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On Jaffard domains

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0 Introduction

Let R be a commutative ring with identity and let $X_1, X_2, \ldots, X_r$ be algebraically independent indeterminates over R. It is known that if R has finite Krull dimension, then

$$r+\dim(R) \leq \dim(R[X_1, X_2, \ldots, X_r]) \leq r+(r+1)\dim(R).$$

These inequalities have been used to characterize *dimension sequences* (i.e., sequences of nonnegative integers $\{a_k : k \geq 0\}$ for which there exists a ring A such that $\dim(A[X_1, \ldots, X_k]) = a_k$ for all $k \geq 0$). Indeed, it has been shown that a strictly increasing sequence of positive integers $\{a_k : k \geq 0\}$ is a dimension sequence if and only if $k a_k \leq (k+1) a_{k-1} + 1$ for all $k \geq 1$. For results of this kind, see Arnold-Gilmer ([AG 1], [AG 2]) and Parker [P].

Moreover, Krull [K] has shown that if R is any finite-dimensional Noetherian ring, then $\dim(R[X_1, \ldots, X_r]) = r+\dim(R)$ for all $r \geq 1$ (cf. also [S 1, Theorem 9]). Seidenberg subsequently proved the same equality in case R is any finite-dimensional Prüfer domain [S 2, Theorem 4]. To unite and extend such results on Krull dimension of polynomial rings, Jaffard [J 3] (and, previously, in the notes [J 1], [J 2]) introduced and studied *valuative dimension*:

Definition-Theorem 0.1. *Let R be a domain which is not a field, K the quotient field of R, L an algebraic extension field of K, and n a positive integer. Then the following conditions are equivalent:*

(i) *Each (L-) valuation overring of R has dimension at most n and there exists a (L-) valuation overring of R having dimension n;*

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(ii) *Each (L-) overring of R has dimension at most n and there exists an (L-) overring of R having dimension n;*

(iii) $\dim(R[X_1, \ldots, X_n]) = 2n$;

(iv) $\dim(R[X_1, \ldots, X_r]) = r + n$ *for all* $r \geq n-1$.

If the above conditions hold, R is said to have valuative dimension n (in short, $\dim_v(R) = n$*). If there exists no positive integer n satisfying* (i)–(iv)*, R is said to have infinite valuative dimension (in short,* $\dim_v(R) = \infty$*). For the sake of completeness, each field is assigned valuative dimension* 0.

Theorem 01 was essentially established by Jaffard [J3, Chapitre IV], who proved the variant of (iv) with $r \geq n$. The above improvement in (iv) is due to Arnold [A, Theorem 6]. A convenient reference for this material is Gilmer [G1, Section 30].

It is clear that $\dim(R) \leq \dim_v(R)$ for any domain R. As noted in [G1, Exercise 17, page 372], it is easy to construct a domain R such that $\dim_v(R) - \dim(R)$ is any preassigned nonnegative integer. Moreover, there exists R such that $\dim(R) < \infty$ and $\dim_v(R) = \infty$. These examples will be reprised in Examples 3.1.

To honor Jaffard, we make the following

Definition 0.2. *A domain R is said to be a Jaffard domain if* $\dim(R) = \dim_v(R) < \infty$; *equivalently (cf.* [G1, *Corollary* 30.12]*), if* $\dim(R) < \infty$ *and* $\dim(R[X_1, \ldots, X_r]) = r + \dim(R)$ *for each* $r \geq 0$.

The above results of Krull and Seidenberg may now be restated as follows. If R is a finite-dimensional domain which is either Noetherian or a Prüfer domain, then R is a Jaffard domain. We next list additional important families of examples that motivate this article's study of Jaffard domains.

(0.3) Each finite-dimensional universally catenarian domain is a Jaffard domain.

(0.3) follows from the fact that the class of all (not necessarily Noetherian) universally catenarian domains is the largest class of catenarian domains which is stable under factor domains and localizations and is such that its members satisfy the altitude formula (cf. Bouvier-Dobbs-Fontana [BDF1, Theorem 5.1] and Kabbaj [Ka2, Corollaire 1.8]). It is noteworthy that if a finite-dimensional domain R is either Cohen-Macaulay or Prüfer or of valuative dimension 1 or of global dimension 2, then R is universally catenarian (cf. [MM], [BF], [BDF1], [BDF2]).

(0.4) Each finite-dimensional stably strong S-domain is a Jaffard domain.

Recall that a stably strong S-domain is a domain R such that $R[X_1, \ldots, X_n]$ is a strong S-domain for all nonnegative integers n. The study of this class of rings was initiated by Malik-Mott [MM] and recently developed further by Kabbaj [Ka1], [Ka2]. Examples of strong S-domains which are not stably strong S-domains have been given by Brewer-Montgomery-Rutter-Heinzer [BMRH].

Notice that (0.4) generalizes (0.3). Indeed, if a finite-dimensional domain R is universally catenarian (for instance, Prüfer or Noetherian, then [BDF1, Theorem 2.4] shows that R is a stably strong S-domain. For an example of a Jaffard domain which is not a stably strong S-domain, see [MM, Example 3.11].

Remark 0.5. A domain R is said to satisfy the *altitude inequality formula* if

$$ht(P)+\mathrm{t.d.}(k(P)/k(P\cap R))\leq ht(P\cap R)+\mathrm{t.d.}(S/R)$$

for each finite-type R-algebra S containing R and each $P\in\mathrm{Spec}(S)$. It is well known that Noetherian domains satisfy the altitude inequality formula. More generally, Kabbaj [Ka2, Theorem 1.6] recently showed that any stably strong S-domain also satisfies the altitude inequality formula. In addition, we claim, generalizing (0.4), that:

(0.6) Each finite-dimensional domain R satisfying the altitude inequality formula is a Jaffard domain.

To prove (0.6), note under the present hypotheses that $ht(P[X_1, \ldots, X_r])=ht(P)$ for each $P\in\mathrm{Spec}(R)$ and each family of indeterminates $\{X_1, X_2, \ldots, X_r\}$ (cf. [Ka2, Lemme 1.4]). Thus (cf. [G1, Theorem 30.18 and Corollary 30.19]), $\dim(R[X_1, \ldots, X_r])=r+\dim(R)$; i.e., $\dim(R)=\dim_v(R)$.

The above results are summarized in the following diagram of implications concerning finite-dimensional domains:

$$\begin{array}{ccc}
\text{Noetherian} & & \text{Prüfer domain} \\
\Downarrow & & \Downarrow \\
\text{stably strong } S\text{-domain} & \Leftarrow & \text{universally caterarian domain} \\
\Downarrow & & \Downarrow \\
\text{altitude inequality formula} & \Leftarrow & \text{altitude formula} \\
\Downarrow & & \\
\text{Jaffard domain} & &
\end{array}$$

In Example 3.2, we shall give an example of a Jaffard domain which does not satisfy the altitude inequality formula.

Section 1 addresses the possible transfer of the Jaffard property for integral extensions, localizations, monoid domains, and Nagata rings. Section 2 finds necessary and sufficient conditions for certain pullbacks to be Jaffard domains. These are used to develop the examples collected in Section 3. These examples illuminate the earlier sections' results, which in some cases are shown to be best-possible.

1 Transfer results for Jaffard domains

We begin by recording the fact that integral extensions preserve and reflect the Jaffard property.

Proposition 1.1. *Let $R \subset S$ be an integral extension of domains. Then S is a Jaffard domain if and only if R is a Jaffard domain.*

Proof. It suffices to note, via integrality, that $\dim(S) = \dim(R)$ and $\dim_v(S) = \dim_v(R)$ (cf. [G1, (11.8) and Proposition 30.13]). ∎

We next show that the Jaffard property is stable under adjunction of indeterminates.

Proposition 1.2. (a) (cf. [G1, Corollary 30.12]) *If R is a Jaffard domain, then $R[X_1, \ldots, X_r]$ is also a Jaffard domain for each positive integer r.*

(b) *Let R be a domain with $\dim_v(R) = n < \infty$. Then $R[X_1, \ldots, X_r]$ is a Jaffard domain for each positive integer $r \geq n-1$.*

Proof. (a) By a remark of Jaffard [J3, Théorème 2, page 60], $\dim_v(R[X_1, \ldots, X_r]) = r + \dim_v(R)$ for each positive integer r (even if R is not a Jaffard domain). Now, with R Jaffard and $n = \dim(R) = \dim_v(R)$,

$$r + n \leq \dim(R[X_1, \ldots, X_r]) \leq \dim_v(R[X_1, \ldots, X_r]) = r + \dim_v(R) = r + n$$

so that $R[X_1, \ldots, X_r]$ is Jaffard.

(b) Combining the above-cited remark of Jaffard with Theorem 0.1(iv), we have

$$\dim_v(R[X_1, \ldots, X_r]) = r + n = \dim(R[X_1, \ldots, X_r])$$

for all $r \geq n-1$. Hence, $R[X_1, \ldots, X_r]$ is Jaffard. ∎

Remark 1.3. (a) Using Proposition 1.2(b), one easily produces a Jaffard domain R and a prime $P \in \operatorname{Spec}(R)$ such that R/P is not a Jaffard domain. (This gives a positive answer to a question of Jaffard [J3, page 68].) Indeed, a suitable R can be built from a non-Jaffard domain A of finite valuative dimension n as follows. Let r be the largest nonnegative integer such that $A[X_1, \ldots, X_r]$ is not Jaffard. (Notice $r < n-1$ by Proposition 1.2(b)). Then $R = A[X_1, \ldots, X_{r+1}]$, and $P = X_{r+1}R$ have the asserted properties. A different construction illustrating the same phenomenon will be given in Example 3.7(b).

(b) Apropos of Proposition 1.2(b), Example 3.4 will present a non-Jaffard domain with $\dim_v(R) = n < \infty$ and a positive integer $r < n-1$ such that $R[X_1, \ldots, X_r]$ is a Jaffard domain.

(c) Using Proposition 1.2(a), it is easy to show that R is a Jaffard domain if and only if $R[X]$ is a Jaffard domain such that $\dim(R[X]) = 1 + \dim(R)$.

It will be shown in Example 3.2 that a localization of a Jaffard domain need not be a Jaffard domain. (This is in contrast with the stability of the classes of Noetherian domains, Prüfer domains, universally catenarian domains, and stably strong S-domains under localization.) This motivates the following:

Definition 1.4. *A domain R is said to be locally Jaffard if R_P is a Jaffard domain for each $P \in \operatorname{Spec}(R)$.*

For each domain R, $\dim_v(R)=\sup\{\dim_v(R_P)\colon P\in\operatorname{Spec}(R)\}$. As an easy consequence, we have:

Proposition 1.5. *Let R be a domain with $\dim_v(R)<\infty$. Then:*

(a) *R is locally Jaffard if and only if $S^{-1}R$ is a Jaffard domain for each multiplicative subset S of R.*

(b) *If R is locally Jaffard, then R is a Jaffard domain.*

Remark 1.6. One cannot delete the hypothesis that $\dim_v(R)<\infty$ in Proposition 1.5. Indeed, consider Nagata's example [N2, page 203] of an infinite-dimensional Noetherian domain each of whose localizations at a maximal ideal is finite-dimensional. Being Noetherian (and hence locally finite-dimensional), this domain is locally Jaffard. However, if is non-Jaffard since its (valuative) dimension is infinite.

We next state a characterization of locally Jaffard domains.

Proposition 1.7. [BDF 1, Proposition 9.3]) *Let R be a domain with $\dim_v(R)<\infty$. Then R is locally Jaffard if and only if*

$$ht(N)+\text{t.d.}(\underline{k}(N)/\underline{k}(N\cap R))\le ht(N\cap R)$$

for each valuation overring (V,N) of R.

We next consider some contexts for which the Jaffard property is a local property. First, recall that a domain R is said to be equicodimensional if all its maximal ideals have the same height.

