SYSTEMS OF DIFFERENTIAL EQUATIONS

Systems of first order linear differential equations with constant coefficients were used in §7.1 to motivate the introduction of eigenvalues and eigenvectors, but now we can delve a little deeper. For constants a_(ij) , the goal is to solve the following system for the unknown functions u_i (t).

$$\begin{aligned} u_1' &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n, & & u_1(0) = c_1, \\ u_2' &= a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n, & \text{with} \quad & u_2(0) = c_2, \\ &\vdots & & \vdots \\ u_n' &= a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n, & & u_n(0) = c_n. \end{aligned} \qquad (7.4.1)$$

Since the scalar exponential provides the unique solution to a single differential equation $u'(t) = \alpha u(t)$ with $u(0) = c$ as $u(t) = e^{\alpha t}c$, it's only natural to try to use the matrix exponential in an analogous way to solve a system of differential equations. Begin by writing (7.4.1) in matrix form as $\mathbf{u}' = \mathbf{A}\mathbf{u}$, $\mathbf{u}(0) = \mathbf{c}$, where

$$\mathbf{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

If $\mathbf{A}$ is diagonalizable with $\sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, then (7.3.6) guarantees

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\mathbf{G}_1 + e^{\lambda_2 t}\mathbf{G}_2 + \cdots + e^{\lambda_k t}\mathbf{G}_k. \qquad (7.4.2)$$

The following identities are derived from properties of the $\mathbf{G}_i$'s given on p. 517.

- $de^{\mathbf{A}t}/dt = \sum_{i=1}^{k} \lambda_i e^{\lambda_i t}\mathbf{G}_i = \left(\sum_{i=1}^{k} \lambda_i \mathbf{G}_i\right)\left(\sum_{i=1}^{k} e^{\lambda_i t}\mathbf{G}_i\right) = \mathbf{A}e^{\mathbf{A}t}.$ (7.4.3)
- $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$ (by a similar argument). (7.4.4)
- $e^{-\mathbf{A}t}e^{\mathbf{A}t} = e^{\mathbf{A}t}e^{-\mathbf{A}t} = \mathbf{I} = e^{\mathbf{0}}$ (by a similar argument). (7.4.5)

Equation (7.4.3) insures that $\mathbf{u} = e^{\mathbf{A}t}\mathbf{c}$ is *one* solution to $\mathbf{u}' = \mathbf{A}\mathbf{u}$, $\mathbf{u}(0) = \mathbf{c}$. To see that $\mathbf{u} = e^{\mathbf{A}t}\mathbf{c}$ is the *only* solution, suppose $\mathbf{v}(t)$ is another solution so

that $\mathbf{v}' = \mathbf{A}\mathbf{v}$ with $\mathbf{v}(0) = \mathbf{c}$. Differentiating $e^{-\mathbf{A}t}\mathbf{v}$ produces

$$\frac{d\left[e^{-\mathbf{A}t}\mathbf{v}\right]}{dt} = e^{-\mathbf{A}t}\mathbf{v}' - e^{-\mathbf{A}t}\mathbf{A}\mathbf{v} = \mathbf{0}, \quad \text{so} \quad e^{-\mathbf{A}t}\mathbf{v} \text{ is constant for all } t.$$

At $t = 0$ we have $e^{-\mathbf{A}t}\mathbf{v}\big|_{t=0} = e^{\mathbf{0}}\mathbf{v}(0) = \mathbf{I}\mathbf{c} = \mathbf{c}$, and hence $e^{-\mathbf{A}t}\mathbf{v} = \mathbf{c}$ for all t. Multiply both sides of this equation by $e^{\mathbf{A}t}$ and use (7.4.5) to conclude $\mathbf{v} = e^{\mathbf{A}t}\mathbf{c}$. Thus $\mathbf{u} = e^{\mathbf{A}t}\mathbf{c}$ is the unique solution to $\mathbf{u}' = \mathbf{A}\mathbf{u}$ with $\mathbf{u}(0) = \mathbf{c}$.

Finally, notice that $\mathbf{v}_i = \mathbf{G}_i\mathbf{c} \in N(\mathbf{A} - \lambda_i\mathbf{I})$ is an eigenvector associated with λ_i, so that the solution to $\mathbf{u}' = \mathbf{A}\mathbf{u}$, $\mathbf{u}(0) = \mathbf{c}$, is

$$\mathbf{u} = e^{\lambda_1 t}\mathbf{v}_1 + e^{\lambda_2 t}\mathbf{v}_2 + \cdots + e^{\lambda_k t}\mathbf{v}_k, \tag{7.4.6}$$

and this solution is completely determined by the eigenpairs (λ_i, v_i). It turns out that u also can be expanded in terms of any complete set of independent eigenvectors. Let’s summarize what’s been said so far.

