FOUR FUNDAMENTAL SUBSPACES

Range Spaces

The ***range of a matrix*** $\mathbf{A} \in \Re^{m \times n}$ is defined to be the subspace $R(\mathbf{A})$ of $\Re^m$ that is generated by the range of $f(\mathbf{x}) = \mathbf{Ax}$. That is,

$$R(\mathbf{A}) = \{\mathbf{Ax} \mid \mathbf{x} \in \Re^n\} \subseteq \Re^m.$$

Similarly, the range of $\mathbf{A}^T$ is the subspace of $\Re^n$ defined by

$$R\left(\mathbf{A}^T\right) = \{\mathbf{A}^T\mathbf{y} \mid \mathbf{y} \in \Re^m\} \subseteq \Re^n.$$

Because $R(\mathbf{A})$ is the set of all "images" of vectors $\mathbf{x} \in \Re^m$ under transformation by $\mathbf{A}$, some people call $R(\mathbf{A})$ the ***image space*** of $\mathbf{A}$.

Column and Row Spaces

For $\mathbf{A} \in \Re^{m \times n}$, the following statements are true.

- $R(\mathbf{A}) =$ the space spanned by the columns of $\mathbf{A}$ (column space).
- $R\left(\mathbf{A}^T\right) =$ the space spanned by the rows of $\mathbf{A}$ (row space).
- $\mathbf{b} \in R(\mathbf{A}) \iff \mathbf{b} = \mathbf{Ax}$ for some $\mathbf{x}$. (4.2.3)
- $\mathbf{a} \in R\left(\mathbf{A}^T\right) \iff \mathbf{a}^T = \mathbf{y}^T\mathbf{A}$ for some $\mathbf{y}^T$. (4.2.4)

Problem: Describe $R(\mathbf{A})$ and $R\left(\mathbf{A}^T\right)$ for $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

Equal Ranges

For two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same shape:

- $R\left(\mathbf{A}^T\right) = R\left(\mathbf{B}^T\right)$ if and only if $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. (4.2.5)
- $R(\mathbf{A}) = R(\mathbf{B})$ if and only if $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$. (4.2.6)

Proof. To prove (4.2.5), first assume $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$ so that there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{B}$. To see that $R\left(\mathbf{A}^T\right) = R\left(\mathbf{B}^T\right)$, use (4.2.4) to write

$$\begin{aligned} \mathbf{a} \in R\left(\mathbf{A}^T\right) &\iff \mathbf{a}^T = \mathbf{y}^T\mathbf{A} = \mathbf{y}^T\mathbf{P}^{-1}\mathbf{PA} \quad \text{for some } \mathbf{y}^T \\ &\iff \mathbf{a}^T = \mathbf{z}^T\mathbf{B} \quad \text{for } \mathbf{z}^T = \mathbf{y}^T\mathbf{P}^{-1} \\ &\iff \mathbf{a} \in R\left(\mathbf{B}^T\right). \end{aligned}$$

Conversely, if $R\left(\mathbf{A}^T\right) = R\left(\mathbf{B}^T\right)$, then

$$span\left\{\mathbf{A}_{1*}, \mathbf{A}_{2*}, \ldots, \mathbf{A}_{m*}\right\} = span\left\{\mathbf{B}_{1*}, \mathbf{B}_{2*}, \ldots, \mathbf{B}_{m*}\right\},$$

so each row of $\mathbf{B}$ is a combination of the rows of $\mathbf{A}$, and vice versa. On the basis of this fact, it can be argued that it is possible to reduce $\mathbf{A}$ to $\mathbf{B}$ by using only row operations (the tedious details are omitted), and thus $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. The proof of (4.2.6) follows by replacing $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A}^T$ and $\mathbf{B}^T$. ■

Problem: Determine whether or not the following sets span the same subspace:

$$\mathcal{A} = \left\{\begin{pmatrix}1\\2\\2\\3\end{pmatrix}, \begin{pmatrix}2\\4\\1\\3\end{pmatrix}, \begin{pmatrix}3\\6\\1\\4\end{pmatrix}\right\}, \qquad \mathcal{B} = \left\{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\}.$$

Solution: Place the vectors as *rows* in matrices $\mathbf{A}$ and $\mathbf{B}$, and compute

$$\mathbf{A} = \begin{pmatrix}1 & 2 & 2 & 3\\2 & 4 & 1 & 3\\3 & 6 & 1 & 4\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & 0 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{pmatrix} = \mathbf{E_A}$$

and

$$\mathbf{B} = \begin{pmatrix}0 & 0 & 1 & 1\\1 & 2 & 3 & 4\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & 0 & 1\\0 & 0 & 1 & 1\end{pmatrix} = \mathbf{E_B}.$$

Hence $span\{\mathcal{A}\} = span\{\mathcal{B}\}$ because the nonzero rows in $\mathbf{E_A}$ and $\mathbf{E_B}$ agree.

