Evaluation of real integrals

let $K(x,y)$ be a quotient polynomial in x and y, e.g.

$$\frac{x^3y - 2xy^2 + y}{x^4 - y}.$$

replace x by $\cos(\theta)$ and y by $\sin(\theta)$.

Problem: Evaluate $\int_0^{2\pi} K[\cos(\theta), \sin(\theta)]\, d\theta$

transfer the integral to complex integral over a unit circle, then use the residuel thm to evaluate the complex integral

$$\text{let} \quad \gamma(\theta) = e^{i\theta} \qquad 0 \le \theta \le 2\pi$$

$$z = e^{i\theta} \Rightarrow \bar{z} = e^{-i\theta} = 1/z$$

$$\Rightarrow \cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + 1/z\right)$$

$$\text{and } \cos(\theta) = 1/2i \left[e^{i\theta} - e^{-i\theta}\right] = \frac{1}{2i}\left[z - \frac{1}{z}\right]$$

$$\text{on } \gamma \Rightarrow dz = i e^{i\theta} d\theta = iz\, d\theta$$

$$\Rightarrow d\theta = \frac{1}{iz}\, dz$$

$$\Rightarrow \oint_\gamma \overbrace{K\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]\frac{dz}{iz}}^{f(z)} = \int_0^{2\pi} K[\cos(\theta), \sin(\theta)]\, d\theta$$

$$= 2\pi i \sum_P \mathrm{Res}(f, P). \quad (\text{by Residue Thm}).$$

where P = poles of $f(z)$ enclosed by the unit circle, $f(z)$ has no singularities on the unit circle.

Ex: Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2(\theta)}{2+\cos(\theta)}\, d\theta$

$$\Rightarrow K(x,y) = \frac{y^2}{2+x}$$

$$\Rightarrow x = \cos(\theta) = \frac{1}{2}\left(z+\frac{1}{z}\right), \quad y = \sin(\theta) = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$f = K\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\cdot\frac{1}{iz}$$

$$= \frac{\left[(1/2i)(z-1/z)\right]^2}{2+\frac{1}{2}(z+1/z)}\cdot\frac{1}{iz} = \frac{i}{2}\,\frac{z^4-2z^2+1}{z^2(z^2+4z+1)}$$

f has double poles at zero and simple poles at zero's of (z^2+4z+1) which are $(-2+\sqrt{3})$ and $(-2-\sqrt{3})$.
Note that $(-2-\sqrt{3})$ is not enclosed by γ.

$$\Rightarrow \int_0^{2\pi}\frac{\sin^2\theta}{2+\cos\theta}\,d\theta = 2\pi i\left[\operatorname{Res}(f,0)+\operatorname{Res}(f,-2+\sqrt{3})\right]$$

$$\operatorname{Res}(f,-2+\sqrt{3}) = \frac{i}{2}\left[\frac{z^4-2z^2+1}{2z(z^2+4z+1)+z^2(2z+4)}\right]_{z=-2+\sqrt{3}}$$

$$= \frac{i}{2}\,\frac{42-24\sqrt{3}}{-12+7\sqrt{3}}.$$

$$\operatorname{Res}(f,0) = \lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z\to 0}\frac{d}{dz}\,\frac{i}{2}\,\frac{z^4-2z^2+1}{z^2+4z+1} = -2i$$

$$\Rightarrow \int_0^{2\pi}\frac{\sin^2\theta}{2+\cos\theta}\,d\theta = \frac{90-52\sqrt{3}}{12-7\sqrt{3}}\,\pi.$$

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Evaluation of $\int_{-\infty}^{\infty} [P(x)/q(x)]\, dx$

1- $P(x)$ and $q(x)$ polynomials with real cofficients and no common factors.

2- $q(x) \neq 0,\ \forall x \in \mathbb{R}$, i.e. has no real zeros.

3. $\deg(q) - \deg(P) \geq 2$

this conditions are sufficient to ensure the convergence of this improper integral.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{P(x)}{q(x)}\, dx = 2\pi i \sum_{j=1}^{m} \operatorname{Res}\left(\frac{P(z)}{q(z)}, z_j\right)$$

Ex: Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^6+64}\, dx$

$\Rightarrow P(z) = 1$ & $q(z) = z^6+64$ (has no real roots)

- $\deg(q) - \deg(P) = 6$

the zeros of z^6+64 are sixth roots of (-64).

$$-64 = 64\, e^{i(\pi+2n\pi)} \quad \text{(polar form)}$$

the six roots of (-64) are

$$2\, e^{i(\pi+2n\pi)/6} \qquad n = 0, 1, 2, \cdots, 5$$

the three roots in the upper half-plane are

$$z_1 = 2e^{\pi i/6},\ z_2 = 2e^{\pi i/2} = 2i \ \ \& \ z_3 = 2e^{5\pi i/6}$$

$$\Rightarrow \operatorname{Res}\left(\frac{1}{z^6+64}, 2e^{i\pi/6}\right) = \frac{1}{6(2e^{i\pi/6})^5} = \frac{1}{192}\, e^{-5i\pi/6}$$

$$\operatorname{Res}\left(\frac{1}{z^6+64}, 2i\right) = \frac{1}{6(2i)^5} = \frac{-i}{192}, \text{ and}$$

$$\text{Res}\left(\frac{1}{z^6+64},\, 2e^{5\pi i/6}\right) = \frac{2\pi i}{192}\left[e^{-5\pi i/6} - i + e^{-\pi i/6}\right]$$

$$= \frac{\pi i}{96}\left[\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right) - i + \cos\left(\frac{\pi}{6}\right) - i\sin(\pi/6)\right]$$

Now $\cos\left(\frac{5\pi}{6}\right) + \cos(\pi/6) = 0$

& $\sin(5\pi/6) = \sin(\pi/6) = 1/2$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^6+64}\,dx = \frac{\pi i}{96}\left(-2\frac{i}{2}\right) = \frac{\pi}{96\,\cancel{48}}.$$