Norms, Inner products and orthogonality

Def: Euclidean vector norm:

for a vector $x_{n\times 1}$ the euclidean norm of x is

- $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \sqrt{x^T x}$ if $x \in \mathbb{R}^{n\times 1}$.
- $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \sqrt{x^* x}$ if $x \in \mathbb{C}^{n\times 1}$.

Example: if

$$u = \begin{pmatrix} 0 \\ -1 \\ 2 \\ -2 \\ 4 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} i \\ 2 \\ 1-i \\ 0 \\ 1+i \end{pmatrix} \quad \text{then}$$

$$\|u\| = \sqrt{\sum u_i^2} = \sqrt{u^T u} = \sqrt{0+1+4+4+16} = 5,$$

$$\|v\| = \sqrt{\sum |v_i|^2} = \sqrt{v^* v} = \sqrt{1+4+2+0+2} = 3.$$

Note: (1) if $z = a+ib$, then $\bar{z} = a-ib$ and $|z| = \sqrt{\bar{z}z}$ $= \sqrt{a^2+b^2} \Rightarrow |z|^2 = a^2+b^2 = \bar{z}z$ which is a real number.

(2) Euclidean norm guarantees that $\forall$ scalar α,

$\|x\| \geq 0$, $\|x\| = 0$ iff $x=0$, and $\|\alpha x\| = |\alpha|\,\|x\|$.

(3) if $x \neq 0$, we can normalize x by setting $u = x/\|x\|$, since

$$\|u\| = \left\|\frac{x}{\|x\|}\right\| = \frac{1}{\|x\|}\|x\| = 1.$$

Standard inner product:

the scalar terms defined by:

$$x^T y = \sum_{i=1}^{n} x_i y_i \in \mathbb{R} \quad \text{and} \quad x^* y = \sum_{i=1}^{n} \bar{x}_i y_i \in \mathbb{C}$$

are called the standard inner products.

<u>Cauchy-Schwarz Inequality</u>

$$|x^* y| \leq \|x\| \, \|y\| \quad \text{for all } x, y \in \mathbb{C}^{n\times 1}.$$

Equality holds iff $y = \alpha x$ for $\alpha = x^* y / (x^* x)$.

<u>Proof:</u> set $\alpha = x^* y / x^* x = x^* y / \|x\|^2$

if $x = 0$, then there is nothing to prove.

Assume $x \neq 0$ and observe that $x^*(\alpha x - y) = 0$, so

$$0 \leq \|\alpha x - y\|^2 = (\alpha x - y)^*(\alpha x - y) = \bar{\alpha} x^* (\alpha x - y) - y^*(\alpha x - y)$$

$$= - y^*(\alpha x - y) = y^* y - \alpha y^* x = \frac{\|y\|^2 \|x\|^2 - (x^* y)(y^* x)}{\|x\|^2}$$

since $y^* x = \overline{x^* y}$, it follows that

$$(x^* y)(y^* x) = |x^* y|^2, \quad \text{so}$$

$$0 \leq \frac{\|y\|^2 \|x\|^2 - |x^* y|^2}{\|x\|^2},$$

Now $\|x\|^2 > 0 \Rightarrow \|y\|^2 \|x\|^2 - |x^* y|^2 \geq 0$ #.

Triangle inequality

$$\|x+y\| \leq \|x\| + \|y\| \quad \forall\ x, y \in \mathbb{C}^n.$$

Proof: let x & y be column vectors

$$\|x+y\|^2 = (x+y)^*(x+y) = x^*x + x^*y + y^*x + y^*y$$

$$= \|x\|^2 + x^*y + y^*x + \|y\|^2$$

if $z = a+ib \Rightarrow z + \bar{z} = 2a = 2\,\mathrm{Re}(z)$ and $|z|^2 = a^2 + b^2 \geq a^2$, so that $|z| \geq \mathrm{Re}(z)$. Using $y^*x = \overline{x^*y}$ with Cauchy-Schwarz inequality yields:

$$x^*y + y^*x = 2\,\mathrm{Re}(x^*y) \leq 2|x^*y| \leq 2\|x\|\,\|y\|.$$

$$\Rightarrow \|x+y\|^2 \leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

Def (P-Norms)

for $p \geq 1$, the p-norm of $x \in \mathbb{C}^n$ is defined as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

Thm (Holder's inequality)

if $p > 1$ and $q > 1$ are real numbers s.th. $\frac{1}{q} + \frac{1}{p} = 1$,

then

$$\sum_{i=1}^{n} |x_i \cdot y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}.$$

Proof: Exc.

<u>Thm</u> (Minkowski's inequality) if $p \geq 1$,

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p .$$

it can be shown that the euclidean norm satisfies

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0$ iff $x = 0$

(ii) $\|\alpha x\|_p = |\alpha| \, \|x\|_p$ α scalar.

(iii) $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

In practice only three p-norms are used:

$\|x\|_1 = \sum_{i=1}^{n} |x_i|$ (grid norm)

$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$ (euclidean norm).

and $\|x\|_\infty = \lim_{p \to \infty} \|x\|_p = \max_i |x_i|$

<u>Example:</u> if $x = (3, 4-3i, 1)$, then $\|x\|_1 = 9$,

$\|x\|_2 = \sqrt{35}$ and $\|x\|_\infty = 5$.

<u>General vector norm:</u>

A norm for a real or complex vector space V is a fn $\| * \|$ mapping V into $\mathbb{R}$, that satisfies the following:

1- $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$

2. $\|\alpha x\| = |\alpha| \, \|x\|$ for all scalars α,

3. $\|x+y\| \leq \|x\| + \|y\|$.

<u>Equivalent norms:</u> A sequence $\{x_k\} \subset V$ is said to converge to x (write $x_k \to x$) if $\|x_k - x\| \to 0$.

<u>Proposition:</u> For $\|*\|_a$, $\|*\|_b$ on an n-dimensional space V, exhibit positive constants α and β (depending only on the norms) such that

$$\alpha \leq \frac{\|x\|_a}{\|x\|_b} \leq \beta \qquad \forall \text{ nonzero vectors in } V.$$

<u>Proof:</u> For $S_b = \{y \mid \|y\|_b = 1\}$, let $\mu = \min_{y \in S_b} \|y\|_a > 0$, and

$$\frac{x}{\|x\|_b} \in S_b \Rightarrow \|x\|_a = \|x\|_b \cdot \left\| \frac{x}{\|x\|_b} \right\|_a \geq \|x\|_b \min_{y \in S_b} \|y\|_a$$
$$= \|x\|_b \, \mu.$$

The same argument show there is a $\nu > 0$ s.th. $\|x\|_b \geq \nu \|x\|_a$. Now let $\alpha = \mu$ and $\beta = 1/\nu$ #.

Note that the proposition insures that $\|x_k - x\|_a \to 0$ iff $\|x_k - x\|_b \to 0$

<u>Note</u>: (1) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\, \|x\|_2$,

(2) $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\, \|x\|_\infty$

(3) $\|x\|_\infty \leq \|x\|_1 \leq n\, \|x\|_\infty$.