Differential Equations

If $\mathbf{A}_{n\times n}$ is diagonalizable with $\sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, then the unique solution of $\mathbf{u}' = \mathbf{A}\mathbf{u}$, $\mathbf{u}(0) = \mathbf{c}$, is given by

$$\mathbf{u} = e^{\mathbf{A}t}\mathbf{c} = e^{\lambda_1 t}\mathbf{v}_1 + e^{\lambda_2 t}\mathbf{v}_2 + \cdots + e^{\lambda_k t}\mathbf{v}_k \tag{7.4.7}$$

in which $\mathbf{v}_i$ is the eigenvector $\mathbf{v}_i = \mathbf{G}_i\mathbf{c}$, where $\mathbf{G}_i$ is the i^{th} spectral projector. (See Exercise 7.4.1 for an alternate eigenexpansion.) Nonhomogeneous systems as well as the nondiagonalizable case are treated in Example 7.9.6 (p. 608).

An Application to Diffusion

Important issues in medicine and biology involve the question of how drugs or chemical compounds move from one cell to another by means of diffusion through cell walls. Consider two cells, as depicted in Figure 7.4.1, which are both devoid of a particular compound. A unit amount of the compound is injected into the first cell at time t = 0, and as time proceeds the compound diffuses according to the following assumption.

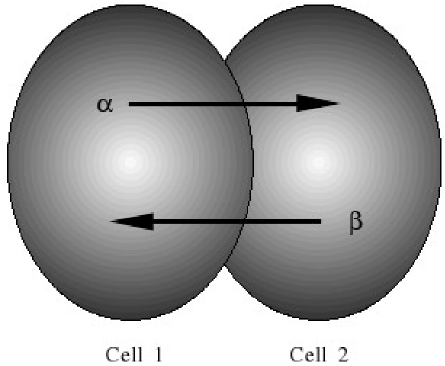

FIGURE 7.4.1

At each point in time the rate (amount per second) of diffusion from one cell to the other is proportional to the concentration (amount per unit volume) of the compound in the cell giving up the compound—say the rate of diffusion from cell 1 to cell 2 is α times the concentration in cell 1, and the rate of diffusion from cell 2 to cell 1 is β times the concentration in cell 2. Assume α, β > 0.

Problem: Determine the concentration of the compound in each cell at any given time t, and, in the long run, determine the steady-state concentrations. Solution: If u_k = u_k (t) denotes the concentration of the compound in cell k at time t, then the statements in the above assumption are translated as follows:

$$\frac{du_1}{dt} = \text{rate in} \; - \; \text{rate out} \; = \beta u_2 - \alpha u_1, \quad \text{where} \quad u_1(0) = 1,$$
$$\frac{du_2}{dt} = \text{rate in} \; - \; \text{rate out} \; = \alpha u_1 - \beta u_2, \quad \text{where} \quad u_2(0) = 0.$$

In matrix notation this system is $\mathbf{u}' = \mathbf{A}\mathbf{u}$, $\mathbf{u}(0) = \mathbf{c}$, where

$$\mathbf{A} = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $\mathbf{A}$ is the matrix of Example 7.3.3 we can use the results from Example 7.3.3 to write the solution as

$$\mathbf{u}(t) = e^{\mathbf{A}t}\mathbf{c} = \frac{1}{\alpha+\beta}\left[\begin{pmatrix} \beta & \beta \\ \alpha & \alpha \end{pmatrix} + e^{-(\alpha+\beta)t}\begin{pmatrix} \alpha & -\beta \\ -\alpha & \beta \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so that

$$u_1(t) = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}e^{-(\alpha+\beta)t} \quad \text{and} \quad u_2(t) = \frac{\alpha}{\alpha+\beta}\left(1 - e^{-(\alpha+\beta)t}\right).$$

In the long run, the concentrations in each cell stabilize in the sense that

$$\lim_{t\to\infty} u_1(t) = \frac{\beta}{\alpha+\beta} \quad \text{and} \quad \lim_{t\to\infty} u_2(t) = \frac{\alpha}{\alpha+\beta}.$$

An innumerable variety of physical situations can be modeled by

$$\mathbf{u}' = \mathbf{A}\mathbf{u},$$

and the form of the solution (7.4.6) makes it clear that the eigenvalues and eigenvectors of A are intrinsic to the underlying physical phenomenon being investigated. We might say that the eigenvalues and eigenvectors of A act as its genes and chromosomes because they are the basic components that either dictate or govern all other characteristics of A along with the physics of associated phenomena.

