Inverses of sums and sensitivity

first note that ~~if~~ rank$(A_{m\times n})=1$ iff there are non-zero columns $u_{m\times 1}$ and $v_{n\times 1}$ s.th.

$$A = u v^T$$

Proof $\Rightarrow$ if rank$(A)=1$, then $\exists$ non-singular matrices P and Q s.th.

$$PAQ = N_1 = e_1 \, e_1^T$$

where e_1 is $m\times 1$ col. vector and e_1^T is $1\times n$, so

$$A = P^{-1} e_1 \, e_1^T Q = (P^{-1})_{*1} \, (Q^{-1})_{1*} = u v^T.$$

$\Leftarrow$ if $A = uv^T \sim e_1 e_1^T = N_1 \Rightarrow$ rank$(A)=1$.

Note 2: $(I + cd^T)^{-1} = I - \dfrac{cd^T}{1+d^Tc}$ if $1+d^Tc \neq 0$.

and where c and d are $n\times 1$ nonzero columns.

Proof: We can verify it by direct multiplication: (i.e)

$$(I+cd^T)\left[I - \frac{cd^T}{1+d^Tc}\right] \qquad \text{let } k = d^Tc = \text{scalar}$$

$$= I + cd^T - \frac{cd^T + cd^Tcd^T}{1+d^Tc}$$

$$= I + cd^T - \frac{cd^T + c\overbrace{(d^Tc)}^{\text{equals } k}d^T}{1+d^Tc} = I + cd^T - \frac{cd^T + kcd^T}{1+k}$$

$$= I + cd^T - cd^T = I$$

$\divideontimes$.

Thm (Sherman–Morrison Formula)

1. If $A_{n\times n}$ is nonsingular and if c and d are $n\times 1$ columns such that $1+d^TA^{-1}c \neq 0$, then the sum

$$(A+cd^T)^{-1} = A^{-1} - \frac{A^{-1}cd^TA^{-1}}{1+d^TA^{-1}c}$$

2. (Sherman–Morrison–Woodbury formula)

If C and D are $n\times k$ s.th. $(I+D^TA^{-1}C)^{-1}$ exist

then $(A+CD^T)^{-1} = A^{-1} - A^{-1}C(I+D^TA^{-1}C)^{-1}D^TA^{-1}$.

Proof (1) $(A+cd^T)^{-1} = [A(I+A^{-1}cd^T)]^{-1}$

$$= (I+A^{-1}cd^T)^{-1}A^{-1}$$

$$= \left(I - \frac{A^{-1}cd^T}{1+d^TA^{-1}c}\right)A^{-1} = A^{-1} - \frac{A^{-1}cd^TA^{-1}}{1+d^TA^{-1}c}$$

this is Sherman–Morrison rank one update formula, since that $\operatorname{rank}(cd^T)=1$ when $c\neq 0 \neq d$.

(2) Since $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & C \\ D^T & -I \end{pmatrix}\begin{pmatrix} I & 0 \\ D^T & I \end{pmatrix} = \begin{pmatrix} A+CD^T & 0 \\ 0 & -I \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} I & 0 \\ -D^T & I \end{pmatrix}\begin{pmatrix} A^{-1}+A^{-1}CS^{-1}D^TA^{-1} & -A^{-1}CS^{-1} \\ -S^{-1}D^TA^{-1} & S^{-1} \end{pmatrix}\begin{pmatrix} I & -C \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} (A+CD^T)^{-1} & 0 \\ 0 & -I \end{pmatrix}$$

where $S = -(I + D^T A^{-1} C)$. Comparing the upper-left-hand blocks produces

$$(A + CD^T)^{-1} = A^{-1} - A^{-1}C(I + D^T A^{-1} C)^{-1} D^T A^{-1} \quad \star.$$

Example: Start with A and A^{-1} given below. Update A by adding 1 to a_{21}, and then use the Sherman-Morrison formula to update A^{-1}.

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \quad \& \quad A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}.$$

Solution: The updated matrix is:

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$= A + e_2 e_1^T.$$

Applying the Sherman-Morrison formula yields:

$$B^{-1} = A^{-1} - \frac{A^{-1} e_2 e_1^T A^{-1}}{1 + e_1^T A^{-1} e_2} = A^{-1} - \frac{[A^{-1}]_{*2} [A^{-1}]_{1*}}{1 + [A^{-1}]_{12}}$$

$$= \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} - \frac{\begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \end{pmatrix}}{1 - 2} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}.$$