Proposition 1.8. *Let R be a finite-dimensional equicodimensional catenarian domain. Then R is locally Jaffard if and only if R is a Jaffard domain.*

Proof. By Proposition 1.5(b), we need only prove the "if" assertion. Since $\dim(A)\le\dim_v(A)$ for all domains A, it is enough to notice that

$$\dim(R)=\dim(R/P)+\dim(R_P)\le\dim_v(R/P)+\dim_v(R_P)\le\dim_v(R)$$

for all $P\in\operatorname{Spec}(R)$. The equality follows from the equicodimensional and catenarian conditions. The second inequality was already known to Jaffard [J3, Proposition 2, page 57]. (In Corollary 2.4, we shall give another proof of this inequality, as a consequence of a general result on pullbacks of Jaffard domains.) ■

Remark 1.9. (a) If one assumes only that the domain R is finite-dimensional and equicodimensional, the previous reasoning shows that R is a Jaffard domain if and only if R_M is Jaffard for each maximal ideal M of R.

(b) If the equivalent conditions in Proposition 1.8 hold, we have, as a by-product of the above proof, that R/P is Jaffard for each $P\in\operatorname{Spec}(R)$. Moreover, that method of proof also establishes that if R is a Jaffard domain and P is a divided prime of R (in the sense of [D], namely $PR_P=P$), then both R/P and R_P are Jaffard domains.

The Jaffard property is also determined locally for low-dimensional domains. Indeed, with R' denoting the integral closure of R, we may rephrase part of [BDF 1, Corollary 6.3] as follows:

Theorem 1.10. *For a one-dimensional domain R, the following conditions are equivalent:*

(i) *R is a Jaffard domain;*

(ii) *R is locally Jaffard;*

(iii) *R is universally catenarian;*

(iv) *$R[X]$ is catenarian;*

(v) *R is a stably strong S-domain;*

(vi) *R is a strong S-domain;*

(vii) *R is an S-domain;*

(viii) *R' is a Prüfer domain.*

Proposition 1.11. *Let R be a two-dimensional equicodimensional domain. Then R is locally Jaffard if and only if R is a Jaffard domain. Moreover, if these conditions hold, then R is a strong S-domain.*

Proof. By Proposition 1.5(b), we need only prove the "if" assertion and the "strong S-"assertion. Now, if R is Jaffard, Theorem 0.1(iv) yields $\dim(R[X])=3$. Under the given hypotheses, an easy case analysis reveals that R is a strong S-domain. Thus, if P is any nonzero nonmaximal prime of R, R_P is a strong S-domain [MM, Corollary 2.4] and, also being one-dimensional, is therefore Jaffard by Theorem 1.10. Moreover, if M is a maximal ideal of R, then R_M is also Jaffard. Indeed,

$$4 \le \dim(R_M[X_1, X_2]) \le \dim(R[X_1, X_2]) = 4$$

whence, by Theorem 0.1(iii), $\dim_v(R_M) = 2 (= \dim(R_M))$. Hence, R is locally Jaffard. ■

Remark 1.12 (a) Unlike the one-dimensional case, a two-dimensional Jaffard domain need not be universally catenarian: consider a Noetherian local (hence, equicodimensional) two-dimensional domain R which is not universally catenarian. Nagata's example [N 1] of such an R has the property that $R[X]$ is not catenarian.

(b) A two-dimensional equicodimensional domain R is a strong S-domain if and only if $\dim(R[X])=3$. However, as Example 3.8 will show, such a domain need not be a Jaffard domain.

In the spirit of Theorem 1.10, we may state a similar result for higher-dimensional going-down domains. It may be proved by combining [BDF 1, Theorem 6.2] with [BDF 2, Theorem 1].

Theorem 1.13. *For a locally finite-dimensional going-down domain R, the following conditions are equivalent:*

(i) *R is locally Jaffard;*

(ii) *R_M is a Jaffard domain for each maximal ideal M of R;*

(iii) *R is universally catenarian;*

(iv) *$R[X]$ is catenarian;*

(v) *R is a stably strong S-domain;*

(vi) *R is a strong S-domain;*

(vii) *R' is universally catenarian;*

(viii) *R' is a Prüfer domain.*

The conditions in Theorem 1.13 may be used, in particular, to characterize the (G) PVD's (in the sense of [DF 1]) which are (locally) Jaffard domains.

One upshot of Example 3.2 will be that the conditions in Theorem 1.13 (under the given hypotheses, imply but) are not equivalent to R being Jaffard.

We have already seen that the Jaffard property is stable under adjunction of indeterminates (Proposition 1.2(a)). We next investigate, more generally, its possible stability under passage to monoid domains. A convenient reference for background on monoid domains is Gilmer [G 2].

Let S be a cancellative (additive) abelian torsion-free monoid with quotient group $G=\langle S\rangle$. (Such an S will be called a "torsionless grading monoid".) Then G is a torsion-free abelian group; and if R is a domain, the semigroup ring $A=R[S]$ is also a domain [G 2, Theorem 8.1]. We next recall the following result of Arnold-Gilmer [AG 3] (cf. also [G 2, Section 21]).

Proposition 1.14. *Let R be a domain and let S be a torsionless grading monoid with quotient group G. If* $\operatorname{rank}(G)=r$, *then,*

$$\begin{aligned}\dim(R[S]) &= \dim(R[G]) = \dim(R[X_1, \ldots, X_r]) \\ &= \dim(R[X_1, X_1^{-1}, \ldots, X_r, X_r^{-1}]).\end{aligned}$$

Theorem 1.17 will establish the valuative dimension analogue of Proposition 1.14. A key lemma towards this goal is:

Lemma 1.15. *If R is a domain, then* $\dim_v(R[X_1, \ldots, X_r]) = \dim_v(R[X_1, X_1^{-1}, \ldots, X_r, X_r^{-1}])$ *for each positive integer r.*

Proof. We shall prove the case $r=1$; the general case may then be easily completed by induction on r. Since $R[X, X^{-1}]$ is an overring of $R[X]$, $\dim_v(R[X, X^{-1}]) \leq \dim_v(R[X])$. For the reverse inequality,

$$\begin{aligned}\dim_v(R[X,X^{-1}]) &\geq \sup\{\dim(S[X,X^{-1}]):\ S \text{ an overring of } R\}\\ &= \sup\{\dim(S[X]):\ S \text{ an overring of } R\}\\ &\geq \sup\{\dim(S)+1:\ S \text{ an overring of } R\}\\ &= \sup\{\dim(S):\ S \text{ an overring of } R\}+1\\ &= \dim_v(R)+1 = \dim_v(R[X]).\end{aligned}$$

We thus have the desired equality. ■

As a direct consequence of Proposition 1.14 and Lemma 1.15, we have:

Corollary 1.16. *Let R be a domain and r a positive integer. Then $R[X_1,\dots,X_r]$ is a Jaffard domain if and only if $R[X_1,X_1^{-1},\dots,X_r,X_r^{-1}]$ is a Jaffard domain.*

We can now give our main result on monoid domains.

Theorem 1.17. *Let R be a domain and let S be a torsionless grading monoid with quotient group G. If $\operatorname{rank}(G)=r$, then $\dim_v(R[S])=\dim_v(R[G])=\dim_v(R[X_1,\dots,X_r])=\dim_v(R[X_1,X_1^{-1},\dots,X_r,X_r^{-1}])$. Moreover, if $\dim_v(R)$ and r are each finite, then this common value is $\dim_v(R)+r$.*

Proof. We may assume that r is finite. Let F be a finitely generated (free abelian) subgroup of G with $\operatorname{rank}(F)=r$. Let $A=R[S]$, $B=R[G]$ and $C=R[F]$. Then B is integral over C since G/F is a torsion group (cf. [G2, Theorem 12.4]). Next, write $C=R[Y_1,Y_1^{-1},\dots,Y_r,Y_r^{-1}]$ for some family of indeterminates $\{Y_1,\dots,Y_r\}$. Let $D=R[Y_1,\dots,Y_r]$. By Proposition 1.14, the domains A, B, C, and D all have the same Krull dimension. Moreover, $\dim_v(B)=\dim_v(C)=\dim_v(D)$, the first equality following by integrality, the second via Lemma 1.15.

As for A, note first that $\dim_v(A)\geq\dim_v(B)$ since B is an overring of A. To establish the reverse inequality, let V be a valuation overring of A. Then for each $1\leq i\leq r$, either $Y_i\in V$ or $Y_i^{-1}\in V$. Hence (by replacing Y_i with Y_i^{-1}, if necessary), we may assume that $D\subset V$. Thus $\dim_v(A)\leq\dim_v(D)$ by Theorem 0.1(i). We thus have $\dim_v(A)=\dim_v(B)=\dim_v(C)=\dim_v(D)$. The final "moreover" assertion follows from an earlier-cited result of Jaffard [J3, Théorème 2, page 60]. ■

As an immediate consequence of Proposition 1.14 and Theorem 1.17, we have the following extension of Corollary 1.16:

Corollary 1.18. *Let R be a domain and let S be a torsionless grading monoid with quotient group G such that $\operatorname{rank}(G)=r<\infty$. Then the following statements are equivalent:*

(i) *$R[S]$ is a Jaffard domain;*

(ii) *$R[G]$ is a Jaffard domain;*

(iii) *$R[X_1,\dots,X_r]$ is a Jaffard domain;*

(iv) *$R[X_1,X_1^{-1},\dots,X_r,X_r^{-1}]$ is a Jaffard domain.*

Our next corollary is the monoid domain analogue of Proposition 1.2.

Corollary 1.19. *Let R be a domain and let S be a torsionless grading monoid with quotient group G.*

(a) *If R is a Jaffard domain, then $R[S]$ is a Jaffard domain if and only if* $\mathrm{rank}(G) < \infty$.

(b) *If* $\dim_v(R) < \infty$, *then $R[S]$ is a Jaffard domain if* $\dim_v(R) - 1 \leq \mathrm{rank}(G) < \infty$.

Proof. The result is clear if $\mathrm{rank}(G)$ is infinite. Hence it suffices to combine Corollary 1.18 with Proposition 1.2.

As a special case of Corollary 1.19(a), we have that for any field K (or, more generally, any finite-dimensional Noetherian domain), a monoid domain $K[S]$ is a Jaffard domain if and only if $\mathrm{rank}(\langle S\rangle) < \infty$.

Remark 1.20. (a) By remark 1.3(b), $R[S]$ may be a Jaffard domain even though $\mathrm{rank}(G) < \dim_v(R) - 1$ and R is not a Jaffard domain. So the converse of Corollary 1.19(b) is false.

(b) Monoid domains may be used to construct Jaffard domains with various ring-theoretic properties. For instance, combining the above remarks with results in [G2, Chapters II and III], we see for any field K, that $K[\mathbb{Q}^+]$ is a one-dimensional, completely integrally closed, nonnoetherian Jaffard domain.

(c) With the assumption that R and $R[S]$ each have finite valuative dimension, Proposition 1.2(b) and Proposition 1.14 may be used to give another proof of Lemma 1.15 and Theorem 1.17 as follows. Let $r = \mathrm{rank}(G) < \infty$. Then $\dim_v(R[S][X_1, \ldots, X_n]) = \dim_v(R[S]) + n$ for each positive integer $n \geq 1$. If $n \geq \max\{\dim_v(R[S]) - 1, \dim_v(R) - 1\}$, then

$$\begin{aligned}\dim_v(R[S]) + n &= \dim_v(R[S][X_1, \ldots, X_n]) = \dim(R[S][X_1, \ldots, X_n]) \\ &= \dim(R[X_1, \ldots, X_n][S]) = \dim(R[X_1, \ldots, X_n, Y_1, \ldots, Y_r]) \\ &= \dim_v(R) + n + r.\end{aligned}$$

Hence, $\dim_v(R[S]) = \dim_v(R) + r$.

We next investigate the transfer of the Jaffard property for Nagata rings. Let R be a domain with quotient field K and let $\{X_1, \ldots, X_r\}$ be a family of indeterminates. The *Nagata ring* (in r variables and with coefficients in R) is the domain $R(X_1, \ldots, X_r) = S^{-1}R[X_1, \ldots, X_r]$, where $S = \{f \in R[X_1, \ldots, X_r] : c(f) = R\}$.

First, we shall determine the Krull dimension of a Nagata ring.

