Proof. $B = A[Y]$ is a flat A-module. By Lemma 4.3, it suffices to show that the canonical homomorphism φ is surjective. Let I be a t-invertible t-ideal of R. By Corollary 2.4, we can assume that $I = J_1 + XJ_2[X]$ for some t-invertible t-ideal J_2 of B and $J_1 \subset J_2$ a nonzero ideal of A. Let $J = J_2 \cap A$; $J \neq 0$ since $J_1 \neq 0$. By [12, Lemma 3.3 and Prop. 3.2], J is a t-invertible t-ideal of A and $J_2 = J[Y]$. Hence $I = J_1 + XJ[Y][X]$. By applying Lemma 2.1 and using the fact that $J[Y]^{-1} = J^{-1}[Y]$ ([12, Lemma 1.6]), we obtain

$$\begin{aligned} I^{-1} &= J_1^{-1} \cap J[Y]^{-1} + XJ[Y]^{-1}[X] \\ &= J_1^{-1} \cap J^{-1}[Y] + XJ^{-1}[Y][X] \\ &= J^{-1} + XJ^{-1}[Y][X], \end{aligned}$$

so that $I = (I^{-1})^{-1} = JR$. Hence φ is surjective.

From the above results, the following natural question arises: Assume that B is integrally closed and a flat A-module. Does the canonical isomorphism $Cl(A + XB[X]) \cong Cl(A)$ always hold? The answer is negative, in general, as is shown by the following examples.

THEOREM 4.10 Let A be an integral domain and α an element of some extension domain of A such that $\alpha \notin qf(A)$, $\alpha^2 \in A$, and α is not a unit in $B = A[\alpha]$. Let $R = A + XB[X]$ and $I = \alpha^2 A + \alpha XB[X]$. Then
(1) I is a divisorial ideal of R.
(2) I is a t–invertible t–ideal of R.
(3) I is not extended from A.

Proof. Since $\alpha^2 \in A$ and $\alpha \notin qf(A)$, then $B = A + A\alpha$ and it is a free A-module with basis $\{1, \alpha\}$.
(1) By applying Lemma 2.1 to the ideal I, we get

$$I^{-1} = \alpha^{-2}A \cap \alpha^{-1}B + \alpha^{-1}XB[X].$$

On the other hand, $A \cap \alpha B = A \cap (A\alpha + A\alpha^2) = A\alpha^2$. Thus $\alpha^{-2}A \cap \alpha^{-1}B = A$, and hence $I^{-1} = A + \alpha^{-1}XB[X]$. Thus

$$\begin{aligned} I_v &= A \cap \alpha B + \alpha XB[X] \\ &= \alpha^2 A + \alpha XB[X] \\ &= I. \end{aligned}$$

(2) It suffices to show that I and I^{-1} are v–finite and I is v–invertible. First we show that $I = (\alpha^2, \alpha X)$. It is obvious that $(\alpha^2, \alpha X) \subset I$. For the reverse inclusion, let $f \in I$. We have $f = \alpha^2 a + \alpha X g(X)$ for some $a \in A$ and some $g \in B[X]$. Then g has the form $g = b + c\alpha + Xh(X)$, where $b, c \in A$ and $h \in B[X]$. Thus $f = \alpha^2(a + cX) + \alpha X(b + Xh(X))$; hence $f \in (\alpha^2, \alpha X)$. In (1), we have shown that $I^{-1} = A + \alpha^{-1}XB[X]$. Hence $I^{-1} = \alpha^{-2}I = (1, \alpha^{-1}X)$. It remains to show that I is v–invertible. We have

$$II^{-1} = (\alpha^2, \alpha X)(1, \alpha^{-1}X) = (\alpha^2, \alpha X, X^2).$$

Let $u \in qf(R)$ such that $(\alpha^2, \alpha X, X^2) \subset uR$. Since $\alpha^2 \in uR$, we can assume that $u = \frac{\alpha^2}{f}$ for some $f \in R$. Since $X^2 \in \frac{\alpha^2}{f}R$, then $X^2 f = \alpha^2 g$ for some $g \in R$; so g has the form $g = X^2 h$ for some $h \in B[X]$. Thus $f = \alpha^2 h$; so that $\alpha^2 h(0) = f(0) \in A$. Hence $h(0) \in A$ and $h \in R$. Thus $1 = uh \in uR$ and $(II^{-1})_v = R$.
(3) If $I = uJR$ for some $u \in qf(R)$ and some ideal J of A, then $u \in qf(A)$. Hence $\alpha^2 A = uJ$ and $\alpha B = uJB$; so $B = \alpha B$, a contradiction.

EXAMPLE 4.11 To construct simple examples illustrating Theorem 4.10, let's consider $A = Z$ and let $d \in Z$, $d \neq -1$ and not a square. Then $\alpha = \sqrt{d}$ satisfies the conditions of Theorem 4.10. The cases where $d \equiv 2, 3 \pmod 4$ and square-free give examples with B integrally closed.