For example, consider the long-run behavior of a physical system that can be modeled by

$$\mathbf{u}' = \mathbf{A}\mathbf{u},$$

We usually want to know whether the system will eventually blow up or will settle down to some sort of stable state. Might it neither blow up nor settle down but rather oscillate indefinitely? These are questions concerning the nature of the limit

$$\lim_{t\to\infty} \mathbf{u}(t) = \lim_{t\to\infty} \mathrm{e}^{\mathbf{A}t}\mathbf{c} = \lim_{t\to\infty} \left(\mathrm{e}^{\lambda_1 t}\mathbf{G}_1 + \mathrm{e}^{\lambda_2 t}\mathbf{G}_2 + \cdots + \mathrm{e}^{\lambda_k t}\mathbf{G}_k\right)\mathbf{c},$$

and the answers depend only on the eigenvalues. To see how, recall that for a complex number $\lambda = x + iy$ and a real parameter $t > 0$,

$$\mathrm{e}^{\lambda t} = \mathrm{e}^{(x+\mathrm{i}y)t} = \mathrm{e}^{xt}\mathrm{e}^{\mathrm{i}yt} = \mathrm{e}^{xt}(\cos yt + \mathrm{i}\sin yt). \qquad (7.4.8)$$

The term $\mathrm{e}^{\mathrm{i}yt} = (\cos yt + \mathrm{i}\sin yt)$ is a point on the unit circle that oscillates as a function of t, so $|\mathrm{e}^{\mathrm{i}yt}| = |\cos yt + \mathrm{i}\sin yt| = 1$ and $\left|\mathrm{e}^{\lambda t}\right| = |\mathrm{e}^{xt}\mathrm{e}^{\mathrm{i}yt}| = |\mathrm{e}^{xt}| = \mathrm{e}^{xt}$. This makes it clear that if $\mathrm{Re}(\lambda_i) < 0$ for each i, then, as $t \to \infty$, $\mathrm{e}^{\mathbf{A}t} \to \mathbf{0}$, and $\mathbf{u}(t) \to \mathbf{0}$ for every initial vector $\mathbf{c}$. Thus the system eventually settles down to zero, and we say the system is *stable*. On the other hand, if $\mathrm{Re}(\lambda_i) > 0$ for some i, then components of $\mathbf{u}(t)$ may become unbounded as $t \to \infty$, and we say the system is *unstable*. Finally, if $\mathrm{Re}(\lambda_i) \le 0$ for each i, then the components of $\mathbf{u}(t)$ remain finite for all t, but some may oscillate indefinitely, and this is called a *semistable* situation. Below is a summary of stability.

Stability

Let $\mathbf{u}' = \mathbf{A}\mathbf{u}$, $\mathbf{u}(0) = \mathbf{c}$, where $\mathbf{A}$ is diagonalizable with eigenvalues λ_i.

- If $\text{Re}(\lambda_i) < 0$ for each i, then $\lim_{t\to\infty} e^{\mathbf{A}t} = \mathbf{0}$, and $\lim_{t\to\infty} \mathbf{u}(t) = \mathbf{0}$ for every initial vector $\mathbf{c}$. In this case $\mathbf{u}' = \mathbf{A}\mathbf{u}$ is said to be a ***stable system,*** and $\mathbf{A}$ is called a ***stable matrix.***
- If $\text{Re}(\lambda_i) > 0$ for some i, then components of $\mathbf{u}(t)$ can become unbounded as $t \to \infty$, in which case the system $\mathbf{u}' = \mathbf{A}\mathbf{u}$ as well as the underlying matrix $\mathbf{A}$ are said to be ***unstable.***
- If $\text{Re}(\lambda_i) \le 0$ for each i, then the components of $\mathbf{u}(t)$ remain finite for all t, but some can oscillate indefinitely. This is called a ***semistable*** situation.

Predator–Prey Application

Consider two species of which one is the predator and the other is the prey, and assume there are initially 100 in each population. Let u_1(t) and u_2(t) denote the respective population of the predator and prey species at time t, and suppose their growth rates are given by

$$u_1' = u_1 + u_2,$$
$$u_2' = -u_1 + u_2.$$

Problem: Determine the size of each population at all future times, and decide if (and when) either population will eventually become extinct.