Proposition 1.21. *If R is a domain, then* $\dim(R(X_1, \ldots, X_r)) = \dim(R[X_1, \ldots, X_r]) - r$ *for each positive integer r. In particular, if R is a Jaffard domain, then* $\dim(R(X_1, \ldots, X_r)) = \dim(R)$ *for each positive integer r.*

Proof. By [G1], Proposition 33.1], $\dim(R(X_1, \ldots, X_r)) = \sup\{ht(M[X_1, \ldots, X_r]) : M$ a maximal ideal of $R\}$. The first assertion now follows from the fact that $\dim(R[X_1, \ldots, X_r]) = \sup\{ht(M[X_1, \ldots, X_r]) : M$ a maximal ideal of $R\} + r$ [AG1, Corollary 2.10]. The second assertion then follows from Proposition 1.2(b).

Next, we shall determine the valuative dimension of a Nagata ring.

Proposition 1.22. *If R is a domain, then* $\dim_v(R(X_1, \ldots, X_r)) = \dim_v(R)$ *for each positive integer* r.

Proof. If $\dim_v(R) = \infty$, then $\dim_v(R(X_1, \ldots, X_r)) = \infty$ also. In general,

$$\begin{aligned}\dim_v(R) &= \dim_v(R') = \dim_v((R')^b) \leq \dim_v(R(X_1, \ldots, X_r)) \\ &\leq \dim_v(R[X_1, \ldots, X_r]) = \dim_v(R) + r\end{aligned}$$

(cf. [G1, Proposition 32.16 and Theorem 33.3]). We may thus assume that both $\dim_v(R)$ and $\dim_v(R(X_1, \ldots, X_r))$ are finite. By Proposition 1.21, for any other family of indeterminates $\{Y_1, \ldots, Y_s\}$, $\dim(R(X_1, \ldots, X_r, Y_1, \ldots, Y_s)) = \dim(R[X_1, \ldots, X_r, Y_1, \ldots, Y_s]) - (r+s)$. Thus, by Theorem 0.1(iv), if $r+s \geq \dim_v(R) - 1$, then $\dim(R(X_1, \ldots, X_r, Y_1, \ldots, Y_s)) = (\dim_v(R) + (r+s)) - (r+s) = \dim_v(R)$.

Next, recall that $R(Y_1, \ldots, Y_m)(Z_1, \ldots, Z_n) = R(Y_1, \ldots, Y_m, Z_1, \ldots, Z_n)$ for any family of indeterminates $\{Y_1, \ldots, Y_m, Z_1, \ldots, Z_n\}$ (cf. [N2, 6.14 and Exercise 1, pages 18–19] or [An, Lemma]). Then, reasoning as above, if $s \geq \dim_v(R(X_1, \ldots, X_r)) - 1$, then $\dim_v(R(X_1, \ldots, X_r, Y_1, \ldots, Y_s)) = \dim(R(X_1, \ldots, X_r)[Y_1, \ldots, Y_s]) - s = \dim_v R(X_1, \ldots, X_r))$. Thus, comparing the above two calculations of $\dim(R(X_1, \ldots, X_r, Y_1, \ldots, Y_s))$, we have $\dim_v(R(X_1, \ldots, X_r)) = \dim_v(R)$. ■

Corollary 1.23. *Let R be a domain with* $\dim_v(R) < \infty$. *Then:*

(a) *If R is a Jaffard domain, then $R(X_1, \ldots, X_r)$ is a Jaffard domain for each positive integer r.*

(b) *For each positive integer r, the following three statements are equivalent:*

(i) $R[X_1, \ldots, X_r]$ *is a Jaffard domain;*

(ii) $R(X_1, \ldots, X_r)$ *is a Jaffard domain;*

(iii) $\dim(R(X_1, \ldots, X_r)) = \dim(R(X_1, \ldots, X_r, X_{r+1}, \ldots, X_n))$ *for each positive integer* $n > r$.

(c) *$R(X_1, \ldots, X_r)$ is a Jaffard domain for each positive integer* $r \geq \dim_v(R) - 1$.

Proof. (a) This follows immediately from Propositions 1.21 and 1.22.

(b) By Propositions 1.21 and 1.22, $\dim(R(X_1, \ldots, X_r)) = \dim_v(R(X_1, \ldots, X_r)) \Leftrightarrow \dim_v(R) = \dim(R[X_1, \ldots, X_r]) - r \Leftrightarrow \dim_v(R[X_1, \ldots, X_r]) = \dim(R[X_1, \ldots, X_r])$. It follows that (i) and (ii) are equivalent.

Moreover, by Proposition 1.21, (ii)$\Rightarrow$(iii). Conversely, suppose that (iii) holds. In general, we have as in the proof of Proposition 1.22 that if $n \geq r + \dim_v(R) - 1$, then $\dim(R(X_1, \ldots, X_r, X_{r+1}, \ldots, X_n)) = \dim_v(R(X_1, \ldots, X_r))$. Hence, by (iii), $\dim_v(R(X_1, \ldots, X_r)) = \dim(R(X_1, \ldots, X_r))$, and so (ii) holds.

(c) This follows by combining Proposition 1.2(b) and the implication (i)$\Rightarrow$(ii) established in (b) above. ■

We define the *N(agata)-dimension sequence* of finite-dimensional domain R to be $\{b_k : k \geq 0\}$, where $b_0 = \dim(R)$ and $b_k = \dim(R(X_1, \dots, X_k))$ for each positive integer k. By Proposition 1.21, $\{b_k : k \geq 0\}$ is a nondecreasing sequence of nonnegative integers bounded above by $\dim_v(R)$. Moreover, $b_k = a_k - k$ for each $k \geq 0$, where $\{a_k : k \geq 0\}$ is the dimension sequence of R. If $\dim_v(R) < \infty$, then Corollary 1.23(c) and Proposition 1.22 combine to yield that $b_k = \dim_v(R)$ for all $k \geq \dim_v(R) - 1$.

Corollary 1.24. *Let R be a domain with $\dim(R) < \infty$.*

(1) *Let $\{b_k : k \geq 0\}$ be the N-dimension sequence of R. Then:*

(a) *$\{b_k : k \geq 0\}$ is eventually constant if and only if $\dim_v(R) < \infty$.*

(b) *Let r be a positive integer. Then $b_k = b_r$ for all $k \geq r$ if and only if $R[X_1, \dots, X_r]$ is a Jaffard domain.*

(2) *Let $\{b_k : k \geq 0\}$ be a sequence of nonnegative integers. Then there is a domain R with $\{b_k : k \geq 0\}$ as its N-dimension sequence if and only if $\{b_k : k \geq 0\}$ is nondecreasing and $kb_k \leq (k+1)b_{k-1}$ for each positive integer k.*

Proof. (1) This follows easily from Corollary 1.23(b).

(2) $\{b_k : k \geq 0\}$ is an N-dimension sequence if and only if $\{a_k = b_k + k : k \geq 0\}$ is a dimension sequence. However, by [P, Theorem 2], $\{a_k : k \geq 0\}$ is a dimension sequence if and only if $\{a_k : k \geq 0\}$ is strictly increasing and $ka_k \leq (k+1)a_{k-1} + 1$ for each positive integer k. By elementary algebra, this is equivalent to $\{b_k : k \geq 0\}$ being nondecreasing and $kb_k \leq (k+1)b_{k-1}$ for each positive integer k. ■

2 Pullbacks and Jaffard domains

In this section, we determine necessary and sufficient conditions for certain "pullback-type" constructions to be Jaffard domains. As a special case, we determine when the $D + M$ construction yields a Jaffard domain.

First, we consider pullbacks of commutative rings

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k \end{array}$$

where T is a domain, φ is a homomorphism from T onto a field k with $\ker(\varphi) = M$, D is a proper subring of k, and $R = \varphi^{-1}(D)$. For the convenience of the reader, we next recall several properties of such Cartesian diagrams.

Lemma 2.1. (a) *$M = (R : T)$ and $R/M \cong D$.*

(b) Spec(R) *is homeomorphic to the topological amalgamated sum* $\mathrm{Spec}(T) \coprod_{\mathrm{Spec}(k)} \mathrm{Spec}(D)$.

(c) *If T is quasilocal, then M is a divided prime ideal of R, and so each prime ideal of R is comparable with M. If, in addition, k is the quotient field of D, then $R_M = T$.*

(d) *If T is quasilocal, then* $\dim(R) = \dim(D) + \dim(T)$.

(e) *For each $P \in \mathrm{Spec}(R)$ with $M \not\subset P$, there is a unique $Q \in \mathrm{Spec}(T)$ such that $Q \cap R = P$, and this Q satisfies $T_Q = R_P$.*

(f) *If S is a flat R-algebra, then the diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & S \otimes_R D \arrow[d, hook] \\ S \otimes_R T \arrow[r, two heads] & S \otimes_R k \end{tikzcd}$$

induced by applying $S \otimes_R \bullet$ to the given pullback diagram is also a pullback diagram.

(g) *If $P \in \mathrm{Spec}(R)$ and $P \supset M$, then there is a unique $Q \in \mathrm{Spec}(D)$ such that $P = \varphi^{-1}(Q)$. Moreover, the following diagram of canonical homomorphisms*

$$\begin{tikzcd} R_P \arrow[r] \arrow[d, hook] & D_Q \arrow[d, hook] \\ T_M \arrow[r, two heads] & k \end{tikzcd}$$

is a pullback diagram.

(h) *T is integral over R if and only if D is a field and k is algebraic over D.*

Proof. For (a), (b), and (e), apply appropriate parts of [F1, Theorem 1.4]. Direct calculations easily establish (c), while (d) is a consequence of [F1, Proposition 2.1(5)]. For (f), cf. [BF, Lemma 2]. (g) follows by direct calculation, using the fact that $MT_M \subset R_P$. Finally, for (h), apply (F1, Corollary 1.5(5)]. ■

We shall begin with the "local case." Let (T, M, k) denote a quasilocal domain T with maximal ideal M and residue field $k = T/M$. Furthermore, let φ: $T \to k$ be the canonical surjection and put $R = \varphi^{-1}(D)$, where D is a proper subring of k. We thus have a pullback diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, hook] & D \arrow[d, hook] \\ T \arrow[r, two heads, "\varphi"] & k \end{tikzcd}$$

For this situation, Theorem 2.6 will present necessary and sufficient conditions for R be to a Jaffard domain. As a first step in this direction, Proposition 2.3 will handle the case in which D has quotient field k. But first a key step:

Lemma 2.2. *If R is a domain and M a divided prime ideal of R, then*

$$ht(Q[X_1, \ldots, X_r]) = ht(Q[X_1, \ldots, X_r]/M[X_1, \ldots, X_r]) + ht(M[X_1, \ldots, X_r])$$

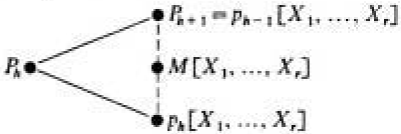
for each positive integer r and each prime ideal Q of R such that $Q \supset M$.

Proof. $ht(Q) = ht(Q/M) + ht(M)$ since M is divided. Thus the equality holds when $ht(Q)$ is infinite. Hence we may assume that all the primes have finite height. Let $d = ht(Q[X_1, \ldots, X_r])$ and choose

$$P_d = Q[X_1, \ldots, X_r] \supset P_{d-1} \supset \ldots \supset P_1 \supset P_0 = (0),$$

a (saturated) chain of prime ideals of $R[X_1, \ldots, X_r]$. By [G1, Corollary 30.19], this chain may be chosen to be a special chain of primes (in the sense that $(P_k \cap R)[X_1, \ldots, X_r]$ is in the chain for each $0 \le k \le d$). If $P_k \cap R = M$ for some $1 \le k \le d$, then we have the desired equality. We show next that this is indeed the case.

If not, then, since M is divided, we have



for some $1 \le h \le d$, where $p_i = P_i \cap R$. Let $T = R_M$ and $I = P_k T[X_1, \ldots, X_r] + MT[X_1, \ldots, X_r]$ $(= P_k T[X_1, \ldots, X_r] + M[X_1, \ldots, X_r]$ since M is divided).

