

# Matrix Norms

## Frobenius matrix norm:

The Frobenius matrix norm of  $A \in \mathbb{C}^{m \times n}$  is defined by:

$$\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2 = \sum_{i'} \|A_{i'}\|_2^2 = \sum_{j'} \|A_{j'}\|_2^2 = \text{Trace}(A^*A)$$

Example:  $A = \begin{bmatrix} 2 & -1 \\ -4 & -2 \end{bmatrix}$ ,  $\|A\|_F = [2^2 + (-1)^2 + (-4)^2 + (-2)^2]^{\frac{1}{2}} = 5$ .

Properties: (i)  $\|Ax\|_2 \leq \|A\|_F \|x\|_2$ .

Proof:  $\|Ax\|_2 = \sum_{i'} |A_{i'}x|^2 \leq \sum_{i'} \|A_{i'}\|_2^2 \|x\|_2^2$   
 $= \|A\|_F^2 \|x\|_2^2$ .

We say that the Frobenius matrix norm  $\|\cdot\|_F$  and the euclidean vector norm  $\|\cdot\|_2$  are compatible.

(ii)  $\|AB\|_F \leq \|A\|_F \|B\|_F$  [Submultiplicative property].

Proof:  $\|AB\|_F^2 = \sum_{j'} \|[AB]_{j'}\|_2^2 = \sum_{j'} \|AB_{j'}\|_2^2$

$$\leq \sum_{j'} \|A\|_F^2 \|B_{j'}\|_2^2 = \|A\|_F^2 \|B\|_F^2$$

Def (General matrix norms).

A matrix norm is a function  $\|\cdot\|$  from the set of all complex matrices into  $\mathbb{R}$  that satisfies the following:

(i)  $\|A\| \geq 0$  and  $\|A\| = 0$  iff  $A = 0$

(ii)  $\|\alpha A\| = |\alpha| \|A\|$   $\alpha$  is a scalar.

(iii)  $\|A+B\| \leq \|A\| + \|B\|$  for matrices of the same size.

(iv)  $\|AB\| \leq \|A\| \|B\|$ .

Exc Verify that Frobenius norm satisfies the above def. .

### (Induced matrix norm)

(\*) A vector norm that is defined on  $\mathbb{C}^p$  for  $p=m, n$  induces a matrix norm on  $\mathbb{C}^{m \times n}$  by setting:

$$\|A\| = \max_{\|x\|=1} \|Ax\| \quad \text{for } A \in \mathbb{C}^{m \times n}, x \in \mathbb{C}^{n \times 1}.$$

(\*)  $\|Ax\| \leq \|A\| \|x\|$  (i.e. compatible).

(\*) If  $A$  is non-singular,  $\min_{\|x\|=1} \|Ax\| = \frac{1}{\|A^{-1}\|}$ .

We have to show that the induced matrix <sup>norm</sup> is a matrix norm, but first note that the induced matrix norm can be define as:

$$(*) \quad \|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\text{Proof: } \frac{\|Ax\|}{\|x\|} = \left\| \frac{1}{\|x\|} Ax \right\| = \left\| A \cdot \frac{x}{\|x\|} \right\|$$

$$\text{where } \left\| \frac{x}{\|x\|} \right\| = 1.$$

Now let us verify that the induced matrix norm satisfy the conditions of the matrix norm.

(\*)  $\|Ax\| \geq 0$  and  $\|x\| > 0 \Rightarrow \frac{\|Ax\|}{\|x\|} \geq 0$  and  $\Rightarrow$ .

