Unitary and orthogonal matrices:

<u>Def</u>

- A unitary matrix is defined to be a complex matrix $U_{n\times n}$ whose columns (or rows) constitute an orthonormal basis for $\mathbb{C}^n$.
- An orthogonal matrix is defined to be a real matrix $P_{n\times n}$ whose columns (or rows) constitute an orthonormal basis for $\mathbb{R}^n$.

<u>Note</u> (1) Unitary matrix means $U^*U = I$

<u>Exc</u>: If U is unitary matrix find its inverse.

(2) Unitary matrix doesn't change the length of a vector.

$$\|Ux\|^2 = (Ux)^* Ux = x^* U^* U x = x^* x = \|x\|^2 \quad \forall x \in \mathbb{C}^n. \quad (1)$$

Conversely if (1) hold then U must be unitary

$$\|Ux\|^2 = \|x\|^2 \quad \forall x \in \mathbb{C}^n$$

$$\Rightarrow \quad x^* U^* U x = x^* x \quad \forall x \in \mathbb{C}^n.$$

$$\Rightarrow \quad e_i^T U^* U e_j = e_i^T e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\Rightarrow (U_{*i})^* U_{*j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Ex: (1) The matrix $P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{pmatrix}$

is an orthogonal matrix.

(2) The matrix $U = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$ is unitary.