More about Rank

Problem: Show that $rank\,(\mathbf{A}+\mathbf{B}) \le rank\,(\mathbf{A}) + rank\,(\mathbf{B})$.

Solution: Observe that

$$R\,(\mathbf{A}+\mathbf{B}) \subseteq R\,(\mathbf{A}) + R\,(\mathbf{B})$$

because if $\mathbf{b} \in R\,(\mathbf{A}+\mathbf{B})$, then there is a vector $\mathbf{x}$ such that

$$\mathbf{b} = (\mathbf{A}+\mathbf{B})\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x} \in R\,(\mathbf{A}) + R\,(\mathbf{B}).$$

Recall from (4.4.5) that if $\mathcal{M}$ and $\mathcal{N}$ are vector spaces such that $\mathcal{M} \subseteq \mathcal{N}$, then $\dim \mathcal{M} \le \dim \mathcal{N}$. Use this together with formula (4.4.19) for the dimension of a sum to conclude that

$$\begin{aligned} rank\,(\mathbf{A}+\mathbf{B}) &= \dim R\,(\mathbf{A}+\mathbf{B}) \le \dim \Big(R\,(\mathbf{A}) + R\,(\mathbf{B}) \Big) \\ &= \dim R\,(\mathbf{A}) + \dim R\,(\mathbf{B}) - \dim \Big(R\,(\mathbf{A}) \cap R\,(\mathbf{B}) \Big) \\ &\le \dim R\,(\mathbf{A}) + \dim R\,(\mathbf{B}) = rank\,(\mathbf{A}) + rank\,(\mathbf{B}). \end{aligned}$$

Rank of a Product

If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then

$$rank\,(\mathbf{AB}) = rank\,(\mathbf{B}) - \dim N\,(\mathbf{A}) \cap R\,(\mathbf{B}). \qquad (4.5.1)$$

Proof. Start with a basis $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ for $N\,(\mathbf{A}) \cap R\,(\mathbf{B})$, and notice $N\,(\mathbf{A}) \cap R\,(\mathbf{B}) \subseteq R\,(\mathbf{B})$. If $\dim R\,(\mathbf{B}) = s + t$, then, as discussed in Example 4.4.5, there exists an extension set $\mathcal{S}_{ext} = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t\}$ such that $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_s, \mathbf{z}_1, \ldots, \mathbf{z}_t\}$ is a basis for $R\,(\mathbf{B})$. The goal is to prove that $\dim R\,(\mathbf{AB}) = t$, and this is done by showing $\mathcal{T} = \{\mathbf{A}\mathbf{z}_1, \mathbf{A}\mathbf{z}_2, \ldots, \mathbf{A}\mathbf{z}_t\}$ is a basis for $R\,(\mathbf{AB})$. $\mathcal{T}$ spans $R\,(\mathbf{AB})$ because if $\mathbf{b} \in R\,(\mathbf{AB})$, then $\mathbf{b} = \mathbf{ABy}$ for some $\mathbf{y}$, but $\mathbf{By} \in R\,(\mathbf{B})$ implies $\mathbf{By} = \sum_{i=1}^{s} \xi_i \mathbf{x}_i + \sum_{i=1}^{t} \eta_i \mathbf{z}_i$, so

$$\mathbf{b} = \mathbf{A} \left(\sum_{i=1}^{s} \xi_i \mathbf{x}_i + \sum_{i=1}^{t} \eta_i \mathbf{z}_i \right) = \sum_{i=1}^{s} \xi_i \mathbf{A}\mathbf{x}_i + \sum_{i=1}^{t} \eta_i \mathbf{A}\mathbf{z}_i = \sum_{i=1}^{t} \eta_i \mathbf{A}\mathbf{z}_i.$$

$\mathcal{T}$ is linearly independent because if $\mathbf{0} = \sum_{i=1}^{t} \alpha_i \mathbf{A}\mathbf{z}_i = \mathbf{A} \sum_{i=1}^{t} \alpha_i \mathbf{z}_i$, then $\sum_{i=1}^{t} \alpha_i \mathbf{z}_i \in N\,(\mathbf{A}) \cap R\,(\mathbf{B})$, so there are scalars β_j such that

$$\sum_{i=1}^{t} \alpha_i \mathbf{z}_i = \sum_{j=1}^{s} \beta_j \mathbf{x}_j \quad \text{or, equivalently,} \quad \sum_{i=1}^{t} \alpha_i \mathbf{z}_i - \sum_{j=1}^{s} \beta_j \mathbf{x}_j = \mathbf{0},$$

and hence the only solution for the α_i 's and β_i 's is the trivial solution because $\mathcal{B}$ is an independent set. Thus $\mathcal{T}$ is a basis for $R\,(\mathbf{AB})$, so $t = \dim R\,(\mathbf{AB}) = rank\,(\mathbf{AB})$, and hence

$$rank\,(\mathbf{B}) = \dim R\,(\mathbf{B}) = s + t = \dim N\,(\mathbf{A}) \cap R\,(\mathbf{B}) + rank\,(\mathbf{AB}). \quad \blacksquare$$