Solution: Write the system as $\mathbf{u}' = \mathbf{A}\mathbf{u}$, $\mathbf{u}(0) = \mathbf{c}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}.$$

The characteristic equation for $\mathbf{A}$ is $p(\lambda) = \lambda^2 - 2\lambda + 2 = 0$, so the eigenvalues for $\mathbf{A}$ are $\lambda_1 = 1 + \mathrm{i}$ and $\lambda_2 = 1 - \mathrm{i}$. We know from (7.4.7) that

$$\mathbf{u}(t) = e^{\lambda_1 t}\mathbf{v}_1 + e^{\lambda_2 t}\mathbf{v}_2 \quad (\text{where } \mathbf{v}_i = \mathbf{G}_i\mathbf{c}\,) \tag{7.4.9}$$

is the solution to $\mathbf{u}' = \mathbf{A}\mathbf{u}$, $\mathbf{u}(0) = \mathbf{c}$. The spectral theorem on p. 517 implies $\mathbf{A} - \lambda_2\mathbf{I} = (\lambda_1 - \lambda_2)\mathbf{G}_1$ and $\mathbf{I} = \mathbf{G}_1 + \mathbf{G}_2$, so $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{c} = (\lambda_1 - \lambda_2)\mathbf{v}_1$ and $\mathbf{c} = \mathbf{v}_1 + \mathbf{v}_2$, and consequently

$$\mathbf{v}_1 = \frac{(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{c}}{(\lambda_1 - \lambda_2)} = 50 \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \mathbf{c} - \mathbf{v}_1 = 50 \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

With the aid of (7.4.8) we obtain the solution components from (7.4.9) as

$$u_1(t) = 50\left(\lambda_2 e^{\lambda_1 t} + \lambda_1 e^{\lambda_2 t}\right) = 100e^t(\cos t + \sin t)$$

and

$$u_2(t) = 50\left(\lambda_1 e^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t}\right) = 100e^t(\cos t - \sin t).$$

The system is unstable because $\mathrm{Re}(\lambda_i) > 0$ for each eigenvalue. Indeed, $u_1(t)$ and $u_2(t)$ both become unbounded as $t \to \infty$. However, a population cannot become negative–once it's zero, it's extinct. Figure 7.4.2 shows that the graph of $u_2(t)$ will cross the horizontal axis before that of $u_1(t)$.

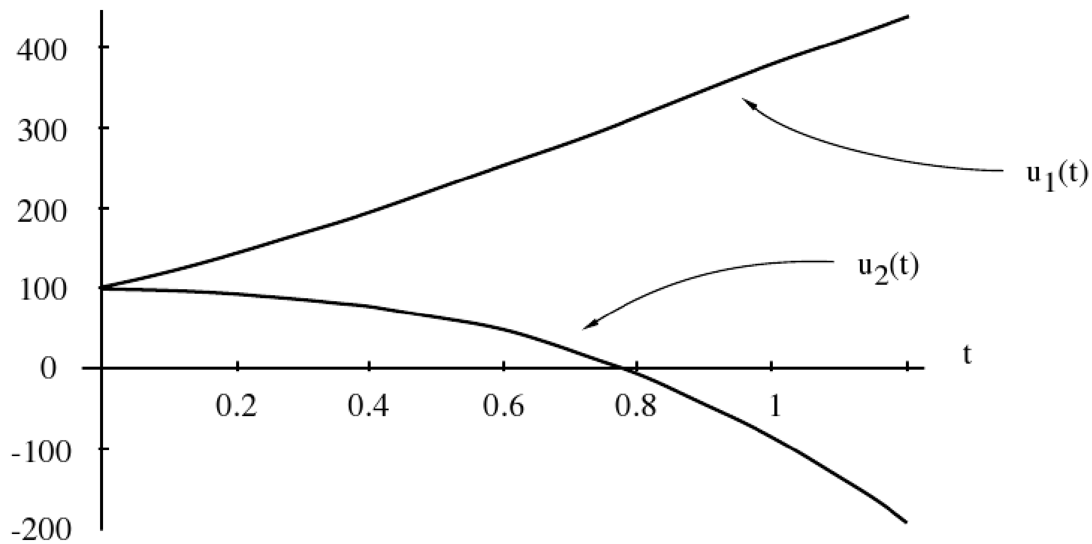

FIGURE 7.4.2

Therefore, the prey species will become extinct at the value of t for which $u_2(t) = 0$—i.e., when

$$100e^t(\cos t - \sin t) = 0 \implies \cos t = \sin t \implies t = \frac{\pi}{4}.$$