If $I \neq T[X_1, \ldots, X_r]$, then we may apply the gluing result [BDF2, Lemma] essentially as in the proof of [BDF2, Theorem 2]. (The lemma applies by taking $X = \mathrm{Spec}(R_M[X_1, \ldots, X_r])$, $Y = \mathrm{Spec}((R/M)[X_1, \ldots, X_r])$, $Z = \mathrm{Spec}(\underline{k}(M)[X_1, \ldots, X_r])$, $x = (P_k)_{R \setminus M}$, and $y = P_{k+1}/M[X_1, \ldots, X_r]$.) The upshot is that there is a prime ideal P' of $R[X_1, \ldots, X_r]$ such that $P_k \subset P' \subset P_{k+1}$ and $P' \cap R = M$: this contradicts the fact that P_k and P_{k+1} are adjacent prime ideals. On the other hand, if $I = T[X_1, \ldots, X_r]$, then

$$1 \in I \cap R[X_1, \ldots, X_r] = (P_k T[X_1, \ldots, X_r] \cap R[X_1, \ldots, X_r]) + M[X_1, \ldots, X_r] = P_k + M[X_1, \ldots, X_r] \subset P_{k+1}:$$

also a contradiction. Thus $P_k \cap R = M$ for some $1 \le k \le d$ and we have the desired equality. ■

Proposition 2.3. *Let (T, M, k) be a quasilocal domain and $\varphi: T \to k$ the canonical surjection. Let $R = \varphi^{-1}(D)$, where D is a proper subring of k with quotient field k. Then:*

(a) $\dim(R[X_1, \ldots, X_r]) = \dim(D[X_1, \ldots, X_r]) + \dim(T[X_1, \ldots, X_r]) - \dim(k[X_1, \ldots, X_r])$ *for each positive integer r.*

(b) $\dim_v(R) = \dim_v(D) + \dim_v(T)$.

(c) *R is a Jaffard domain if and only if D and T are each Jaffard domains.*

Proof. (a) By Lemma 2.1(d), $\dim(R) < \infty$ if and only if both $\dim(D)$ and $\dim(T)$ are finite. Hence $\dim(R[X_1, \ldots, X_r]) < \infty$ if and only if both $\dim(D[X_1, \ldots, X_r])$ and $\dim(T[X_1, \ldots, X_r])$ are finite. We may assume that each domain is finite-dimensional.

Let Q be a maximal ideal of $R[X_1, \ldots, X_r]$ such that $\dim(R[X_1, \ldots, X_r]) = ht(Q) = ht(q[X_1, \ldots, X_r]) + r$, where $q = Q \cap R$ is a maximal ideal of R. (Such a Q exists by [AG1, Corollary 2.9].) Note that $M \subset q$ by Lemma 2.1(c). Thus $\bar{q} = q/M$ and $\bar{Q} = Q/M[X_1, \ldots, X_r]$ are maximal ideals of D and $D[X_1, \ldots, X_r]$ respectively. Also, by Lemma 2.2, $ht(q[X_1, \ldots, X_r]) = ht(q[X_1, \ldots, X_r]/M[X_1, \ldots, X_r]) + ht(M[X_1, \ldots, X_r])$. Hence

$$\begin{aligned}\dim(R[X_1, \ldots, X_r]) = {} & ht(q[X_1, \ldots, X_r]/M[X_1, \ldots, X_r]) \\ & + ht(M[X_1, \ldots, X_r]) + r.\end{aligned}$$

Next, notice that $M[X_1, \ldots, X_r]$ has the same height in $R[X_1, \ldots, X_r]$ and $T[X_1, \ldots, X_r]$ since $T = R_M$ by Lemma 2.1(c). Thus [AG1, Corollary 2.10] yields that $ht(M[X_1, \ldots, X_r]) + r = \dim(T[X_1, \ldots, X_r])$. Finally, by another application of [AG1, Corollary 2.9], $ht(q[X_1, \ldots, X_r]/M[X_1, \ldots, X_r]) + r = ht(\bar{Q}) = \dim(D[X_1, \ldots, X_r])$, the last equality following from the choice of Q. We thus have the desired equality.

(b) First suppose that $\dim_v(R) < \infty$. Then $\dim(D) + \dim(T) = \dim(R) < \infty$, and so both $\dim(D)$ and $\dim(T)$ are finite. In addition, $\dim_v(T) < \infty$ since T is an overring of R. Next, we shall observe that $\dim_v(D) < \infty$. If B is an n-dimensional overring of D, then $A = \varphi^{-1}(B)$ is an overring of R, and Lemma 2.1(d) yields $\dim(A) = n + \dim(T)$. Hence, $\dim_v(D) \leq \dim(A) \leq \dim_v(R) < \infty$.

Let r be a positive integer such that $r \geq \max\{\dim_v(R), \dim_v(D), \dim_v(T)\} - 1$. Then by Theorem 0.1(iv),

$$\dim(R[X_1, \ldots, X_r]) = \dim_v(R) + r,$$

$$\dim(D[X_1, \ldots, X_r]) = \dim_v(D) + r,$$

and

$$\dim(T[X_1, \ldots, X_r]) = \dim_v(T) + r.$$

Then by (a), $\dim_v(R) + r = (\dim_v(D) + r) + (\dim_v(T) + r) - r$, yielding (b) in case $\dim_v(R) < \infty$.

To complete the proof of (b), we show that $\dim_v(R) < \infty$ whenever $\dim_v(D)$ and $\dim_v(T)$ are both finite. Let r be a positive integer such that $r \geq \max\{\dim_v(D), \dim_v(T)\} - 1$. Then by (a) and Theorem 0.1(iv), $\dim(R[X_1, \ldots, X_r]) = \dim(D[X_1, \ldots, X_r]) + \dim(T[X_1, \ldots, X_r]) - r = (\dim_v(D) + r) + (\dim_v(T) + r) - r = (\dim_v(D) + \dim_v(T)) + r$. Hence, $\dim_v(R) < \infty$ by another appeal to Theorem 0.1(iv).

(c) Since $\dim(R)=\dim(D)+\dim(T)$ and $\dim(B)\le\dim_v(B)$ for any domain B, (c) follows directly from (b). ■

As an application of Proposition 2.3, we obtain a proof of the following result of Jaffard (cf. [J3, Proposition 2, page 57]). This result was used in the proof of Proposition 1.8.

Corollary 2.4. *Let R be a domain. Then $\dim_v(R)\ge\dim_v(R/P)+\dim_v(R_P)$ for each prime ideal P of R.*

Proof. Let k be R_P/PR_P, the quotient field of R/P, let $\varphi: R_P\to k$ be the canonical surjection, and let $T=\varphi^{-1}(R/P)$. The Proposition 2.3 gives $\dim_v(T)=\dim_v(R_P)+\dim_v(R/P)$. However, since T is an overring of R, we also have $\dim_v(T)\le\dim_v(R)$, which now gives the asserted inequality.

The next step in determining when a pullback $R=\varphi^{-1}(D)$, constructed from the quasilocal domain (T, M, k) and an arbitrary proper subring D of k, is a Jaffard domain is to determine when the pullback $\varphi^{-1}(F)$ is a Jaffard domain for F a subfield of k.

Proposition 2.5. *Let (T, M, k) be a quasilocal domain which is not a field and let $\varphi: T\to k$ be the canonical surjection. Let $R=\varphi^{-1}(F)$, where F is a subfield of k. Then:*

(a) $\dim_v(R)=\dim_v(T)+\mathrm{t.d.}(k/F)$.

(b) *R is a Jaffard domain if and only if T is a Jaffard domain and k is algebraic over F.*

Proof. Since $\dim_v(R)$ is infinite if either $\dim_v(T)$ or $\mathrm{t.d.}(k/F)$ is infinite, we may assume that $\dim_v(T)$ and $d=\mathrm{t.d.}(k/F)$ are each finite. We show next that we may also assume that K is purely transcendental extension of F. Let $\{Y_1, \dots, Y_d\}$ be a transcendence basis for k over F, and put $L=F(Y_1, \dots, Y_d)$. Since k is algebraic over L, Lemma 2.1(h) yields that T is integral over $A=\varphi^{-1}(L)$. Thus $\dim_v(A)=\dim_v(T)$. Since $\mathrm{t.d.}(L/F)=\mathrm{t.d.}(k/F)$ also, we may replace T with A and k with L; i.e. we may take k purely transcendental over F.

We shall prove the equality asserted in (a) by induction on $d=\mathrm{t.d.}(k/F)$. The induction basis concerns the case $d=0$, and this is evident, for then $F=k$ and $R=T$.

Next, suppose $d=1$. Write $k=F(Y)$ and $Y=\varphi(y)$ for some $y\in T\setminus M$. We claim that $R[y]=\varphi^{-1}(F[Y])$. The "$\subset$" inclusion is clear. For the reverse containment, consider any $x\in T$ with $\varphi(x)\in F[Y]$. There exists $z\in R[y]$ with $\varphi(z)=\varphi(x)$. Then $x-z\in\ker(\varphi)=M\subset R$, whence $x\in z+R\subset R[y]$, proving the claim. Similarly, $R[y^{-1}]=\varphi^{-1}(F[Y^{-1}])$. By Proposition 2.3(b), $\dim_v(R[y])=\dim_v(F[Y])+\dim_v(T)=1+\dim_v(T)$ and, similarly, $\dim_v(R[y^{-1}])=1+\dim_v(T)$. Moreover, $\dim_v(R)=\max\{\dim_v(R[y]), \dim_v(R[y^{-1}])\}$, since any valuation overring of R is also a valuation overring of either $R[y]$ or $R[y^{-1}]$. Thus $\dim_v(R)=\dim_v(T)+1$.

Suppose now that the equality asserted in (a) holds whenever the transcendence degree is less than d. As above, $k=F(Y_1, \ldots, Y_d)$. Let $K=F(Y_1, \ldots, Y_{d-1})$ and $B=\varphi^{-1}(K)$. By the induction hypothesis, $\dim_v(R)=\dim_v(B)+\text{t.d.}(K/F) = \dim_v(B)+(d-1)$. Also, by the case in the preceding paragraph, $\dim_v(B) = \dim_v(T)+1$. Hence, as asserted, $\dim_v(R)=\dim_v(T)+d$.

(b) By Lemma 2.1(d), $\dim(R)=\dim(k)+\dim(T)=\dim(T)$. Hence (b) follows immediately from (a) above. ■

By combining Propositions 2.3 and 2.5, we determine when the pullback $R=\varphi^{-1}(D)$ is a Jaffard domain.

Theorem 2.6. *Let (T, M, k) be a quasilocal domain which is not a field and $\varphi: T\to k$ the canonical surjection. Let $R=\varphi^{-1}(D)$, where D is any subring of k. Let F be the quotient field of D. Then:*

(a) $\dim_v(R)=\dim_v(D)+\dim_v(T)+\text{t.d.}(k/F)$.

(b) *R is a Jaffard domain if and only if D and T are each Jaffard domains and k is algebraic over F.*

We next improve the content of Proposition 2.5(a) by bounding the dimension of polynomial rings over certain pullbacks.

Proposition 2.7. *Let (T, M, k) be a quasilocal domain which is not a field, $\varphi: T\to k$ the canonical surjection, F a subfield of k, $R=\varphi^{-1}(F)$, and $d=\text{t.d.}(k/F)$. Then:*

(a) $r+\dim(T)+\min\{d, r\}\le\dim(R[X_1, \ldots, X_r])\le r+\dim_v(T)+d$ *for each positive integer r.*

(b) *If T is a valuation domain, then the left equality in* (a) *holds.*

(c) *If* $r\ge\dim_v(R)-1$, *then both equalities in* (a) *hold.*

(d) *If either T is a valuation domain or* $r\ge\dim_v(R)-1$, *then* $\dim(R[X_1, \ldots, X_r]) =\dim(R)+r+\min\{d, r\}$.

Proof. (a) As in the proof of Proposition 2.5, we may assume that $k=F(Y_1, \ldots, Y_d)$. Next, apply Arnold's formula for the Krull dimension of a polynomial domain [A, Theorem 5, page 320]:

$$\begin{aligned}\dim(R[X_1, \ldots, X_r]) &= r+\sup\{\dim(R[x_1, \ldots, x_r]):\ x_1, \ldots, x_r\in K\}\\ &\ge r+\sup\{\dim(R[t_1, \ldots, t_r]:\ t_1, \ldots, t_r\in T\},\end{aligned}$$

where K is the common quotient field of R and T.