Bounds on the Rank of a Product

If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then

- $rank(\mathbf{AB}) \le \min\{rank(\mathbf{A}),\ rank(\mathbf{B})\}$, (4.5.2)
- $rank(\mathbf{A}) + rank(\mathbf{B}) - n \le rank(\mathbf{AB})$. (4.5.3)

Proof. In words, (4.5.2) says that the rank of a product cannot exceed the rank of either factor. To prove $rank(\mathbf{AB}) \le rank(\mathbf{B})$, use (4.5.1) and write

$$rank(\mathbf{AB}) = rank(\mathbf{B}) - \dim N(\mathbf{A}) \cap R(\mathbf{B}) \le rank(\mathbf{B}).$$

This says that the rank of a product cannot exceed the rank of the right-hand factor. To show that $rank(\mathbf{AB}) \le rank(\mathbf{A})$, remember that transposition does not alter rank, and use the reverse order law for transposes together with the previous statement to write

$$rank(\mathbf{AB}) = rank(\mathbf{AB})^T = rank\left(\mathbf{B}^T\mathbf{A}^T\right) \le rank\left(\mathbf{A}^T\right) = rank(\mathbf{A}).$$

To prove (4.5.3), notice that $N(\mathbf{A}) \cap R(\mathbf{B}) \subseteq N(\mathbf{A})$, and recall from (4.4.5) that if $\mathcal{M}$ and $\mathcal{N}$ are spaces such that $\mathcal{M} \subseteq \mathcal{N}$, then $\dim \mathcal{M} \le \dim \mathcal{N}$. Therefore,

$$\dim N(\mathbf{A}) \cap R(\mathbf{B}) \le \dim N(\mathbf{A}) = n - rank(\mathbf{A}),$$

and the lower bound on $rank(\mathbf{AB})$ is obtained from (4.5.1) by writing

$$rank(\mathbf{AB}) = rank(\mathbf{B}) - \dim N(\mathbf{A}) \cap R(\mathbf{B}) \ge rank(\mathbf{B}) + rank(\mathbf{A}) - n. \quad \blacksquare$$

Block Matrix Multiplication

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are partitioned into submatrices—often referred to as ***blocks***—as indicated below.

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1r} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{s1} & \mathbf{A}_{s2} & \cdots & \mathbf{A}_{sr} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1t} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{r1} & \mathbf{B}_{r2} & \cdots & \mathbf{B}_{rt} \end{pmatrix}.$$

If the pairs $(\mathbf{A}_{ik}, \mathbf{B}_{kj})$ are conformable, then $\mathbf{A}$ and $\mathbf{B}$ are said to be ***conformably partitioned.*** For such matrices, the product $\mathbf{AB}$ is formed by combining the blocks exactly the same way as the scalars are combined in ordinary matrix multiplication. That is, the (i, j)-block in $\mathbf{AB}$ is

$$\mathbf{A}_{i1}\mathbf{B}_{1j} + \mathbf{A}_{i2}\mathbf{B}_{2j} + \cdots + \mathbf{A}_{ir}\mathbf{B}_{rj}.$$

Block multiplication is particularly useful when there are patterns in the matrices to be multiplied. Consider the partitioned matrices

$$\mathbf{A} = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) = \begin{pmatrix} \mathbf{C} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{B} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \end{array}\right) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{C} \end{pmatrix},$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Using block multiplication, the product $\mathbf{AB}$ is easily computed to be

$$\mathbf{AB} = \begin{pmatrix} \mathbf{C} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 2\mathbf{C} & \mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} = \left(\begin{array}{cc|cc} 2 & 4 & 1 & 2 \\ 6 & 8 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

For instance, if A, B, C, X, Y, U, V are n-square matrices, then

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} X & Y \\ U & V \end{pmatrix} = \begin{pmatrix} AX + BU & AY + BV \\ CU & CV \end{pmatrix}.$$

Similar properties are true for columns. Two often-used facts are

$$\det(AB) = \det A \ \det B, \quad A,\ B \in \mathbb{M}_n,$$

and

$$\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \det A \ \det C, \quad A \in \mathbb{M}_n,\ C \in \mathbb{M}_m.$$

Example: Prove that:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det A \ \det(D - CA^{-1}B).$$

Theorem 2.7 *Let A and B be $m \times n$ matrices, and denote by C and D, respectively, the partitioned matrices*

$$C = (I_m,\ I_m), \quad D = \begin{pmatrix} A \\ B \end{pmatrix}.$$

Then

$$\begin{aligned} \operatorname{rank}(A+B) &= \operatorname{rank}(A) + \operatorname{rank}(B) - \dim(\operatorname{Im} D \cap \operatorname{Ker} C) \\ &\quad - \dim(\operatorname{Im} A^* \cap \operatorname{Im} B^*). \end{aligned} \tag{2.8}$$

Theorem 2.3 *Suppose that the partitioned matrix*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible and that the inverse is conformally partitioned as

$$M^{-1} = \begin{pmatrix} X & Y \\ U & V \end{pmatrix},$$

where A, D, X, and V are square matrices. Then

$$\det A = \det V \det M.$$

Find the inverse of M.