Matrix Norms

Frobenius matrix norm:

The Frobenius matrix norm of $A \in \mathbb{C}^{m\times n}$ is defined by:

$$\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2 = \sum_i \|A_{i*}\|_2^2 = \sum_j \|A_{*j}\|_2^2 = \mathrm{Trace}(A^*A)$$

Example: $A = \begin{bmatrix} 2 & -1 \\ -4 & -2 \end{bmatrix}$, $\quad \|A\|_F = \left[2^2 + (-1)^2 + (-4)^2 + (-2)^2\right]^{1/2} = 5.$

Properties: (1) $\|Ax\|_2 \leq \|A\|_F \, \|x\|_2$.

Proof: $$\|Ax\|_2 = \sum_i |A_{i*}x|^2 \leq \sum_i \|A_{i*}\|_2^2 \, \|x\|_2^2$$

$$= \|A\|_F^2 \, \|x\|_2^2 .$$

We say that the Frobenius matrix norm $\|*\|_F$ and the euclidean vector norm $\|*\|_2$ are compatible.

(2) $\|AB\|_F \leq \|A\|_F \, \|B\|_F$ [Submultiplicative property].

Proof: $$\|AB\|_F^2 = \sum_j \|[AB]_{*j}\|_2^2 = \sum_j \|AB_{*j}\|_2^2$$

$$\leq \sum_j \|A\|_F^2 \, \|B_{*j}\|_2^2 = \|A\|_F^2 \, \|B\|_F^2 .$$

Def (General matrix norms).

A matrix norm is a function $\|*\|$ from the set of all complex matrices into $\mathbb{R}$ that satisfies the following:

(i) $\|A\| \geq 0$ and $\|A\| = 0$ iff $A = O$

(ii) $\|\alpha A\| = |\alpha| \, \|A\|$ $\quad \alpha$ is a scalar.

(iii) $\|A + B\| \leq \|A\| + \|B\|$ for matrices of the same size.

(IV) $\|AB\| \leq \|A\| \, \|B\|$.

Exc Verify that Frobenius norm satisfies the above def.

(Induced matrix norm)

(a) A vector norm that is defined on $\mathbb{C}^p$ for $p=m,n$ induces a matrix norm on $\mathbb{C}^{m\times n}$ by setting:

$$\|A\| = \max_{\|x\|=1} \|Ax\| \quad \text{for } A\in \mathbb{C}^{m\times n},\ x\in \mathbb{C}^{n\times 1}.$$

(i) $\|Ax\| \leq \|A\|\ \|x\|$ (i.e. compatible).

(ii) If A is non-singular, $\displaystyle\min_{\|x\|=1} \|Ax\| = \frac{1}{\|A^{-1}\|}$.

We have to show that the induced matrix norm is a matrix norm, but first note that the induced matrix norm can be define as:

$$(*) \qquad \|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$$

Proof: $\displaystyle\frac{\|Ax\|_p}{\|x\|_p} = \left\| \frac{1}{\|x\|_p} Ax \right\| = \left\| A \frac{x}{\|x\|}\right\|$

where $\left\| \frac{x}{\|x\|}\right\| = 1$.

Now let us verify that the induced matrix norm satisfy the conditions of the matrix norm.

(a) $\|Ax\| \geq 0$ and $\|x\| > 0 \Rightarrow \dfrac{\|Ax\|}{\|x\|} \geq 0$ ~~and~~ $\Rightarrow$.