Let $\bar{t}_i=\varphi(t_i)$ for $t_i\in T$. As in the proof of Proposition 2.5, we have $R[t_1, \ldots, t_r] =\varphi^{-1}(F[\bar{t}_1, \ldots, \bar{t}_r])$. Hence by Lemma 2.1(d), $\dim(R[t_1, \ldots, t_r]) =\dim(F[\bar{t}_1, \ldots, \bar{t}_r])+\dim(T)$. Thus $\dim(R[X_1, \ldots, X_r])\ge r+\sup\{\dim(T) +\dim(F[\bar{t}_1, \ldots, \bar{t}_r]): \bar{t}_1, \ldots, \bar{t}_r\in k\}=r+\dim(T)+\min\{r, d\}$.

For the right-hand inequality, $\dim(R[X_1, \ldots, X_r])\le\dim_v(R[X_1, \ldots, X_r]) =\dim_v(R)+r=\dim_v(T)+d+r$, the last equation resulting from Proposition 2.5.

(b) Let $T=V$ be a valuation domain. Then it is easily shown that $R[x]=V[x]$ for each $x\in K\setminus V$. Hence $\sup\{\dim(R[x_1,\ldots,x_r]): x_1,\ldots,x_r\in K\}$ $=\sup\{\dim(R[t_1,\ldots,t_r]): t_1,\ldots,t_r\in V\}$ since $\dim(R)=\dim(V)\geq\dim(W)$ for each overring W of V. Thus, by Arnold's formula, $\dim(R[X_1,\ldots,X_r])=r+\dim(V)$ $+\min\{r,d\}$.

(c) If $r\geq\dim_v(R)-1$, then also $r\geq\dim_v(T)-1$ by Proposition 2.5. Hence $r+\dim(T)=\dim(T[X_1,\ldots,X_r])=\dim_v(T[X_1,\ldots,X_r])=\dim_v(T)+r$, by Proposition 1.2(b). Also, $\min\{d,r\}=d$ because Proposition 2.5 gives $r\geq\dim_v(R)-1$ $=\dim_v(T)+d-1\geq d$, the last step holding since T is not a field. Hence the left- and right-hand terms in (a) are equal, so we have equality.

(d) In either case, $\dim(R)=\dim(T)$ by Lemma 2.1(d). Hence (d) follows from (b) and (c). ■

As a consequence of Proposition 2.7 and Proposition 2.3(a), we obtain a slight generalization of a theorem of Bastida-Gilmer [BG, Theorem 5.4].

Corollary 2.8. *Let (V, M, k) be a valuation domain which is not a field, $\varphi: V\to k$ the canonical surjection, D a subring of k, $R=\varphi^{-1}(D)$, F the quotient field of D, and $d=$ t.d.(k/F). Then*

$$\dim(R[X_1,\ldots,X_r])=\dim(V)+\dim(D[X_1,\ldots,X_r])+\min\{d,r\}$$

for each positive integer r.

Proof. Consider the quasilocal domain $A=\varphi^{-1}(F)$. Then $\dim(A[X_1,\ldots,X_r])$ $=\dim(V)+r+\min\{d,r\}$ by Proposition 2.8, while $\dim(R[X_1,\ldots,X_r])$ $=\dim(D[X_1,\ldots,X_r])+\dim(A[X_1,\ldots,X_r])-\dim(F[X_1,\ldots,X_r])$ by Proposition 2.3(a). The asserted equality follows immediately. ■

As an application, we apply some of our previous results of pseudo-valuation domains (PVD's). Recall that a domain R is a *PVD* if each prime ideal P of R is strongly prime (i.e., $xy\in P$ with x and y in the quotient field of R implies that either $x\in P$ or $y\in P$). PVD's were introduced by Hedstrom-Houston [HH 1] and have been studied extensively by several authors. Recall that a domain R is a *PVD* if and only if R is quasilocal with maximal ideal M which is strongly prime [HH 1, Corollary 1.3 and Theorem 1.4]. Additionally, a quasilocal domain R with maximal ideal M is a PVD if and only if the conductor $(M:M)$ is a valuation domain with maximal ideal M [AD, Proposition 2.5]. Moreover, all PVD's R arise in the following manner as pullbacks [AD, Proposition 2.6]. If R is a PVD with maximal ideal M, set $V=(M:M)$, $k=V/M$, and $F=R/M\subset k$; then $R=\varphi^{-1}(F)$, where $\varphi: V\to k$ is the canonical surjection. Conversely, given a valuation domain (V, M, k), F a subfield of k, and $\varphi: V\to k$ the canonical surjection, then $R=\varphi^{-1}(F)$ is a PVD.

Proposition 2.9. *Let R be a* PVD *with residue field $F=R/M$. Let $V=(M:M)$ be the associated valuation domain of R, with residue field $k=V/M$. Put $d=$ t.d.(k/F).*

Then:

(a) $\dim(R[X_1, \ldots, X_r]) = \dim(R) + r + \min\{d, r\}$.

(b) $\dim_v(R) = \dim(R) + d$.

(c) *R is a Jaffard domain if and only if* $\dim(R) < \infty$ *and* $d = 0$.

(d) *For each positive integer r, $R[X_1, \ldots, X_r]$ is a Jaffard domain if and only if* $\dim(R) < \infty$ *and* $r \geq d$.

Proof. (a) This is an immediate consequence of Corollary 2.8 since $\dim(V) = \dim(R)$. (b) follows immediately from Theorem 2.6(a) since $\dim_v(V) = \dim(V) = \dim(R)$. Evidently, (c) follows immediately from (b).

(d) Suppose $\dim(R) < \infty$. Since (b) gives $\dim_v(R[X_1, \ldots, X_r]) = \dim_v(R) + r = \dim(R) + r + d$, (a) shows that $\dim(R[X_1, \ldots, X_r]) = \dim_v(R[X_1, \ldots, X_r])$ if and only if $d = \min\{d, r\}$, i.e., if and only if $r \geq d$. ■

Remark 2.10. A special case of Proposition 2.9(a) was observed by Hedstrom-Houston [HH2, Theorem 2.5], who showed that if a finite-dimensional PVD, R, is not a strong S-domain, then $\dim(R[X]) = \dim(R) + 2$. Using this and Lemma 2.1(h), we recover [HH2, Remark 2.6]: a finite-dimensional PVD, R, is a strong S-domain if and only if $R' = V$.

We next proceed to generalize the previous "quasilocal" theory. Our first result in this direction is the "global" analogue of Lemma 2.1(d) and Theorem 2.6.

Theorem 2.11. *Let T be a domain with maximal ideal M, $k = T/M$, and $\varphi: T \to k$ the canonical surjection. Let D be a proper subring of k with quotient field F. Put $R = \varphi^{-1}(D)$ and $d = \mathrm{t.d.}(k/F)$. Then*:

(a) $\dim(R) = \max\{\dim(T), \dim(D) + \dim(T_M)\}$.

(b) $\dim_v(R) = \max\{\dim_v(T), \dim_v(D) + \dim_v(T_M) + d\}$.

Proof. $\dim(R) = \max\{\sup\{\dim(R_P)$: $P \in \mathrm{Spec}(R)$, and $P \not\supseteq M\}$, $\sup\{\dim(R_P)$: $P \in \mathrm{Spec}(R)$ and $P \supseteq M\}\}$. If $P \not\supseteq M$, Lemma 2.1(e) gives $R_P = T_Q$ for some $Q \in \mathrm{Spec}(T) \setminus \{M\}$ with $Q \cap R = P$. If $P \supseteq M$, then $\dim(R_P) = \dim(D_Q) + \dim(T_M)$ for some $Q \in \mathrm{Spec}(D)$ with $P = \varphi^{-1}(Q)$, by Lemma 2.1(g) and (d). All the inequalities needed to establish (a) now follow easily.

(b) Argue as above, using Theorem 2.6(a). ■

Corollary 2.12. *With the same hypothesis as in Theorem* 2.11:

(a) *R is a locally Jaffard domain if and only if D and T are each locally Jaffard domains and k is algebraic over F.*

(b) *If T is locally Jaffard domain with $\dim_v(T) < \infty$, D is a Jaffard domain, and k is algebraic over F, then R is a-Jaffard domain.*

Proof. (b) follows from Theorem 2.11 and Proposition 1.5(b). (a) follows from Theorem 2.6 via Lemma 2.1(e) and (g). ■

Remark 2.13. Examples 3.6–3.7 will show that R *can* be a Jaffard domain when any one of the hypotheses in Corollary 2.12(b) fails. However, we note in the context of Corollary 2.12 that if R is equicodimensional and catenarian and has finite valuative dimension, then R is a Jaffard domain if and only if D is a Jaffard domain, T is a locally Jaffard domain, and k is algebraic over F. For a proof, note that $D \cong R/M$ is also equicodimensional and catenarian. Thus, by Proposition 1.8, R and D are both Jaffard domains if and only if they are both locally Jaffard. The assertion therefore follows from Corollary 2.12(a). ■

We next give some concrete applications of the above "global" theory to $D+M$ constructions. Here, T is assumed to be a domain of the form $T=K+M$, where K is a subfield of T and M is a maximal ideal of T. If D is a subring of K, then $R=D+M$ is a subring of T. The classical case arises when $T=V$ is a valuation domain (with maximal ideal M). Since $\dim_v(V)=\dim(V)$, Theorem 2.6 yields:

Proposition 2.14. *Let V be a nontrivial valuation domain of the form $V=K+M$, where K is a field and M is the maximal ideal of V. Let $R=D+M$, where D is a proper subring of K. Let F be the quotient field of D, and let $d=\text{t.d.}(k/F)$. Then:*

(a) $\dim_v(R)=\dim_v(D)+\dim(V)+d$.

(b) *R is a Jaffard domain if and only if D is a Jaffard domain, V is finite-dimensional, and k is algebraic over F.*

A "global" type of $D+M$ construction arises from $T=K[X]$, the polynomial ring over a field K, by considering $M=XT$ and a subring D of K. In this case, neither T nor R is quasilocal. Theorem 2.11 yields:

Proposition 2.15. *Let K be a field, D a subring of K with quotient field F, $R=D+XK[X]$, and $d=\text{t.d.}(k/F)$. Then:*

(a) $\dim(R)=\dim(D)+1$.

(b) $\dim_v(R)=\dim_v(D)+d+1$.

(c) *R is a Jaffard domain if and only if D is a Jaffard domain and k is algebraic over F.*

We remark that Costa-Mott-Zafrullah [CMZ, Corollaries 2.10 and 4.17] obtained parts (a) and (b) of Proposition 2.15 for the special case in which D has quotient field K. In [CMZ], they investigate the more general construction $T^{(S)}=D+XD_S[X]$ where D is a domain and S is a multiplicative subset of D. Rather than seek an analogue of Proposition 2.15 for the $T^{(S)}$ construction, we shall consider, more generally, domains of the form $A=D+XR[X]$, where $D\subset R$ are domains having quotient field K.

Proposition 2.16. *Let $D\subset R$ be domains with a common quotient field K and let $A=D+XR[X]$. Then*

(a) $\dim_v(A)=\dim_v(D)+1$.

(b) *If D is a Jaffard domain, then A is a Jaffard domain.*

Proof. (a) The inclusion of domains $D[X] \subset A \subset D+XK[X]$ yields $\dim_v(D)+1 = \dim_v(D[X]) \geq \dim_v(A) \geq \dim_v(D+XK[X]) = \dim_v(D)+1$, the last step via Proposition 2.15(b). Then (a) is immediate.

(b) First, note that $\dim(D)+1 \leq \dim(A)$ for any domain D. If in addition, D is a Jaffard domain, then $\dim_v(D)+1 = \dim(D)+1 \leq \dim(A) \leq \dim_v(A) = \dim_v(D)+1$ by (a) above. Hence, A is a Jaffard domain. ■

Remark 2.17. In general, the converse of Proposition 2.16(b) is not valid, even for $A = T^{(S)} = D + XD_S[X]$. To see this, consider a domain D such that $D[X]$ is a Jaffard domain but D is not a Jaffard domain, Then, with $S = \{1\}$, $A = T^{(S)} = D[X]$ is a Jaffard domain, but D is not a Jaffard domain. The relationship between $T^{(S)}$ and D will be studied more deeply in a subsequent article.

We next examine a generalization of the situation described in Theorem 2.11 in order to apply the "global" theory developed earlier to two important classes of domains introduced by Gilmer and Nagata. For the rest of Section 2, the reader is assumed to be familiar with the gluing-theoretic generalization of Lemma 2.1 to be found in [F1, Theorem 1.4]. Section 3 can be read independently of the rest of Section 2.

