(24.2) The Residue Thm

Def (Residue)

let f has an isolated singularity at z_0 and Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ in some annulus $0 < |z-z_0| < R$. Then the cofficient c_{-1} is called the residue of f at z_0, and is denoted $\mathrm{Res}(f, z_0)$.

Thm: Γ: closed path, f diff on Γ and all points enclosed by Γ except for $z_1, \ldots, z_n$, which are all isolated singularities of f enclosed by Γ. Then

$$\oint_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$$

Thm If f has a simple at z_0, then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z-z_0) f(z) = c_{-1}.$$

Ex: $f(z) = \sin(z)/z^2$ has a simple pole at 0, and

$$\mathrm{Res}(f, 0) = \lim_{z \to 0} \frac{\sin(z)}{z} = 1$$

if Γ: is any closed path enclosing the origin, then

$$\oint_\Gamma \frac{\sin(z)}{z}\,dz = 2\pi i\,\mathrm{Res}(f, 0) = 2\pi i.$$

Coro. let $f(z) = h(z)/g(z)$, h continuous at z_0 and $h(z) \neq 0$. Suppose g is differentiable at z_0 and has a simple zero there. Then f has a simple pole at z_0 and

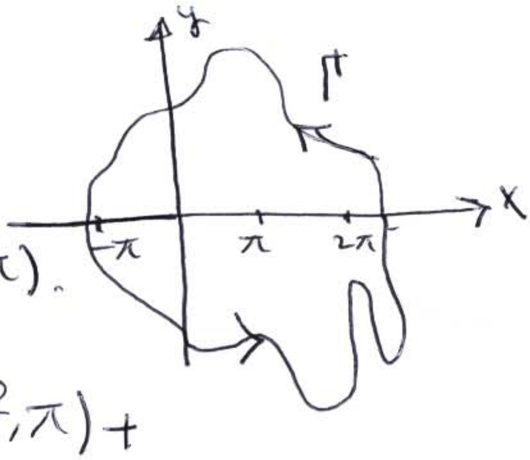
$$\mathrm{Res}(f, z_0) = h(z_0)/g'(z_0).$$

Ex: $f(z) = \dfrac{4iz-1}{\sin(z)} \Rightarrow f$ has a simple pole at π.

$$\mathrm{Res}(f, \pi) = \frac{4i\pi - 1}{\cos(\pi)} = 1 - 4\pi i.$$

and same for the pole at $n\pi$. ay

Ex: Evaluate $\oint \frac{4iz-1}{\sin(z)}\,dz$



Γ is the closed path in the Fig.

$\Rightarrow$ Γ encloses the poles $0, \pi, 2\pi$ and $(-\pi)$.

$$\Rightarrow \oint \frac{4iz-1}{\sin(z)}\,dz = 2\pi i\,[\mathrm{Res}(f,0)+\mathrm{Res}(f,\pi)+\mathrm{Res}(f,2\pi)+\mathrm{Res}(f,-\pi)]$$

$$= 2\pi i\,[-1+(1-4\pi i)+(-1+8\pi i)+(1+4\pi i)] = -16\pi^2.$$

Thm: let f have a pole of order m at z_0, then

$$\mathrm{Res}(f,z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right].$$

Ex: let $f(z) = \frac{\cos(z)}{(z+i)^3}$.

then f has pole at $-i$ of order 3.

$$\Rightarrow \mathrm{Res}(f,-i) = \frac{1}{2!}\lim_{z\to -i}\frac{d^2}{dz^2}\left[(z+i)^3\cdot\frac{\cos(z)}{(z+i)^3}\right]$$

$$= -\tfrac{1}{2}\cos(i).$$

Ex: let $f(z) = \frac{2iz-\cos(z)}{z^3+z}$

Evaluate $\oint_\Gamma f(z)\,dz$, with Γ a closed path doesn't pass ~~any~~ through any singularity of f.

Soln: the singularities of f are simple poles at 0, i and $-i$.

$$\Rightarrow \mathrm{Res}(f,0) = -\frac{\cos(0)}{1} = -\cos(0).$$

$$\mathrm{Res}(f,i) = \frac{2i^2-\cos(i)}{3(i)^2+1} = 1+\tfrac{1}{2}\cos(i)$$

& $$\mathrm{Res}(f,-i) = \frac{3i(-i)-\cos(-i)}{3(-i)^2+1} = -1+\tfrac{1}{2}\cos(i).$$

Now consider the cases

① Γ doesn't enclose any of singularities, then $\oint_\Gamma f(z)dz = 0$ by Cauchy thm.

② Γ enclose 0 but not i or $-i$, then

$$\oint_\Gamma f(z)\,dz = 2\pi i\,\mathrm{Res}(f,0) = -2\pi i.$$

③ Γ enclose i but not 0 or $-i$, then

$$\oint_\Gamma f(z)\,dz = 2\pi i\left[1+\frac{1}{2}\cos(i)\right]$$

④ Γ enclose $-i$ but not 0 or i, then

$$\oint_\Gamma f(z)\,dz = 2\pi i\left[-1+\frac{1}{2}\cos(i)\right]$$

⑤ If Γ encloses 0 and i but not $-i$, then

$$\oint_\Gamma f(z)\,dz = 2\pi i\left[-1+1+\frac{1}{2}\cos(i)\right] = \pi i\cos(i)$$

⑥ If Γ encloses 0 and $-i$ but not i, then

$$\oint_\Gamma f(z)\,dz = 2\pi i\left[-1-1+\frac{1}{2}\cos(i)\right] = 2\pi i\left[-2+\frac{1}{2}\cos(i)\right]$$

⑦ If Γ encloses i and $-i$ but not 0, then

$$\oint_\Gamma f(z)\,dz = 2\pi i\left[1+\frac{1}{2}\cos(i)-1+\frac{1}{2}\cos(i)\right] = 2\pi i\cos(i)$$

⑧ If Γ encloses all three singularities, then

$$\oint_\Gamma f(z)\,dz = 2\pi i\left[-1+1+\frac{1}{2}\cos(i)-1+\frac{1}{2}\cos(i)\right]$$

$$= 2\pi i\left[-1+\cos(i)\right].$$