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \geq 0.$$

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = 0 \quad \text{iff } \|Ax\| = 0 \quad \forall x \in \mathbb{R}$$

$$\text{iff } A = 0.$$

$$\begin{aligned} \text{b) } \|\alpha A\| &= \sup_{x \neq 0} \frac{\|(\alpha A)x\|}{\|x\|} = \sup_{x \neq 0} |\alpha| \frac{\|Ax\|}{\|x\|} \\ &= |\alpha| \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = |\alpha| \|A\|. \end{aligned}$$

$$\begin{aligned} \text{c) } \|A+B\|_p &= \sup_{x \neq 0} \frac{\|(A+B)x\|}{\|x\|} = \sup_{x \neq 0} \frac{\|Ax+Bx\|}{\|x\|} \\ &\leq \sup_{x \neq 0} \left( \frac{\|Ax\| + \|Bx\|}{\|x\|} \right) \leq \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} + \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} \\ &= \|A\| + \|B\|. \end{aligned}$$

$$\begin{aligned} \text{d) } \|AB\| &= \sup_{\|x\|=1} \|(AB)x\| = \sup_{\|x\|=1} \|A(Bx)\| \leq \sup_{\|x\|=1} \|A\| \|Bx\| \\ &= \|A\| \sup_{\|x\|=1} \|Bx\| = \|A\| \|B\|. \end{aligned}$$

### Matrix 2-Norm

⊙ The matrix norm induced by the euclidean vector norm

$$\text{is } \|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sqrt{\lambda_{\max}},$$

where  $\lambda_{\max}$  = largest number  $\lambda$  s.th.  $A^T A - \lambda I$  is singular

⊙ If  $A$  is non-singular,

$$\|A^{-1}\|_2 = \frac{1}{\min_{\|x\|_2=1} \|Ax\|_2} = \frac{1}{\sqrt{\lambda_{\min}}},$$

where  $\lambda_{\min}$  = smallest number  $\lambda$  s.th.  $A^* A - \lambda I$  is singular.

Note  $\lambda_{\min}$  and  $\lambda_{\max}$  are smallest and largest eigenvalue of  $A^* A$ .

Problem Determine the induced norm  $\|A\|_2$  as well as  $\|A^{-1}\|_2$  for the nonsingular matrix

$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{pmatrix}$$

$$\underline{\text{a)}} \quad A^T A - \lambda I = \begin{pmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 3-\lambda \\ 3-\lambda & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -1 & 3-\lambda \\ 0 & -1+(3-\lambda)^2 \end{pmatrix}$$

$\Rightarrow A^T A - \lambda$  is singular if  $-1+(3-\lambda)^2 = 0$

$\Rightarrow \lambda = 2$  or  $\lambda = 4 \Rightarrow \lambda_{\min} = 2$  &  $\lambda_{\max} = 4$ .

$\Rightarrow \|A\|_2 = \sqrt{\lambda_{\max}} = 2$ ,  $\|A^{-1}\|_2 = \frac{1}{\sqrt{\lambda_{\min}}} = \frac{1}{\sqrt{2}} *$ .

Properties of the 2<sup>nd</sup>-norm.

$$(1) \quad \|A\|_2 = \max_{\|x\|_2=1} \max_{\|y\|_2=1} |y^* A x|$$

$$(2) \quad \|A_2\| = \|A^*\|_2$$

$$(3) \quad \|A^* A\|_2 = \|A\|_2^2$$

$$(4) \quad \left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\|_2 = \max \{ \|A\|_2, \|B\|_2 \}$$

$$(5) \quad \|U^* A V\|_2 = \|A\|_2 \quad \text{when } U^* U = I \text{ & } V^* V = I.$$

## Matrix 1-Norm and $\infty$ -Norm

The matrix norms induced by the 1-norm and  $\infty$ -norm are as follows:

$$a) \|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 = \max_j \sum_i |a_{ij}|$$

= the largest absolute column sum.

$$b) \|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}|.$$

= the largest absolute row sum.

Ex: Determine the induced matrix norms  $\|A\|_1$  and  $\|A\|_\infty$

for 
$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{pmatrix}$$

and compare the result with  $\|A\|_2$  and  $\|A\|_F$ .

sol:  $\|A\|_1 = \frac{1}{\sqrt{3}} + \frac{\sqrt{8}}{3} \approx 2.21$  &  $\|A\|_\infty = \frac{4}{\sqrt{3}} \approx 2.31$

$$\|A\|_2 = 2 \quad \text{and} \quad \|A\|_F = \sqrt{\text{Tr}(A^T A)} = \sqrt{6} = 2.45$$

We see that the norms are not equal.