Proposition 2.18. *Let T be a domain with $M_1, \ldots, M_r$ pairwise distinct nonzero maximal ideals of T. For each $1 \leq i \leq r$, let $\varphi_i\colon T \to T/M_i = k_i$ be the canonical surjection; D_i a subring of k_i; $R_i = \varphi_i^{-1}(D_i)$; and $d_i = \mathrm{t.d.}(k_i/D_i)$. Let $\varphi\colon T \to \prod_{i=1}^{r} k_i$ be the canonical surjection induced by the φ_i's. Set $R = \varphi^{-1}\left(\prod_{i=1}^{r} D_i\right)$. Then:*

(a) $R = \bigcap_{i=1}^{r} R_i$.

(b) $\dim(R) = \max\{\dim(T);\ \dim(D_i) + \dim(T_{M_i}) \colon 1 \leq i \leq r\}$.

(c) $\dim_v(R) = \max\{\dim_v(T);\ \dim_v(D_i) + \dim_v(T_{M_i}) + d_i \colon 1 \leq i \leq r\}$.

Proof. (a) follows easily from the definitions involved. The proof of (b) follows directly from the order-theoretic implications of the topological description of $\mathrm{Spec}(R)$ given by [F1, Theorem 1.4]. The proof of (c) is similar to the proof of Theorem 2.11(b), once the following observations have been made. Let $P \in \mathrm{Spec}(R)$. Then either P contains none of $M_1, \ldots, M_r$ or P contains exactly one M_i. In the former case, $R_P = T_Q$ with $Q \in \mathrm{Spec}(T)$, $Q \neq M_1, \ldots, M_r$. In the latter case, P is the image of some $Q_1 \in \mathrm{Spec}(D_i)$. Then by applying [F1, Proposition 1.9] to the pullback definition of R, we have $R_P \cong (D_i)_{Q_1} \times_{k_i} T_{M_i}$. Using Theorem 2.6(a), we can now finish the proof as in Theorem 2.11(b). ■

We apply Proposition 2.18 to a well known class of domains introduced by Gilmer [G3, Section 3].

Corollary 2.19. *Let L be a field, K a subfield of L, and $\{V_i\colon 1 \le i \le r\}$ a finite set of pairwise incomparable finite-dimensional valuation domains of L, each with maximal ideal M_i, such that $V_i = K + M_i$ for each $1 \le i \le r$. For each i, let D_i be a subring of K and $d_i = \mathrm{t.d.}(K/D_i)$. Set $J_i = D_i + M_i$, $J = \bigcap_{i=1}^{r} J_i$, and $P = \bigcap_{i=1}^{r} V_i$. Let $\varphi\colon P \to \prod_{i=1}^{r} K_i = P/\mathrm{Rad}(P)$ be the canonical surjection from the semiquasilocal Prüfer domain P to $\prod_{i=1}^{r} K_i$. Then:*

(a) *The diagram*

$$\begin{tikzcd} J \arrow[r] \arrow[d, hook] & \prod_{i=1}^{r} D_i \arrow[d, hook] \\ P \arrow[r, "\varphi"] & \prod_{i=1}^{r} K_i \end{tikzcd}$$

is a pullback.

(b) *J is a locally Jaffard domain if and only if D_i is a locally Jaffard domain and $d_i = 0$ for each $1 \le i \le r$.*

(c) *If D_i is a Jaffard domain and $d_i = 0$ for each $1 \le i \le r$, then J is a Jaffard domain.*

Proof. (a) is immediate from the definitions. Since P is a Jaffard domain, (b) follows by applying Theorem 2.11 to the isomorphisms noted in the proof of Proposition 2.18(c). Finally, since P is locally Jaffard, (c) is a direct consequence of Proposition 2.18(b), (c). ■

Another important class of domains, suggested by [N2, Example 2.1, page 204], is analyzed next.

Proposition 2.20. *Let $M_1, \ldots, M_r$ be finitely many maximal ideals of a domain T. For each i, $1 \le i \le r$, let $k_i = T/M_i$ and let $\varphi_i\colon T \to k_i$ be the canonical surjection. Let φ be the canonical surjection from T to $T/\bigcap M_i \cong \prod k_i$. Let D be a domain whose quotient field k is contained in $\bigcap k_i$. Consider the diagonal embedding $\Delta\colon k \to k^r = k \times \ldots \times k$, the embedding, $\alpha\colon D \to k$, and the product embedding $\beta\colon k^r \to \Pi k_i$. Put $T_0 = \varphi^{-1}(k)$, $R = \varphi^{-1}(D)$, and $R_i = \varphi_i^{-1}(D)$ for all i. Then:*

(a) $\dim(T_0) = \dim(T)$.

(b) $\dim_v(T_0) = \max\{\dim_v(T);\ \dim_v(T_{M_i}) + \mathrm{t.d.}(k_i/k)\colon 1 \le i \le r\}$.

(c) $\dim(R) = \max\{\dim(T);\ \dim(D) + \dim(T_{M_i})\colon 1 \le i \le r\}$.

(d) $\dim_v(R) = \max\{\dim_v(T);\ \dim_v(D) + \dim_v(T_{M_i}) + \mathrm{t.d.}(k_i{:}k)\colon 1 \le i \le r\}$.

Proof. We have the following pullbacks:

$$\begin{tikzcd}
R \arrow[r] \arrow[d, hook] & D \arrow[d, hook, "\alpha"] \\
T_0 \arrow[r] \arrow[d, hook] & k \arrow[d, hook, "\Delta"] \\
S=\varphi^{-1}(k') \arrow[r] \arrow[d, hook] & k' \arrow[d, hook, "\beta"] \\
T \arrow[r] & \Pi k_i
\end{tikzcd}
\qquad
\begin{tikzcd}
R_i=\varphi_i^{-1}(D) \arrow[r] \arrow[d, hook] & D \arrow[d, hook, "\alpha"] \\
T_i=\varphi_i^{-1}(k) \arrow[r] \arrow[d, hook] & k \arrow[d, hook] \\
T \arrow[r, "\varphi_i"] & k_i
\end{tikzcd}$$

Using the definition of β, we see easily that $S=T_1\cap\ldots\cap T_r$. Moreover, the order-theoretic implication of [F1, Theorem 1.4] is that $\dim(T_0)=\dim(T)=\dim(S)$. In particular, (a) holds. Similarly, by [F1, Theorem 1.4], we have for each i, $1\le i\le r$, that

$$\dim(R_i)=\max\{\dim(T),\dim(D)+\dim(T_{M_i})\};\quad\text{and}$$
$$\dim(R)=\max\{\dim(T);\dim(D)+\dim(T_{M_i}): 1\le i\le r\}.$$

In particular, (c) holds.

Next, since Δ is finite, S is integral over T_0 (cf. [F1, Corollary 1.5(4)]). Hence, $\dim_v(S)=\dim_v(T_0)$. However, Proposition 2.18(c) gives $\dim_v(S)=\max\{\dim_v(T), \dim_v(T_{M_i})+\text{t.d.}(k_i/k): 1\le i\le r\}$ since $\dim_v(k)=0$. Thus, (b) follows. Moreover, (d) follows by combining (b) with Theorem 2.6(a). ■

Proposition 2.20 can be used to derive the following result about another important class of domains. We leave the proof to the reader.

Corollary 2.21. *Let $\{V_i: 1\le i\le r\}$ be a set of finite-dimensional pairwise incomparable valuation domains of a field L. Suppose that each (V_i, M_i) has the form $V_i=K_i+M_i$, where K_i is a subfield of V_i. Let D be a domain with quotient field K contained in each K_i. Set $J_i=D+M_i$ for each $1\le i\le r$, $J=\bigcap_{i=1}^{r} J_i(=D+\bigcap M_i)$, and $P=\bigcap_{i=1}^{r} V_i$. Then:*

(a) *The following diagram of canonical maps*

$$\begin{tikzcd}
J \arrow[r] \arrow[d, hook] & D \arrow[d, hook] \\
P \arrow[r] & \prod_{i=1}^{r} K_i
\end{tikzcd}$$

is a pullback.

(b) *J is a Jaffard domain (resp., locally Jaffard domain) if and only if D is a Jaffard domain (resp., locally Jaffard domain) and* t.d.$(K_i/K)=0$ *for each* $1 \leq i \leq r$.

3 Examples

This section collects some examples showing that several of the results in the earlier sections are best-possible. We begin with a known example which was alluded to in Section 0. This is included both for the sake of completeness and to facilitate the development of some later examples.

Example 3.1. (cf. [G1, Exercise 17(1), page 372]) (a) For each positive integer n, there exists a finite-dimensional (non-Jaffard) domain R such that $\dim_v(R) - \dim(R) = n$.

Indeed, consider a set $\{X_1, \ldots, X_{n+1}\}$ of $n+1$ indeterminates over a field k. Let L denote the field $k(X_1, \ldots, X_n)$. Also, define the valuation domain

$$V = L[X_{n+1}]_{(X_{n+1})} = L + M \quad (\text{with } M = X_{n+1}V) \text{ and the ring } R = k + M.$$

As is well known (cf. [G1, Exercise 12, page 202]), $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ as sets, and so $\dim(R) = \dim(V) = 1$. This also follows from [F1, Proposition 2.1], since $R \cong k \times_L V$. Applying Proposition 2.5(a) to this pullback description of R, we have $\dim_v(R) = \dim_v(V) + \text{t.d.}(L/k) = \dim(V) + n = 1 + n$. Thus, $\dim_v(R) - \dim(R) = n$, as asserted.

(b) There exists a finite-dimensional (non-Jaffard) domain R with $\dim_v(R) = \infty$.

The construction of a suitable R is similar to that in (a). Now, let $\{X_1, X_2, \ldots\}$ be a denumerable set of indeterminates over a field k, let Y be another indeterminate, and put $L = k(\{X_1, X_2, \ldots\})$, $V = L[Y]_{(Y)} = L + M$ (with $M = YV$), and $R = k + M (\cong k \times_L V)$. Arguing as above, we see that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ and $\dim(R) = \dim(V) = 1$; and that $\dim_v(R) = \dim_v(V) + \text{t.d.}(L/k) = 1 + \infty = \infty$. Thus, $\dim_v(R) = \infty$, completing the proof of (b).

(c) One may arrange domains R satisfying the assertions in (a) and (b) but having arbitrary positive Krull dimension m by instead choosing the valuation domain $V = L + M$ to be m-dimensional.

The next example is included for several reasons. Its assertion (a) relates to Propositions 1.5 and 1.6; (b) relates to Theorem 1.13; and (d) shows that the converse of (0.6) is false.

Example 3.2. There exists a two-dimensional Jaffard domain R such that all the following conditions hold:

(a) R is not locally Jaffard;

(b) R is a going-down domain (indeed a GPVD in the sense of [DF1]);

(c) There exists a maximal ideal N of the polynomial ring $R[X]$ such that $ht(N)=3=\dim(R[X])$, although $m=N\cap R$ is a maximal ideal of R with $ht(m)=1$ $(<\dim(R))$;

(d) R does not satisfy the altitude inequality formula. (Hence, R is not stable strong S-domain. In fact, R is not an S-domain.)

The construction begins with a suitable field K and three indeterminates Z_1, Z_2, Z_3 over K. Put $L=K(Z_1, Z_2, Z_3)$. Now,

$$V_1=K(Z_1,Z_2)[Z_3]_{(Z_3)}=K(Z_1,Z_2)+M_1$$

is a (discrete) rank 1 valuation ring of L, with maximal ideal $M_1=Z_3V_1$. For suitable K (for instance, $K=F(T_1, T_2, \ldots)$ with T_i indeterminates over a field F), choose (V, M) to be a rank 1 valuation overring of $K(Z_3)[Z_1, Z_2]$ of the form $V=K_1+M$, with $K_1\cong K(Z_3)$. With φ denoting the canonical surjection $V\to K(Z_3)$, consider the pullback $V_2=\varphi^{-1}(K[Z_3]_{(Z_3)})=K[Z_3]_{(Z_3)}+M$. By the lore of the $D+M$ construction, V_2 is a rank 2 valuation ring of L, and $V_2=K+M_2$ has maximal ideal $M_2=Z_3K[Z_3]_{(Z_3)}+M$ and height 1 prime M.