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} \geq 0.$$

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = 0 \quad \text{iff } \|Ax\| = 0 \quad \forall\ x\in \mathbb{R}$$

$$\text{iff} \quad A = 0.$$

(b) $$\|\alpha A\| = \sup \frac{\|(\alpha A)x\|}{\|x\|} = \sup_{x\neq 0} |\alpha| \frac{\|Ax\|}{\|x\|}$$

$$= |\alpha| \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = |\alpha|\ \|A\|.$$

(c) $$\|A+B\|_p = \sup_{x\neq 0} \frac{\|(A+B)x\|}{\|x\|} = \sup_{x\neq 0} \frac{\|Ax+Bx\|}{\|x\|}$$

$$\leq \sup\left(\frac{\|Ax\|+\|Bx\|}{\|x\|}\right) \leq \sup_{x\neq 0}\frac{\|Ax\|}{\|x\|} + \sup_{x\neq 0}\frac{\|Bx\|}{\|x\|}$$

$$= \|A\| + \|B\|.$$

(d) $$\|AB\| = \sup_{\|x\|_p=1} \|(AB)x\| = \sup_{\|x\|=1} \|A(Bx)\| \leq \sup_{\|x\|=1} \|A\|\ \|Bx\|$$

$$= \|A\| \sup_{\|x\|=1} \|Bx\| = \|A\|\ \|B\|.$$

Matrix 2-Norm

(•) The matrix norm induced by the euclidean vector norm is $$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sqrt{\lambda_{max}},$$

where λ_{max} = largest number λ s.th. $A^TA-\lambda I$ is singular

(•) If A is non-singular,

$$\|A^{-1}\|_2 = \frac{1}{\min_{\|x\|_2=1}\|Ax\|_2} = \frac{1}{\sqrt{\lambda_{min}}},$$

where λ_{min} = smallest number λ s.th. $A^*A-\lambda I$ is singular.

Note λ_{min} and λ_{max} are smallest and largest eigenvalue of A^*A.

Problem Determine the induced norm $\|A\|_2$ as well as $\|A^{-1}\|_2$ for the nonsingular matrix

$$A = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{pmatrix}$$

Sol $$A^T A - \lambda I = \begin{pmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 3-\lambda \\ 3-\lambda & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -1 & 3-\lambda \\ 0 & -1+(3-\lambda)^2 \end{pmatrix}$$

$\Rightarrow$ $A^T A - \lambda$ is singular if $-1+(3-\lambda)^2 = 0$

$\Rightarrow \lambda = 2$ or $\lambda = 4 \Rightarrow \lambda_{min} = 2$ & $\lambda_{max} = 4$.

$\Rightarrow \|A\|_2 = \sqrt{\lambda_{max}} = 2$, $\|A^{-1}\|_2 = \frac{1}{\sqrt{\lambda_{min}}} = \frac{1}{\sqrt{2}}$ ✱.

Properties of the 2^{nd} – norm.

(1) $$\|A\|_2 = \max_{\|x\|_2=1} \max_{\|y\|_2=1} |y^* A x|$$

(2) $\|A_2\| = \|A^*\|_2$.

(3) $\|A^* A\|_2 = \|A\|_2^2$

(4) $$\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\|_2 = \max\{\|A\|_2, \|B\|_2\}$$

(5) $\|U^* A V\|_2 = \|A\|_2$ when $U^* U = I$ & $V^* V = I$.

Matrix 1-Norm and ∞-Norm

The matrix norms induced by the 1-norm and ∞-norm are as follows:

(•) $\|A\|_1 = \max\limits_{\|x\|_1=1} \|Ax\|_1 = \max\limits_{j} \sum\limits_{i} |a_{ij}|$

= the largest absolute column sum.

(•) $\|A\|_\infty = \max\limits_{\|x\|_\infty=1} \|Ax\|_\infty = \max\limits_{i} \sum\limits_{j} |a_{ij}|$.

= the largest absolute row sum.

Ex: Determine the induced matrix norms $\|A\|_1$ and $\|A\|_\infty$ for $A = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{pmatrix}$

and compare the result with $\|A\|_2$ and $\|A\|_F$.

Sol: $\|A\|_1 = \frac{1}{\sqrt{3}} + \frac{\sqrt{8}}{3} \simeq 2.21$ & $\|A\|_\infty = \frac{4}{\sqrt{3}} \simeq 2.31$

$\|A\|_2 = 2$ and $\|A\|_F = \sqrt{Tr(A^TA)} = \sqrt{6} = 2.45$

We see that the norms are not equal.