We have the following corollary of Theorem 4.10.

COROLLARY 4.12 The canonical homomorphism $Cl(A) \to Cl(A + XB[X])$ is not surjective in general even if B is integrally closed.

EXAMPLE 4.13 Let k be a field and $A = k[[S^2]]$, ($\alpha = S$ an indeterminate); so $B = A[S] = k[[S]]$. Let $R = A + XB[X]$. Then $Cl(A) = 0$ and $Cl(R) = Z/2Z$. To see this, let I be a t–invertible t–ideal of R. Since A and B are DVRs, and by applying Corollary 2.4, we can assume that $I = S^{2n}A + S^m XB[X]$. By dividing out suitable powers of S^2, we may assume that $m = 0$ or $m = 1$. For $I = S^{2n}A + XB[X]$; $I^{-1} = A + XB[X] = R$, and hence $I = I_v = R$ and $n = 0$. For $I = S^{2n}A + SXB[X]$; $I^{-1} = A + S^{-1}XB[X]$, and hence $I = I_v = S^2A + SXB[X]$ and $n = 1$. By Theorem 4.10, the only nonzero class in $Cl(R)$ is $[I]$, where $I = S^2A + SXB[X]$, and its order is two. Thus $Cl(R) = Z/2Z$.

We can say more about the surjectivity of the canonical homomorphism $Cl(A) \to Cl(A + XB[X])$; this is a consequence of Theorem 4.10.

COROLLARY 4.14 Let G be an abelian group. Then there exists an extension $A \subset B$ of integral domains such that B is integrally closed, $Cl(A) = G$, and the canonical homomorphism $Cl(A) \to Cl(A + XB[X])$ is not surjective.

Proof. By Claborn's theorem there exists a Dedekind domain D such that $Cl(D) = G$. Let S be an indeterminate, and let $A = D[S^2]$, $B = D[S]$, and $R = A + XB[X] = D[S^2, SX, X]$. We have $Cl(A) = Cl(D) = G$. By Theorem 4.10, the natural homomorphism $Cl(A) \to Cl(R)$ is not surjective. Also, note that in this case, $Pic(R) = Pic(A) = Pic(D)$.

REMARK 4.15 By modifying the hypothesis "α not a unit in B" in Theorem 4.10 to "$a = 1 - \alpha^2$ not a unit in B" and considering the ideal $I = aA + (1 + \alpha)XB[X]$, one can show, using similar arguments, that $I = (a, (1 + \alpha)X)$ and it is a t–invertible t–ideal of R. In this case, if we take $A = Z$ and $\alpha = i$, we have the simple example $R = Z + XZ[i][X]$. See the example below for the class group of this ring.

EXAMPLE 4.16 Let $R = Z + XZ[i][X]$. Then $Cl(R)$ is the direct sum of $Z/2Z$ and a countably infinite number of copies of Z. The $Z/2Z$ summand corresponds to 2, the prime in Z that splits in $Z[i]$ with two associate prime factors, and each of the Z summands corresponds to a positive prime p in Z that splits in $Z[i]$ with two nonassociate prime factors. Let $2 = ab$ in $Z[i]$; a and b are associates. Then for $I = 2Z + aXZ[i][X]$, we have $[I] = -[I] = [2Z + bXZ[i][X]]$ is nonzero. For $p \neq 2$, a positive prime in Z that splits as $p = ab$ in $Z[i]$, we have $-[pZ + aXZ[i][X]] = [pZ + bXZ[i][X]]$ and each has infinite order in $Cl(R)$. These statements and those below follow from the form of a divisorial ideal in R and the formula for I^{-1} for such a divisorial ideal.

We next show that the above classes of ideals generate $Cl(R)$. Such a divisorial ideal I has the form $nZ+aXZ[i][X]$ with $n=ab$. Using the above comments, one can show that any prime divisor p of n in Z that does not split in $Z[i]$ must also divide a. If $p=cd$ splits in $Z[i]$ and p^k exactly divides n, then one can show that either c^k or d^k is exactly the prime power that divides a. Thus $[nZ+aXZ[i][X]] = \sum k_j[p_jZ + c_jXZ[i][X]]$, where $\{p_j = c_jd_j\}$ is the set of positive primes in Z that split in $Z[i]$. We show that the above classes are independent. Assume $\sum k_j[p_jZ + c_jXZ[i][X]] = 0$ in $Cl(R)$. We may assume that each $k_j \geq 0$ (replace c_j by d_j, or conversely, if needed). Thus the corresponding $nZ + aXZ[i][X]$ is principal. One then uses the above comments to show that each k_j is 0 (or for the prime 2, that k_j is even).

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