Scalar Product

Def A vector space V over a field K is called a space with scalar product if each $x, y \in V$, there corresponds a certain number $\langle x, y\rangle \in K$ called scalar product of the vectors x and y, such the following conditions are satisfied:

(1) $\langle x, x\rangle \geq 0$, $\quad \langle x, x\rangle = 0 \Rightarrow x = 0$.

(2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, where $\overline{\langle y, x\rangle}$ = conjugate of $\langle x, y\rangle$.

(3) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(4) $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle$

Example (1) the standard inner product $\langle x, y\rangle = y^T x$ for $x \in \mathbb{R}^{n\times 1}$ and $\langle x, y\rangle = y^* x$ for $\mathbb{C}^{n\times 1}$.

(2) If $A_{n\times n}$ is a nonsingular matrix, then

$$\langle x, y\rangle = y^* A^* A x$$

this is called A-inner product or elliptical inner product.

(3) Consider the vector space of $m\times n$ matrices. The fn. defined by

$$\langle A, B\rangle = \mathrm{Trace}(B^T A) \ \&$$

$$\langle A, B\rangle = \mathrm{Trace}(B^* A)$$

are inner product space of $\mathbb{R}^{m\times n}$ & $\mathbb{C}^{m\times n}$. These are referred to as the standard inner product for matrices.

Proposition: the scalar product $\langle x, y\rangle$ has the following properties

(1) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ (additivity)

(2) $\langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle$

(3) $\langle x, 0\rangle = \langle 0, y\rangle = 0 \quad \forall\, x, y \in V.$

(4) $\langle \lambda x, \lambda y\rangle = |\lambda|^2 \langle x, y\rangle$

Prove:

(1) $\langle x, y+z\rangle = \overline{\langle y+z, x\rangle} = \overline{\langle y, x\rangle + \langle z, x\rangle} =$

$= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \langle x, y\rangle + \langle x, z\rangle.$

(2) $\langle x, \lambda y\rangle = \overline{\langle \lambda y, x\rangle} = \overline{\lambda\langle y, x\rangle} = \bar{\lambda}\,\overline{\langle y, x\rangle}$

$= \bar{\lambda}\langle x, y\rangle.$

(3) $\langle x, 0\rangle = \langle x, 0x\rangle = \bar{0}\langle x, x\rangle = 0.$

(4) $\langle \lambda x, \lambda y\rangle = \lambda\bar{\lambda}\langle x, y\rangle = |\lambda|^2\langle x, y\rangle.$

Proposition: (Cauchy-Schwartz inequality)

Let x and y be vectors in vector space V with scalar product

$\Rightarrow \quad |\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}.$

Proof if $\langle x, y\rangle = 0$ then the inequality hold.

Consider the case $\langle x, y\rangle \neq 0$ and define the fn.

$f(\lambda) = \langle x + \lambda\langle x, y\rangle y,\ x + \lambda\langle x, y\rangle y\rangle$

As for $\lambda \in \mathbb{R}$

$$f(\lambda) = \langle x,x\rangle + \lambda \overline{\langle x,y\rangle} \langle x,y\rangle + \lambda \langle x,y\rangle \langle y,x\rangle + \lambda^2 |\langle x,y\rangle|^2 \langle y,y\rangle$$

$$= \lambda^2 |\langle x,y\rangle|^2 \langle y,y\rangle + 2\lambda |\langle x,y\rangle|^2 + \langle x,x\rangle \geq 0 \quad \forall \lambda \in \mathbb{R}$$

iff $$|\langle x,y\rangle|^4 - |\langle x,y\rangle|^2 \langle x,x\rangle \langle y,y\rangle \leq 0$$

$$\Rightarrow \quad |\langle x,y\rangle|^2 \leq \langle x,x\rangle \langle y,y\rangle \quad ✱$$

The Cauchy-Schwartz inequality makes it possible to define the angle between two vectors by the scalar product.

<u>Def</u>: The angle between arbitrary vectors x and y of the vector space with scalar product V is defined by:

$$\cos(\widehat{x,y}) = \frac{\langle x,y\rangle}{\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}}.$$

<u>Problem</u>: Show that for each two complex vectors x and y the equality $\langle x,\bar{y}\rangle = \overline{\langle \bar{x},y\rangle}$ holds.

<u>Problem</u>: The scalar product in the vector space $P_n[\alpha,\beta]$ of polynomials of at most degree n with real coefficients on $[\alpha,\beta]$ is defined by

$$\langle x,y\rangle = \int_\alpha^\beta x(t)\,y(t)\,dt.$$

Find the angle between the polynomials $x = t-1$ and $y = t^2+1$.

<u>Ex</u>: Determine the angle between $x = \begin{pmatrix} -4 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix}$.

Proposition: A space with scalar product V is a normed space with the norm

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Proof: $\|x\| = 0 \Leftrightarrow \sqrt{\langle x,x\rangle} = 0 \leftrightarrow \langle x,x\rangle = 0 \Leftrightarrow x = 0$

$$\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{|\lambda|^2 \langle x,x\rangle} = |\lambda| \sqrt{\langle x,x\rangle} = |\lambda| \, \|x\|$$

$$\|x+y\| = \sqrt{\langle x+y, x+y\rangle} = \sqrt{\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle}$$

$$= \sqrt{\|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2} = \sqrt{\|x\|^2 + 2R\langle x,y\rangle + \|y\|^2}$$

$$\leqslant \sqrt{\|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2} \leqslant \sqrt{\|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2}$$

$$\leqslant \sqrt{(\|x\| + \|y\|)^2} = \|x\| + \|y\|.$$

$R\langle x,y\rangle$ = means real part of $\langle x,y\rangle$.

