# Moore-Penrose Pseudoinverse

**Thm:** let $A_{m\times n}$ matrix over a field $k$ (field of real or complex numbers). Then there is a unique matrix $A^+$ over $k$ such that:

P1. $AA^+A = A$ P2. $A^+AA^+ = A^+$

P3. $(AA^+)^* = AA^+$ P4. $(A^+A)^* = A^+A$

$A^+$ is called the Pseudoinverse of $A$.

Note that the inverse of non-singular matrix satisfies all four Penrose properties. Also a right or left inverses satisfies no fewer than three of the four properties.

**Ex:** Consider $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Verify directly that $A^+ = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \end{bmatrix}$ satisfies P1-P4.

Another characterization of $A^+$ is given the following theorem. While not generally suitable for computer implementation this characterization can be useful for hand calculation of small examples.

**Thm** let $A \in \mathbb{R}^{m\times n}$, then

$$A^+ = \lim_{\delta\to 0} (A^TA + \delta^2 I)^{-1} A^T$$

$$= \lim_{\delta\to 0} A^T (AA^T + \delta^2 I)^{-1}.$$

**Example:** for any scalar $\alpha$: $\alpha^+ = \begin{cases} \alpha^{-1} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases}$

Example: (a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^{+} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{+} = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$.

## Properties and application:

[1] Suppose that $B$ and $C$ are two $n\times m$ matrices satisfying the pseudoinverse properties: (i.e.)

a) $BA = (BA)^T$, $CA = (CA)^T$

b) $AB = (AB)^T$, $AC = (AC)^T$

(c) $ABA = A$, $ACA = A$

(d) $BAB = B$, $CAC = C$

then $B = C$ [uniqueness].

Proof:

$$AB = B^T A^T = B^T A^T C^T A^T = ABC^T A^T = ABAC = AC$$

Similarly: $BA = CA$

Now $B = BAB = BAC = CAC = C$ ※.

[2] if $A^* A = 0$, then $A = 0$.

Proof: the $j$th diagonal element of $A^*A$ is

$$\sum_{i=1}^{m} (a^*)_{ji}\, a_{ij} = \sum_{i=1}^{m} \overline{a_{ij}}\, a_{ij} = 0$$

But since $x\bar{x} \geq 0$ $\forall x$ and equals zero only for $x = 0 \Rightarrow a_{ij} = 0$ $\forall j \Rightarrow A = 0$.

[3] if $BAA^* = CAA^* \Rightarrow BA = CA$.

Proof:

$$BAA^* = CAA^* \leftrightarrow BAA^* - CAA^* = (BA - CA)A^* = 0 \leftrightarrow$$

$$(BA-CA)A^*(B-C)^* = 0 \leftrightarrow (BA-CA)(BA-CA)^* = 0 \rightarrow BA - CA = 0$$

$$\Rightarrow BA = CA.$$

④ If the matrix $A$ is invertible, the pseudoinverse and the concide, i.e $A^{+} = A^{-1}$.

⑤ the pseudoinverse of the pseudoinverse is the orginal matrix, i.e $(A^{+})^{+} = A$.

⑥ $(A^{*})^{+} = (A^{+})^{*}$

⑦ $(\alpha A)^{+} = \alpha^{-1} A^{+}$ for $\alpha \neq 0$.

⑧ <u>full-rank:</u>

- If the columns of $A$ are linearly independent, then $A^{*}A$ is invertible $\Rightarrow$ $A^{+} = (A^{*}A)^{-1} A^{*}$

  it follows that $A^{+}$ is a left inverse of $A$: $A^{+}A = I$

- If the rows of $A$ are linearly independent, then $AA^{*}$ is invertible $\Rightarrow$ $A^{+} = A^{*}(AA^{*})^{-1}$

  it follows that $A^{+}$ is a right inverse of $A$: $AA^{+} = I$.

[i.e $A^{+}A = I_n$ iff rank $(A) = r = n$, where $A_{m \times n}$ and $AA^{+} = I_m$ iff $r = m$].

<u>Proof</u>

$\Rightarrow$ if $A^{+}A = I_n \Rightarrow \text{rank}(A^{+}A) = n$, but $\text{rank}(A^{+}A)$ equals $\text{rank}(A) = n$.

$\Leftarrow$ if $A_{m \times n}$ with $\text{rank}(A) = r = n$, then $A^{+}A$ is $n \times n$ matrix is non-singular gives $A^{+}A = I_n$. #

what is left $\text{rank}(A^{*}A^{+}) = \text{rank}(A)$, since

$R(A) = R(AA^{+}A) \subset R(AA^{+}) \subset R(A)$

Exc: Show that $\mathcal{N}(AA^+) = \mathcal{N}(A)$ and

$$R[(A^+A)^*] = R(A^*).$$

Note: $(AB)^+ \neq B^+A^+$

as an example consider $A = [0\ 1]$ and $B = \begin{bmatrix}1\\1\end{bmatrix}$,

then $(AB)^+ = 1^+ = 1$

while $B^+A^+ = [\frac{1}{2}\ \ \frac{1}{2}]\begin{bmatrix}0\\1\end{bmatrix} = 1/2$.

Thm: $(AB)^+ = B^+A^+$ iff

1) $R(BB^TA^T) \subseteq R(A^T)$ and

2) $R(A^TAB) \subseteq R(B)$

Proof Exc.