Condition number of a matrix

Some time we need the inversion of $(I-A)$, which is not always exist. However if the entries of A are small enough in magnitude $(\lim_{n\to\infty} A^n = 0)$, then

$$(I-A)(I+A+\cdots+A^{n-1}) = I - A^n \longrightarrow I \quad \text{as } n\to\infty$$

Neuman Series (geometric series)

If $\lim_{n\to\infty} A^n = 0$, then $I-A$ is nonsingular and

$$(I-A^{-1}) = I + A + A^2 + \cdots = \sum_{k=0}^{\infty} A^k$$

A first order approximation is $(I-A)^{-1} \simeq I + A$ *.

There is no useful formula for $(A+B)^{-1}$ in general.

But if A^{-1} exist and the entries of B are small enough in magnitude to insure $\lim_{n\to\infty}(A^{-1}B)^n = 0$, then

$$(A+B)^{-1} = \left[A(I+A^{-1}B)\right]^{-1} = \left[I-(-A^{-1}B)\right]^{-1}A^{-1}$$

$$= \left(\sum_{k=0}^{\infty}\left[-A^{-1}B\right]^k\right)A^{-1}.$$

using first order approximation

$$(A+B)^{-1} \simeq A^{-1} - A^{-1}BA^{-1}.$$

the effect of small perturbation B is magnified by multiplication with A^{-1}, so if A^{-1} has large entries, small perturbation in A can produce large perturbations in the resulting inverse.

Let $\|A\|_\infty = \max_i \sum_j |a_{ij}| \Rightarrow$.

$$\|A^{-1} - (A+B)^{-1}\| \simeq \|A^{-1}BA^{-1}\| \leq \|A^{-1}\|\,\|B\|\,\|A^{-1}\|$$

$$\Rightarrow \frac{\|A^{-1} - (A+B)^{-1}\|}{\|A^{-1}\|} \leq \|A^{-1}\| \, \|B\| = \|A^{-1}\| \, \|A\| \left\{ \frac{\|B\|}{\|A\|} \right\}$$

the left term is the relative change in the inverse and $\|B\|/\|A\|$ is the relative change in A.

<u>Def</u> For a fixed matrix norm the value

$$k(A) = \|A\| \, \|A^{-1}\|$$

is called the condition number corresponding to the regular square matrix $A \in \mathbb{R}^{n\times n}$. For a singular square matrix we will define $k(A) = +\infty$

If $k(A)$ is relativly small then the matrix A is well-conditioned matrix, but if $k(A)$ is great then an ill conditioned matrix.

<u>Note</u>: $k_F(A)$: the condition number corresponding to the Frobenius norm and $k_p(A)$ the condition number corresponding to the p-norm

<u>Exc</u>: Show that if $A \in \mathbb{R}^{n\times n}$, then

$$\frac{1}{n} k_2(A) \leq k_1(A) \leq n \, k_2(A).$$

$$\frac{1}{n} k_\infty(A) \leq k_2(A) \leq n \, k_\infty(A).$$

$$\frac{1}{n^2} k_1(A) \leq k_\infty(A) \leq n^2 k_1(A).$$

<u>Remark</u>: Since $k_p(A) = \|A\|_p \|A^{-1}\|_p \geq \|A A^{-1}\|_p$
$= \|I\|_p = 1.$

then always $k_p(A) \geq 1$.

Exc: Calculate $k_1(A)$, $k_2(A)$ and $k_\infty(A)$ if

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

sol

$$A^{-1} = \begin{bmatrix} -3/4 & -1/2 & -1/4 \\ -1/2 & -1 & -1/2 \\ -1/4 & -1/2 & -3/4 \end{bmatrix}$$

then $\|A\|_1 = \|A\|_\infty = 4$, $\|A\|_2 = 2+\sqrt{2}$.

$\|A^{-1}\|_1 = \|A^{-1}\|_\infty = 2$ and $\|A^{-1}\|_2 = 1+\frac{1}{2}\sqrt{2}$.

$\Rightarrow$ $k_1(A) = k_\infty(A) = 2$ and

$$k_2(A) = \frac{1}{2}(2+\sqrt{2})^2 \quad \#$$

In linear system $Ax=b$, where A is non-singular matrix. If A is perturbed to produce $(A+B)\tilde{x} = b$

$\Rightarrow$ $x = A^{-1}b$ & $\tilde{x} = (A+B)^{-1}b$

$\Rightarrow$ $x - \tilde{x} = A^{-1}b - (A+B)^{-1}b$

$$\simeq A^{-1}b - (A^{-1} - A^{-1}BA^{-1})b = A^{-1}Bx$$

$\Rightarrow$ $\|x-\tilde{x}\| \leq \|A^{-1}\| \, \|B\| \, \|x\|$

So the relative change is

$$\frac{\|x-\tilde{x}\|}{\|x\|} \leq \|A^{-1}\| \, \|B\|.$$

$$\Rightarrow \frac{\|x-\tilde{x}\|}{\|x\|} \leq \|A^{-1}\| \, \|A\| \left\{ \frac{\|B\|}{\|A\|} \right\}$$

$$= k(A) \left\{ \frac{\|B\|}{\|A\|} \right\}.$$

Example
$$\begin{pmatrix} 0.835 & 0.667 \\ 0.333 & 0.266 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.168 \\ 0.067 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 0.835 & 0.667 \\ 0.333 & 0.266 \end{pmatrix} \quad \& \quad A^{-1} = \begin{pmatrix} -266000 & 667000 \\ 333000 & -835000 \end{pmatrix}$$

then the condition number for A is

$$k(A) = \|A\| \|A^{-1}\| = (1.502)(116860) \approx 1.7 \times 10^6$$

Because $k(A)$ is of order 10^6 the relative change in the solution can be about a million times larger than the relative change in A. Therefor we must consider A and the associated linear system to be ill conditioned.