We show next that V_1 and V_2 are incomparable. If not, it would follow from the one-dimensionality of V_1 that $V_2\subset V_1$. Then we would have $V_1=(V_2)_M$, whence $Z_3V_1=M_1=M(V_2)_M=M$ and $1=Z_3Z_3^{-1}\in MV=M$, a contradiction. Thus, V_1 and V_2 are incomparable. Since V_1 and V_2 both has quotient field L, we now see from [N2, Theorem 11.11] that $S=V_1\cap V_2$ is a two-dimensional Prüfer domain with exactly two maximal ideals, say m_1 and m_2, denoted so that $S_{m_1}=V_1$ and $S_{m_2}=V_2$.

Next, choose k to be any subfield of $K(Z_1, Z_2)$ such that $\mathrm{t.d.}(K(Z_1, Z_2)/k)=1$. With $\varphi_1\colon V_1\to K(Z_1, Z_2)$ denoting the canonical surjection, consider the pullback $A=\varphi_1^{-1}(k)=k+M_1$. Next, with $\psi\colon S\to S/m_1$ $(\cong V_1/M_1\cong K(Z_1, Z_2))$ denoting the canonical surjection, consider the pullback $R=\psi^{-1}(k)$. Since $V_1=S_{m_1}$, it follows that $R=A\cap S$, whence $R=A\cap V_2$. Moreover, by applying [F1, Theorem 1.4] to the pullback definition of R, we infer that the canonical map $\operatorname{Spec}(S)\to\operatorname{Spec}(R)$ is a homeomorphism, and hence an order-isomorphism. Thus, $\dim(R)=\dim(S)=2$.

Moreover, Theorem 2.11 gives

$$\begin{aligned}\dim_v(R)&=\sup\{\dim_v(S), \dim_v(S_{m_1})+\dim_v(k)+\mathrm{t.d.}((S/m_1)/k)\}\\&=\sup\{\dim(S), \dim(V_1)+0+\mathrm{t.d.}(K(Z_1,Z_2)/k)\}\\&=\sup\{2, 1+0+1\}=2.\end{aligned}$$

Hence, R is a (two-dimensional) Jaffard domain. We now proceed to verify (a)-(d).

(a) Recall that $R=A\cap V_2$. Since V_1 and V_2 are incomparable, it follows from the proof of [DF, Example 2.5] that if $n_1=m_1\cap R$ and $n_2=m_2\cap R$ denote the maximal ideals of R, then $R_{n_1}=A$ and $R_{n_2}=V_2$. Now,

$$\dim(R_{n_1})=\dim(k+M_1)=\dim(k)+\dim(V_1)=0+1=1,$$

but Proposition 2.5(a) gives

$$\dim_v(R_{n_1}) = \dim_v(k + M_1) = \dim_v(V_1) + \text{t.d.}(K(Z_1, Z_2)/k) = 1 + 1 = 2.$$

Hence, R_{n_1} is not a Jaffard domain, and so R is not locally Jaffard.

(b) As in [DF, Examples 3.2(a)], $R = A \cap V_2$ is a GPVD. Hence (cf. [DF, page 156 and Corollary 2.3] and [D, Proposition 2.1]), R is a going-down domain.

(c) Since $S_{m_1} = V_1$, we have $M_1 \cap S = m_1$ and, in particular, $m_1 \subset M_1 \subset A$. As $R = A \cap V_2$, it follows that $m_1 \subset R$. Since $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is an order-isomorphism, $m_1 = m_1 \cap R$ is a height 1 maximal ideal of R. In fact, $m_1 = n_1$, and so $R_{m_1} = A$. Next note, since $m_1 R_{m_1} = M_1$, that

$$ht(m_1[X]) = ht(m_1 R_{m_1}[X]) = ht(M_1[X]).$$

We proceed to show that each of these heights is 2. Now, since $K(Z_1, Z_2)$ is not algebraic over k, $A' \neq V_1$. Thus, applying [HH2, Remark 2.6 and Theorem 2.5] to the pseudo-valuation domain A, we have $\dim(A[X]) = \dim(A) + 2 = 3$; hence, $ht(M_1[X]) = 2$. (cf. also Remark 2.10.) Moreover, since R is Jaffard, $\dim(R[X]) = 1 + \dim(R) = 3$. Of course, the height 2 prime $m_1[X]$ is not a maximal ideal; let N be any prime of $R[X]$ properly containing $m_1[X]$. Evidently, $ht(N) = 3$ and $N \cap R = m_1$. Thus, (c) has been established, with $m = m_1$.

(d) The altitude inequality formula fails if we consider $S = R[X]$ and $P = N$ (as in (c)). Indeed,

$$\begin{aligned} &ht(N) + \text{t.d.}(\underline{k}(N)/\underline{k}(N \cap R)) \geq 3 + 0 = 3 \\ &> 1 + 1 = ht(m_1) + 1 = ht(N \cap R) + \text{t.d.}(R[X]/R). \end{aligned}$$

For the parenthetical conclusion, recall from (c) that $ht(m_1) = 1$ and $ht(m_1[X]) = 2$. ∎

Examples 3.4 and 3.5 will relate to the sufficient condition in Proposition 1.2(b). First, it will be helpful to establish the following result.

Proposition 3.3. *Let $X_1, \ldots, X_r$ be finitely many indeterminates over a domain R which is not a field. Assume $n = \dim_v(R) < \infty$. Then:*

(a) *If $R[X_1, \ldots, X_r]$ is a Jaffard domain, then*

$$r \geq \frac{n - \dim(R)}{\dim(R)}.$$

(b) *Assume* $\dim(R) = 1$. *Then $R[X_1, \ldots, X_r]$ is a Jaffard domain if and only if $r \geq n - 1$.*

Proof. (a) $r + n = \dim_v(R[X_1, \ldots, X_r]) = \dim(R[X_1, \ldots, X_r])$ is, by a result recalled in Section 0, at most $r(1 + \dim(R)) + \dim(R)$. Thus,

$$r + n \leq r + r\dim(R) + \dim(R),$$

which immediately yields (a).

(b) The "if" assertion is a special case of Proposition 1.2(b), while the "only if" assertion is a special case of (a). ■

Example 3.4. For each positive integer r, there exists a finite-dimensional non-Jaffard domain R such that r in the least positive integer m for which the polynomial ring $R[X_1, \ldots, X_m]$ is a Jaffard domain.

For a suitable construction, take R as in Example 3.1(a), with $\dim(R)=1$ and $\dim_v(R)=r+1$. As $r+1\neq 1$, R is not a Jaffard domain. The final assertion follows directly from Proposition 3.3(b), since $n=r+1$. ■

Example 3.5. There exists a non-Jaffard domain T with $n=\dim_v(T)<\infty$ and a positive integer $r<n-1$ such that the polynomial ring $T[X_1, \ldots, X_r]$ is a Jaffard domain.

To begin the construction, take R as in Example 3.1(a), with $\dim(R)=1$ and $\dim_v(R)=3$. Consider $r+1$ indeterminates $Y, Y_1, \ldots, Y_r$ over R, and put $T=R[Y]$. In the notation of Example 3.1(a), $R'\neq V$ since L is not algebraic over k. Thus, by [HH2, Remark 2.6 and Theorem 2.5] or Remark 2.10, $\dim(T)=\dim(R)+2=3$. Moreover, by Arnold's result (see Theorem 0.1(iv)), $r\geq\dim_v(R)-1=2$ implies that $\dim(R[Y_1, \ldots, Y_r])=r+\dim_v(R)=r+3$. Thus, $r\geq 1$ implies $\dim(T[Y_1, \ldots, Y_r])=\dim(R[Y, Y_1, \ldots, Y_r])=(r+1)+3=r+4$.

Next, we compute a key valuative dimension. Note that

$$\dim_v(T)=\dim_v(R[Y])=1+\dim_v(R)=4>3=\dim(T).$$

In particular, T is not a Jaffard domain. It will therefore suffice to prove that $T[Y_1]$ is a Jaffard domain, since $1<3=\dim_v(T)-1$. We have seen that $\dim(T[Y_1])=1+4=5$, and so we need only prove $\dim_v(T[Y_1])=5$. This, in turn, follows, for

$$\dim_v(T[Y_1])=\dim_v(R[Y, Y_1])=2+\dim_v(R)=2+3=5. \quad ■$$

Examples 3.6–3.7 are designed to show that none of hypotheses in Corollary 2.12(b) is a necessary condition. Specifically, we consider a maximal ideal M of a domain T, the residue field $k=T/M$, the canonical surjection $\varphi: T\to k$, a subring D of k, and the pullback $R=\varphi^{-1}(D)$. We shall show that R *can* be a Jaffard domain in each of the following situations:

(3.6): T is a non-Jaffard domain, $\dim_v(T)<\infty$, D is a Jaffard domain, and k is algebraic over D;

(3.7(a)): T and D are each Jaffard domains and k is not algebraic over D;

(3.7(b)): T is a Jaffard domain, D is a non-Jaffard domain, $\dim_v(D)<\infty$, and k is not algebraic over D.

(3.7(c)): T is a Jaffard domain, D is a non-Jaffard domain, $\dim_v(D)<\infty$, and k is algebraic over D.

Examples 3.6. Let U, V be two indeterminates over a field K. Define

$$V_1 = K(U)[V]_{(V)}, \quad V_2 = K(V)[U]_{(U)}, \quad P = V_1 \cap V_2,$$
$$A = K + UK(V)[U]_{(U)}, \quad \text{and} \quad T = V_1 \cap A.$$

As in Example 3.2, [N2, Theorem 11.11] shows that P is a one-dimensional Prüfer domain with two maximal ideals, $M_1 = VK(U)[V]_{(V)} \cap P$ and $M_2 = UK(V)[U]_{(U)} \cap P$; and that $P_{M_1} = V_1$ and $P_{M_2} = V_2$. Similarly, [DF, Examples 2.5 and 3.2(a)] yields that T is one-dimensional with but two maximal ideals, $m_1 = M_1 \cap T$ and $m_2 = M_2 \cap T$; and that $T_{m_1} = V_1$ and $T_{m_2} = A$. Now, by Proposition 2.5(a), $\dim_v(T_{m_2}) = \dim_v(A) = \dim_v(V_2) + \text{t.d.}(K(V)/K) = 1 + 1 = 2 > 1 = \dim(T_{m_2})$. Thus, T is not locally Jaffard. Indeed, T is not a Jaffard domain since $\dim(T) = 1$ is less than

$$\dim_v(T) = \max\{\dim_v(T_{m_1}), \dim_v(T_{m_2})\} = \max\{1, 2\} = 2.$$

Now, consider $M = m_1 \in \operatorname{Spec}(T)$ and the canonical surjection φ from T onto $k = T/M \cong V_1/V(V_1) \cong K(U)$. Put $D = K[U]$ and $R = \varphi^{-1}(D)$. Of course, D is a Jaffard domain and k is (the quotient field of, and hence) algebraic over D. It remains only to prove that R is a Jaffard domain. For this, note first via Theorem 2.11(a) that $\dim(R) = \max\{\dim(T), \dim(T_M) + \dim(D)\} = \max\{1, 1+1\} = 2$. Next, via Theorem 2.11(b), we have

$$\dim_v(R) = \max\{\dim_v(T), \dim_v(T_M) + \dim_v(D) + \text{t.d.}(k/D)\}$$
$$= \max\{2, \dim_v(V_1) + 1 + 0\} = \max\{2, 2\} = 2. \quad \blacksquare$$

Example 3.7. (a) Let V and W be incomparable valuation rings of a suitable field K, with $n = \dim(V) \geq 3$ and $\dim(W) = 1$. Then, by [N2, Theorem 11.11], $T = V \cap W$ is an n-dimensional Prüfer domain with but two maximal ideals, say M_1 and M, denoted so that $T_{M_1} = V$ and $T_M = W$. Let $\varphi\colon T \to T/M = k$ be the canonical surjection. We further require that k be "suitable" as follows: k has a subfield F and a subring D such that $\dim(D) = 1 = \dim_v(D)$, F is the quotient field of D, and $\text{t.d.}(k/F) = 1$. Put $R = \varphi^{-1}(D)$.

