Derivative and integrals of trigonometric functions

$f(x)$	$f'(x)$	$\int f(x)\,dx$
$\sin x$	$\cos x$	$-\cos x + C$
$\cos x$	$-\sin x$	$\sin x + C$
$\tan x$	$\sec^2 x$	$-\ln\lvert\cos x\rvert + C$
$\cot x$	$-\csc^2 x$	$\ln\lvert\sin x\rvert + C$
For Completeness:		
($\sec x$	$\sec x \tan x$	$\ln\lvert\sec x + \tan x\rvert + C$)
($\csc x$	$-\csc x \cot x$	$\ln\lvert\csc x - \cot x\rvert + C$)

Ex1: find $\int \sin 3x\,dx$

Sol: by substitution let $u = 3x$

$\Rightarrow du = 3dx$

$$\Rightarrow \int \sin 3x\,dx = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = \frac{-\cos 3x}{3} + C$$

Remark: $\int \sin ax\,dx = \frac{-\cos ax}{a} + C$; $\int \cos ax\,dx = \frac{\sin ax}{a} + C$; and so on.

Exc: Find $\int \cos 7x\,dx$

Note $\sin^2 x + \cos^2 x = 1$.

Ex2: $\int \frac{\cos x}{1 - \sin^2 x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$

Ex2: Find $\int x^3 \cos(x^4+2)\,dx$

Sol: let $u = x^4+2$

$\Rightarrow du = 4x^3\,dx$.

$$\Rightarrow \int x^3 \cos(x^4+2)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C$$

$$= \frac{1}{4}\sin(x^4+2) + C.$$

Ex3: $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

let $u = \sqrt{x}$ $\Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$

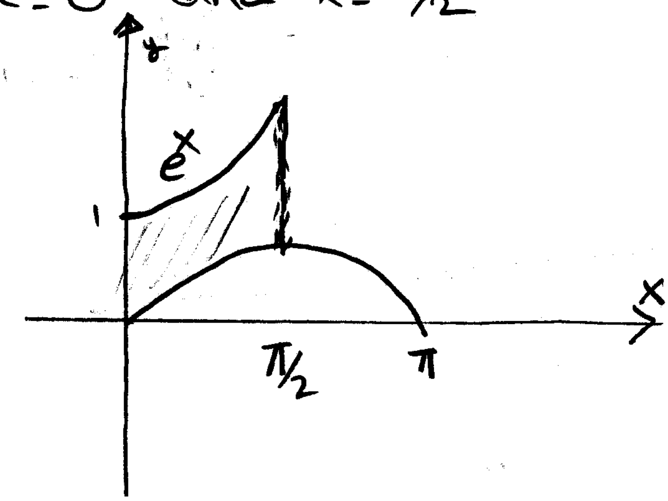
$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos u\,du = 2\sin u + C$$

$$= 2\sin\sqrt{x} + C.$$

Exc Find $\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$

Ex: Find the area of the region bounded by the curves $y = \sin x$, $y = e^x$, $x = 0$ and $x = \pi/2$

Sol $A = \int_0^{\pi/2} (e^x - \sin x)\,dx$



$$= e^x + \cos x \Big]_0^{\pi/2}$$

$$= (e^{\pi/2} + \cos \pi/2) - (e^0 + \cos 0)$$

$$= e^{\pi/2} - 2$$

Integration by parts

Ex: Calculate $\int x \cos x \, dx$

Sol: let $u = x$ $\qquad dv = \cos x \, dx$

$du = dx \qquad v = \sin x$

$$\Rightarrow \int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C$$

Exc Find ① $\int x \sin x \, dx$ ② $\int x^2 \cos x \, dx$

Ex: find $\int e^x \cos x \, dx$

Sol let $u = \cos x \qquad dv = e^x \, dx$

$du = -\sin x \qquad v = e^x$

$$\Rightarrow \int e^x \cos(x) \, dx = e^x \cos x + \int e^x \sin x \, dx$$

we use integration by parts again, with

$u = \sin x \qquad dv = e^x \, dx$

$du = \cos x \, dx \qquad v = e^x$

$$\Rightarrow \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

Putting these together we get

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$\Rightarrow 2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x + C_1$$

$$\Rightarrow \int e^x \cos x \, dx = \frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C.$$

Remark: To evaluate (a) $\int \sin mx \cos nx\, dx$ (b) $\int \sin mx \sin nx\, dx$ or $\int \cos mx \cos nx\, dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

(b) $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

(c) $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

Ex: Evaluate $\int \sin 4x \cos 5x\, dx$

Sol

$$\int \sin 4x \cos 5x\, dx = \frac{1}{2}\int \left(\sin(-x) + \sin 9x\right) dx$$

$$= \frac{1}{2}\int (-\sin x + \sin 9x)\, dx$$

$$= \frac{1}{2}\left(\cos x - \frac{\sin 9x}{9}\right) + C.$$

Exc ① $\int \sin 5x \sin 2x\, dx$ ② $\int \cos 7\theta \cos 5\theta\, d\theta$.

Half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Ex: Evaluate $\int_0^{\pi} \sin^2 x\, dx$

Sol

$$\int_0^{\pi} \sin^2 x\, dx = \frac{1}{2}\int_0^{\pi} (1 - \cos 2x)\, dx = \left[\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\right]_0^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{\sin 2\pi}{2}\right) - \frac{1}{2}\left(0 - \frac{\sin 0}{2}\right) = \pi/2.$$

Exc Find $\int \cos^2 x\, dx$.

Remark: Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd $(n = 2k+1)$.

save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$, i.e.

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

then substitute $u = \sin x$

(b) If the power of sine is odd $(m = 2k+1)$

save one sine factor and use $\sin^2 x = 1 - \cos^2 x$, i.e.

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

then substitute $u = \cos x$.

For Completeness:

(c) If the powers of both sine and cosine are even use half angle identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$; $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

It is helpful to use $\sin x \cos x = \frac{1}{2} \sin 2x$.

Ex: $\int \sin^5 x \cos^2 x \, dx$

Sol: $\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^2 x \sin x$

substituting $u = \cos x$, we have $du = -\sin x \, dx$

$$\Rightarrow \int \sin^5 x \cos^2 x \, dx = \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx$$

$$= -\int (1 - u^2)^2 u^2 \, du = -\int (u^2 - 2u^4 + u^6) \, du$$

$$= -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right) + C = \frac{-\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C.$$

Exe: ⓐ Find $\int \sin^4 x \, dx$ ⓑ $\int \cos^5 x \sin^4 x \, dx$.