Variation of parameters

The linear second-order DE:

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = g(x).$$

we begin by putting the equation into the standard form:

$$\boxed{y'' + P(x)\, y' + Q(x)\, y = f(x).} \longrightarrow (1)$$

we seek a solution of the form

$$y_p = u_1(x)\, y_1 + u_2(x)\, y_2,$$

where y_1 and y_2 form a fundamental set of solutions on I associated with the homogeneous form of (1). The functions $u_1(x)$ and $u_2(x)$ can be found as follows:

$$u_1' = \frac{W_1}{W} = \frac{-y_2 f(x)}{W} \quad \& \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$.

<u>Example:</u> Solve $y'' - 4y' + 4y = (x+1)e^{2x}$

<u>Solution:</u> $r^2 - 4r + 4 = (r-2)^2 = 0$ Chara. eqn.

$$\Rightarrow \quad y_c = C_1 e^{2x} + C_2 x e^{2x} \Rightarrow y_1 = e^{2x} \;\&\; y_2 = x e^{2x}$$

next compute the Wronskian:

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

Since the given DE is already in the standard form $\Rightarrow$ $f(x) = (x+1)e^{2x} \Rightarrow$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x} \quad \text{and}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

so that $u_1' = -\dfrac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x$ ~~and~~

and $u_2' = \dfrac{(x+1)e^{4x}}{e^{4x}} = x+1.$

$\Rightarrow u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$ and $u_2 = \frac{1}{2}x^2 + x$. Hence

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

and $y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}.$

Example 2: Solve $4y'' + 36y = \csc 3x$.

Solution: put the eqn. in the standard form

$$y'' + 9y = \frac{1}{4}\csc 3x$$

Chara. eqn $r^2 + 9 = 0 \Rightarrow r_1 = 3i$ & $r_2 = -3i$.

$\Rightarrow y_c = c_1\cos 3x + c_2\sin 3x$

$\Rightarrow$ using $y_1 = \cos 3x$, ~~and~~ $y_2 = \sin 3x$ and $f(x) = \frac{1}{4}\csc 3x$.

we obtain

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3.$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}, \quad W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

$$\Rightarrow u_1' = \frac{W_1}{W} = -\frac{1}{12} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$$

gives $u_1 = -\frac{1}{12}x$ and $u_2 = \frac{1}{36}\ln|\sin 3x|.$

Thus $y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}\sin 3x \ln|\sin 3x|.$

$\Rightarrow y = \cdots$