Spanning the Row Space and Range

Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{U}$ be any row echelon form derived from $\mathbf{A}$. Spanning sets for the row and column spaces are as follows:

- The nonzero rows of $\mathbf{U}$ span $R\left(\mathbf{A}^T\right)$. (4.2.7)
- The basic columns in $\mathbf{A}$ span $R\left(\mathbf{A}\right)$. (4.2.8)

Problem: Determine spanning sets for $R(\mathbf{A})$ and $R(\mathbf{A}^T)$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

Solution: Reducing $\mathbf{A}$ to any row echelon form $\mathbf{U}$ provides the solution—the basic columns in $\mathbf{A}$ correspond to the pivotal positions in $\mathbf{U}$, and the nonzero rows of $\mathbf{U}$ span the row space of $\mathbf{A}$. Using $\mathbf{E_A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ produces

$$R(\mathbf{A}) = span\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad R(\mathbf{A}^T) = span\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Nullspace

- For an $m \times n$ matrix $\mathbf{A}$, the set $N(\mathbf{A}) = \{\mathbf{x}_{n\times 1} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$ is called the ***nullspace*** of $\mathbf{A}$. In other words, $N(\mathbf{A})$ is simply the set of all solutions to the homogeneous system $\mathbf{Ax} = \mathbf{0}$.
- The set $N(\mathbf{A}^T) = \{\mathbf{y}_{m\times 1} \mid \mathbf{A}^T\mathbf{y} = \mathbf{0}\} \subseteq \Re^m$ is called the ***left-hand nullspace*** of $\mathbf{A}$ because $N(\mathbf{A}^T)$ is the set of all solutions to the left-hand homogeneous system $\mathbf{y}^T\mathbf{A} = \mathbf{0}^T$.

Problem: Determine a spanning set for $N(\mathbf{A})$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

Solution: $N(\mathbf{A})$ is merely the general solution of $\mathbf{Ax} = \mathbf{0}$, and this is determined by reducing $\mathbf{A}$ to a row echelon form $\mathbf{U}$. As discussed in §2.4, any such $\mathbf{U}$ will suffice, so we will use $\mathbf{E_A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. Consequently, $x_1 = -2x_2 - 3x_3$, where x_2 and x_3 are free, so the general solution of $\mathbf{Ax} = \mathbf{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

In other words, $N(\mathbf{A})$ is the set of all possible linear combinations of the vectors

$$\mathbf{h}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix},$$

and therefore $span\{\mathbf{h}_1, \mathbf{h}_2\} = N(\mathbf{A})$. For this example, $N(\mathbf{A})$ is the plane in $\Re^3$ that passes through the origin and the two points $\mathbf{h}_1$ and $\mathbf{h}_2$.

Zero Nullspace

If $\mathbf{A}$ is an $m \times n$ matrix, then

- $N(\mathbf{A}) = \{\mathbf{0}\}$ if and only if $rank(\mathbf{A}) = n$; (4.2.10)
- $N(\mathbf{A}^T) = \{\mathbf{0}\}$ if and only if $rank(\mathbf{A}) = m$. (4.2.11)

Proof. We already know that the trivial solution $\mathbf{x} = \mathbf{0}$ is the only solution to $\mathbf{Ax} = \mathbf{0}$ if and only if the rank of $\mathbf{A}$ is the number of unknowns, and this is what (4.2.10) says. Similarly, $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ has only the trivial solution $\mathbf{y} = \mathbf{0}$ if and only if $rank(\mathbf{A}^T) = m$. Recall from (3.9.11) that $rank(\mathbf{A}^T) = rank(\mathbf{A})$ in order to conclude that (4.2.11) holds. ∎

Summary

The four fundamental subspaces associated with $\mathbf{A}_{m\times n}$ are as follows.

- The range or column space: $R(\mathbf{A}) = \{\mathbf{Ax}\} \subseteq \Re^m$.
- The row space or left-hand range: $R(\mathbf{A}^T) = \{\mathbf{A}^T\mathbf{y}\} \subseteq \Re^n$.
- The nullspace: $N(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$.
- The left-hand nullspace: $N(\mathbf{A}^T) = \{\mathbf{y} \mid \mathbf{A}^T\mathbf{y} = \mathbf{0}\} \subseteq \Re^m$.

Let $\mathbf{P}$ be a nonsingular matrix such that $\mathbf{PA} = \mathbf{U}$, where $\mathbf{U}$ is in row echelon form, and suppose $rank(\mathbf{A}) = r$.

- Spanning set for $R(\mathbf{A})$ = the basic columns in $\mathbf{A}$.
- Spanning set for $R(\mathbf{A}^T)$ = the nonzero rows in $\mathbf{U}$.
- Spanning set for $N(\mathbf{A})$ =the $\mathbf{h}_i$'s in the general solution of $\mathbf{Ax} = \mathbf{0}$.
- Spanning set for $N(\mathbf{A}^T)$ = the last $m - r$ rows of $\mathbf{P}$.

If $\mathbf{A}$ and $\mathbf{B}$ have the same shape, then

- $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \iff N(\mathbf{A}) = N(\mathbf{B}) \iff R(\mathbf{A}^T) = R(\mathbf{B}^T)$.
- $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} \iff R(\mathbf{A}) = R(\mathbf{B}) \iff N(\mathbf{A}^T) = N(\mathbf{B}^T)$.