Proposition: In the normed space with scalar product the parallelogram rule:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \quad \text{holds.}$$

Proof $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$$

$$= 2\left(\|x\|^2 + \|y\|^2\right).$$

Orthogonal vectors

Def: The vectors x and y of the vector space with scalar product V are called orthogonal if $\langle x, y\rangle = 0$. We write $x \perp y$ to indicate the orthogonality of vectors x and y. A vector x of the vector space V is called orthogonal to the set $Y \subset V$ if $x \perp y \ \forall y \in Y$.

Problem: Find all vectors that are orthogonal both to the vector $a = [4 \ 0 \ 6 \ -2 \ 0]^T$ and $b = [2 \ 1 \ -1 \ 1 \ 1]^T$.

Def The sets Y and Z ~~are called~~ of the vector space V are called orthogonal if $y \perp z \ \forall y \in Y$ and $\forall z \in Z$.

Def: A vector space with complex scalar product is called a Hilbert space H if it turns out to be complete with respect to the convergence by the $\|x\| = \sqrt{\langle x, x\rangle}$.

Ex: The space $\mathbb{C}^n$ with $\langle x, y\rangle = \sum_{k=1}^{n} \xi_k \overline{\eta_k}$ is a Hilbert space.

Proposition: Orthogonality of vectors in the vector space with scalar product V has the following properties:

(1) $x \perp x$ iff $x = 0$.

(2) $x \perp y$ iff $y \perp x$.

(3) if $x \perp \{y_1, \ldots, y_k\} \Rightarrow x \perp (y_1 + \cdots + y_k)$.

(4) $x \perp y \Rightarrow x \perp \lambda y \quad \forall \lambda \in K$ (field).

(5) $x \perp y_n \ (n = 1, 2, \ldots)$ if $y_n \to y \Rightarrow x \perp y$ (only in Hilbert space)

Proof (1) $x \perp x \Leftrightarrow \langle x, x\rangle = 0 \Leftrightarrow x = 0$.

(2) $x \perp y \Leftrightarrow \langle x, y\rangle = 0 \Leftrightarrow \overline{\langle y, x\rangle} = 0 \Leftrightarrow \langle y, x\rangle = 0$

$\Leftrightarrow y \perp x$

(3) $x \perp \{y_1, \dots, y_k\} \leftrightarrow x \perp y_1$ & $\dots$ $x \perp y_k \Leftrightarrow$

$\langle x, y_1\rangle = 0, \quad \dots \quad, \langle x, y_k\rangle = 0 \Rightarrow$

$\langle x, y_1\rangle + \langle x, y_2\rangle + \dots + \langle x, y_k\rangle = 0$

$\Leftrightarrow \langle x, y_1 + y_2 + \dots + y_k\rangle = 0 \Leftrightarrow x \perp (y_1 + \dots + y_k)$.

(4) $x \perp y \leftrightarrow \langle x, y\rangle = 0 \leftrightarrow \overline{\lambda}\langle x, y\rangle = 0 \quad \forall \lambda \in K \Leftrightarrow$

$\langle x, \lambda y\rangle = 0 \quad \forall \lambda \in K \Leftrightarrow x \perp \lambda y$.

(5) $x \perp y_n \quad \forall n \in \mathbb{N}$ and $y_n \longrightarrow y \Leftrightarrow \langle x, y_n\rangle = 0$

and $\quad \|y_n - y\| \to 0$

$\Rightarrow \langle x, y_n\rangle = 0$ & $\left|\langle x, y_n\rangle - \langle x, y\rangle\right| = \left|\langle x, y_n - y\rangle\right|$

$\leq \|x\| \, \|y_n - y\| \longrightarrow 0$

$\Rightarrow \langle x, y\rangle = 0 \Leftrightarrow x \perp y$.

Orthonormal sets

Def: $\mathcal{B} = \{u_1, \dots, u_n\}$ is called an orthonormal set whenever $\|u_i\| = 1$ for each i and $u_i \perp u_j$ for all $i \neq j$. In other words:

$$\langle u_i, u_j \rangle = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}.$$

Example: The set $\mathcal{B}' = \left\{ u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\}$ is a set of mutually orthogonal vectors because $u_i^T u_j = 0$ for $i \neq j$, but $\mathcal{B}'$ is not an orthonormal set. However, it is easy to convert an orthogonal set into orthonormal set by simply normalizing each vector. Since $\|u_1\| = \sqrt{2}$, $\|u_2\| = \sqrt{3}$ and $\|u_3\| = \sqrt{6}$, it follows that $\mathcal{B} = \left\{ \frac{u_1}{\sqrt{2}}, \frac{u_2}{\sqrt{3}}, \frac{u_3}{\sqrt{6}} \right\}$ is orthonormal.

Thm: (1) Every orthonormal set is linearly independent.
(2) Every orthonormal set of n-vectors from n-dim space V is orthonormal basis for V.

Proof: Suppose that $\mathcal{B} = \{u_1, \dots, u_n\}$ orthonormal set. If $0 = \alpha_1 u_1 + \dots + \alpha_n u_n \Rightarrow$ $\quad 0 = \langle 0, u_i \rangle$.

$$0 = \langle \alpha_1 u_1 + \dots + \alpha_n u_n, u_i \rangle$$
$$= \alpha_1 \langle u_1, u_i \rangle + \dots + \alpha_n \langle u_n, u_i \rangle = \alpha_i \|u_i\|^2$$
$$= \alpha_i \quad \text{for each } i \quad ✲.$$