We claim that R is a Jaffard domain. To see this, note via Theorem 2.11(a), (b) that $\dim(R) = \max\{n, 1+1\} = n$ and $\dim_v(R) = \max\{\dim_v(T), \dim_v(T_M) + \dim_v(D) + \text{t.d.}(k/D)\} = \max\{n, 1+1+1\} = n$.

It remains only to observe that T and D are each Jaffard domains and k is not algebraic over D.

(b) Suppose we alter the construction in (a) by taking $n \geq 4$ and $\dim_v(D) = 2$ (keeping all the other conditions, including $\dim(D) = 1$). The only change in the announced conclusion for (a) is that (the new) D is not a Jaffard domain. In particular, (the new) R is a Jaffard domain. Now, consider $P = M \cap R \in \operatorname{Spec}(R)$. In fact, $P = \ker(\varphi) = M$ and $R/M \cong D$. Thus, we consider another example of a phenomenon noted in Remark 1.3(a): a factor domain of a Jaffard domain need not be a Jaffard domain. Such examples should be contrasted with Remark 1.9(b).

(c) It suffices to alter the above construction by taking k algebraic over F. We leave the details to the reader. ■

Next, we give an example that relates to Proposition 1.11 and was promised in Remark 1.12(b).

Example 3.8. There exists a two-dimensional quasilocal (hence, equicodimensional) strong S-domain R which is not a Jaffard domain.

The construction is essentially that of [BMRH, Example 3], which we assume that the reader has at hand. However, we modify the construction of V^* in that example. (This is done in order to be sure that $V=K+M_1$, a fact used in the proof that R/P is a DVR.) Specifically, consider $s=\sum_{n=1}^{\infty} T^{n!}\in K[[T]]$. Since s is known to be transcendental over $K(T)$, the assignment $Y_1\mapsto T$, $Y_2\mapsto s$ gives a K-algebra monomorphism $K[Y_1, Y_2]\to K[[T]]$. This induces an embedding φ: $K(Y_1, Y_2)\to K((T))$ of fields. Put $V^*=\varphi^{-1}(K[[T]])$. since $K[[T]]$ is a rank 1 valuation domain of $K((T))$ and since φ is an injection, we see that V^* is a discrete rank 1 valuation overring of $K[Y_1, Y_2]$. (To see that V^* is discrete rank 1, just notice that its value group, $(K(Y_1, Y_2)\setminus\{0\})/U(V^*)$, embeds in the value group of $K[[T]]$.) Evidently, Y_1 is in the maximal ideal, say N, of V^*. However, it is easy to see from the injectivity of φ that $V^*=K+Y_1V^*$; in particular, $N=Y_1V^*$ and $V^*=K+N$. Now, we may resume the approach in [BMRH]. Letting f: $V_1\to k(V_1)\cong K(Y_1, Y_2)$ be the canonical surjection, we consider the pullback $V=f^{-1}(V^*)$. Of course, (V, M_1) is a rank 2 valuation domain of $K(Y_1, Y_2, Y_3)$. Moreover, since $V^*=K+N$ is a homomorphic image of V, a simple calculation shows that $V=K+M_1$. The construction of R and verification of its asserted properties, including the fact that $\dim_v(R)=3>2=\dim(R)$, now continues as in [BMRH, page 4], line 18 – end of Example 3]. ■

Our next example will show that the inequalities in Proposition 2.7(a) can be strict.

Example 3.9. There exist a quasilocal finite-dimensional domain (T, M, k), with canonical surjection φ: $T\to k$, and a subfield F of k such that $R=\varphi^{-1}(F)$ and $d=\text{t.d.}(k/F)<\infty$ satisfy

$$r+\dim(T)+\min\{d, r\}<\dim(R[X_1, \ldots, X_r])<r+\dim_v(T)+d$$

for some positive integer r.

For the construction, consider at least five indeterminates $Y_1, \ldots, Y_d, U, V, Z, W$ over a field F. Define $k=F(Y_1, \ldots, Y_d)$, $R_1=k(U, V, Z)[W]_{(W)}$, $S=k(U)[V]_{(V)}$, $A=k(U, V)+WR_1$, $B=k+VS$, $R_2=S+WR_1$, and $R_3=k+VS+WR_1(=B+WR_1)$. Thus, each square in the diagram

$$\begin{tikzcd}
R_3 \arrow[r, two heads] \arrow[d, hook] & B \arrow[r, two heads] \arrow[d, hook] & k \arrow[d, hook] \\
R_2 \arrow[r, two heads] \arrow[d, hook] & S \arrow[r, two heads] \arrow[d, hook] & k(U) \\
A \arrow[r, two heads] \arrow[d, hook] & k(U,V) \arrow[d, hook] & \\
R_1 \arrow[r, two heads] & k(U,V,Z) &
\end{tikzcd}$$

is a pullback. By applying Lemma 2.1(d) and Theorem 2.6(a), we have the following conclusions:

$\dim(R_1)=\dim_v(R_1)=1=\dim(S)=\dim_v(S)$;

$\dim(A)=0+1=1$;

$\dim_v(A)=0+1+1=2$;

$\dim(R_2)=\dim(S)+\dim(R_1)=2$;

$\dim_v(R_2)=1+1+1=3$;

$\dim(B)=0+1=1$;

$\dim_v(B)=0+1+1=2$;

$\dim(R_3)=\dim(k)+\dim(R_2)=2$;

$\dim_v(R_3)=0+3+1=4$.

Let T denote the quasilocal domain R_3, let $\varphi\colon T\to k$ be the canonical surjection, and consider the pullback $R=\varphi^{-1}(F)$. By Lemma 2.1(d), $\dim(R)=0+2=2$. By Theorem 2.6(a),

$$\dim_v(R)=\dim_v(F)+\dim_v(T)+\text{t.d.}(k/F)=0+4+d=4+d.$$

In addition, the P^{n}VD theory in [F2, Theorem 2.1] (with $e=$ "d" $=n+1=2$) gives $\dim(T[X])=d+e+1=5$. Similarly, $\dim(R[X])=5$.

Now, restrict to $r=1$ (and $d\geq 1$). The desired inequality

$$r+\dim(T)+\min\{d,r\}<\dim(R[X])<1+\dim_v(T)+d$$

asserts

$$1+2+\min\{d,1\}<5<1+4+d,$$

which is evident. ■

Remark 3.10. (a) We shall say that a domain R is a *stably S-domain* in case $R[X_1,\dots,X_r]$ is an S-domain for each positive integer r. The proof of [Ka2, Lemme 1.4, (i)⇒(iii)] also shows that if a domain R satisfies the altitude inequality formula, then R is a stably S-domain. In (b), we shall show that "stably S-domain" is not implied by, and does not imply, "Jaffard domain".

(b) By Example 3.2(d), a Jaffard domain need not be a stably S-domain. Moreover, one can show that a stably S-domain need not be a Jaffard domain. To do this, it is enough to take R to be a two-dimensional PVD with $\dim_v(P)=3$. The proof follows easily from the fact [HH 1, Proposition 2.6] that R_P is a valuation domain for the unique height 1 prime ideal P of R.

(c) It is interesting to note the following result. If R is a Jaffard domain with exactly one prime ideal of height 1, then R is a stably S-domain.

For the proof, we must show that if $P \in \operatorname{Spec}(R[X_1, \ldots, X_n])$ has height 1, then $P[X_{n+1}]$ also has height 1. To do this, let $p = P \cap R$. Since R is a Jaffard domain, there are only two cases: either $p=0$ or $P=p[X_1, \ldots, X_n]$. In the former case, the assertion is clear, by viewing matters in $K[X_1, \ldots, X_{n+1}]$, where K is the quotient field of R. The latter case follows easily from the observation that $\dim(R[X_1, \ldots, X_{n+1}]) = \dim(R)+n+1$.

References

[An] D.D. Anderson, Some remarks on the ring $R(X)$, Comment. Math. Univ. St. Pauli 26 (1977), 137–140.

[AD] D.F. Anderson and D.E. Dobbs, Pairs of rings with the same prime ideals, Canad. J. Math. 32 (1980), 362–384.

[A] J.T. Arnold, On the dimension theory of overrings of an integral domain, Trans. Amer. Math. Soc. 138 (1969), 313–326.

[AG1] J.T. Arnold and R. Gilmer, The dimension sequence of a commutative ring, Amer. J. Math. 96 (1974), 385–408.

[AG2] J.T. Arnold and R. Gilmer, Two questions concerning dimension sequences, Arch. Math. 29 (1977), 497–503.

[AG3] J.T. Arnold and R. Gilmer, The dimension theory of commutative semigroup rings, Houston J. Math. 2 (1976), 299–313.

[BG] E. Bastida and R. Gilmer, Overrings and divisorial ideals of rings of the form $D+M$, Michigan Math. J. 20 (1973), 79–95.

[BDF1] A. Bouvier, D.E. Dobbs and M. Fontana, Universally catenarian integral domains, Advances in Math., to appear.

[BDF2] A. Bouvier, D.E. Dobbs and M. Fontana, Two sufficient conditions for universal catenarity, Comm. Algebra 15 (1987), 861–872.

[BF] A. Bouvier and M. Fontana, The catenarian property of the polynomial rings over a Prüfer domain, in Sém. Algèbre P. Dubreil et M.P. Malliavin, Springer Lecture Notes Math. 1146 (1985), 340–354.

[BMRH] J.W. Brewer, P.R. Montgomery, E.A. Rutter and W.J. Heinzer, Krull dimension of polynomial rings, in Springer Lecture Notes Math. 311 (1972), 26–45.

[CMZ] D. Costa, J.L. Mott and M. Zafrullah, The construction $D+XD_S[X]$, J. Algebra 53 (1978), 423–439.

[D] D.E. Dobbs, Divided rings and going-down, Pacific J. Math. 67 (1976), 353–363.

[DF] D.E. Dobbs and M. Fontana, Locally pseudo-valuation domains, Ann. Mat. Pura Appl. 134 (1983), 147–168.

[F1] M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. 123 (1980), 331–355.

[F2] M. Fontana, Sur quelques classes d'anneaux divisés, Rend. Sem. Mat. Fis. Milano 51 (1981), 179–200.

[G1] R. Gilmer, *Multiplicative ideal theory*, Dekker, New York, 1972.

[G2] R. Gilmer, *Commutative semigroup rings*, Chicago Lectures Math., Univ. Chicago Press, Chicago, 1984.

[G3] R. Gilmer, Two constructions of Prüfer domains, J. Reine Angew. Math. 239/240 (1969), 153–162.

[HH1] J.R. Hedstrom and E.G. Houston, Pseudo-valuation domains, Pacific J. Math. 75 (1978), 137–147.

[HH2] J.R. Hedstrom and E.G. Houston, Pseudo-valuation domains, II, Houston J. Math. 4 (1978), 199–207.

[J1] P. Jaffard, Dimension des anneaux de polynômes: Comportement asymptotique de la dimension, C.R. Acad. Sci. Paris A-B 246 (1958), 3199–3200.

[J2] P. Jaffard, Dimension des anneaux de polynômes: La notion de dimension valuative, C.R. Acad. Sci. Paris A-B 246 (1958), 3305–3307.

[J3] P. Jaffard, *Théorie de la dimension dans les anneaux de polynômes*, Mém. Sc. Math. 146, Gauthier-Villars, Paris, 1960.

[Ka1] S. Kabbaj, *S*-domains forts universels, Rapport de Recherce Univ. Lyon I, 1985.

[Ka2] S. Kabbaj, La formule de la dimension pour les *S*-domaines forts universels, Boll. Un. Mat. Ital., to appear.

[K] W. Krull, Jacobsonsche Ringe, Hilbertsche Nullstellensatz, Dimension Theorie, Math. Z. 54 (1951), 354–387.

[MM] S. Malik and J.L. Mott, Strong *S*-domains, J. Pure Appl. Algebra 28 (1983), 249–264.

[N1] M. Nagata, On the chain problem of prime ideals, Nagoya Math. J. 10 (1956), 51–64.

[N2] M. Nagata, *Local rings*, Interscience, New York, 1962.

[P] T. Parker, A number theoretic characterization of dimension sequences, Amer. J. Math. 97 (1975), 308–311.

[S1] A. Seidenberg, A note on dimension theory of rings, Pacific J. Math. 3 (1952), 502–512.

[S2] A. Seidenberg, On the dimension theory of rings, II, Pacific J. Math. 3 (1952), 603–614.

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