Thm: If $\{v_1, \ldots, v_n\}$ is a basis for the vector space V, then each $v \in V$ has unique expression of the form

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, \quad \text{for certain scalars } c_i.$$

Proof: assume there are two expressions for v,

$$v = c_1 v_1 + \cdots + c_n v_n$$
$$v = d_1 v_1 + \cdots + d_n v_n$$

$$\Rightarrow v - v = 0 = (c_1 - d_1) v_1 + \cdots + (c_n - d_n) v_n.$$

By linearly independence of v_i implies that

$$c_i = d_i \quad \forall i \quad \#$$

Def (Dimension)

Let V be finitely generated vector space. If V is non-zero define the dimension of V to be the number of elements in a basis of V.

Remark: The dimension of zero vector space is zero.

Ex: (1) $\dim(\mathbb{R}^n) = n$ (2) $\dim(P_n(\mathbb{R})) = n$

(3) Find the dimension for the nullspace of

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & 1 & 3 & 2 \\ 1 & 5 & 3 & -2 \end{bmatrix}$$

the solution is
$$X = \begin{bmatrix} -4c/3 - 4d/3 \\ -c/3 + 2d/3 \\ c \\ d \end{bmatrix}$$

$$\Rightarrow X = c \begin{bmatrix} -4/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -4/3 \\ 2/3 \\ 0 \\ 1 \end{bmatrix}, \quad c \,\&\, d \text{ scalars}$$

$$\Rightarrow \dim(\text{Nullspace}) = 2.$$

Note: Using elementry row operation put $A_{m\times n}$ in reduced row echelon form with r pivots. Then the general solution of the linear system $AX=0$ will contain $n-r$. This means that the general solution can be written:

$$X = c_1 X_1 + c_2 X_2 + \dots + c_{n-r} X_{n-r}$$

where $X_1, \dots, X_{n-r}$ are particular solutions.

Thm let A be a matrix with n columns and suppose that the number of pivots in the reduced row echelon form of A is r. Then the nullspace of A has dimension $n-r$.

Sum and intersection of subspaces:

If U and W are subspaces of the vector space V, then there are two natural subspaces:

(i) $U \cap W$

(ii) $U + W$

furthermore $U \cup W$ ~~contains~~ is not a subspace because it is not ingeneral closed under addition.

Proof (i) certainly $U \cap W$ contains the zero vector and it is closed ~~under~~ with respect to addition and scalar multiplication since both U and W are.

ii) Clearly $0 + 0 = 0 \in U + W$.

If u_1, u_2 and w_1, w_2 are vectors in U and W, then

$$(u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2) \in U + W,$$

and

$$c(u_1 + w_1) = cu_1 + cw_1 \in U + W$$

So it is a subspace.

Ex: Consider the subspaces $\mathcal{U}$ and W of $\mathbb{R}^4$ consisting of vectors of the form:

$$\begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ d \\ e \\ f \end{bmatrix} \qquad a, b, \ldots, f \text{ are scalars.}$$

then $\mathcal{U} \cap W$ consists of all vectors of the form:

while $\mathcal{U}+W$ equals $\mathbb{R}^4$.

(*) Thm: Let $\mathcal{U}$ and W be subspaces of finitely generated vector space V. Then

$$\dim(\mathcal{U}+W) + \dim(\mathcal{U}\cap W) = \dim(\mathcal{U}) + \dim(W).$$

Direct sum of subspaces:

Let $\mathcal{U}$ and W be two subspaces of a vector space V. Then V is said to be the direct sum of $\mathcal{U}$ and W if

$$V = \mathcal{U} + W \quad \text{and} \quad \mathcal{U} \cap W = 0$$

the notation for the direct sum is

$$V = \mathcal{U} \oplus W$$

Ex: $\mathcal{U}$: subset of $\mathbb{R}^3$ consists of $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$,

W: subset of $\mathbb{R}^3$ consists of $\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$ a, b, c scalars.

then $\mathcal{U}$ and W are subspaces of $\mathbb{R}^3$. In addition $\mathcal{U}+W = \mathbb{R}^3$ and $\mathcal{U} \cap W = 0 \Rightarrow \mathbb{R}^3 = \mathcal{U} \oplus W$.

Note If $V = \mathcal{U} \oplus W$, V = vector space & ($\mathcal{U}$ & W) subspaces of V, then

$$\dim(V) = \dim(\mathcal{U}) + \dim(W) \;\#$$

Thm (Subspace dimension)

If U and W are vector spaces s.th. $U \subseteq W$,

then

(i) $\dim(U) \leq \dim(W)$

(ii) If $\dim(U) = \dim(W) \Rightarrow U = W$.

Proof (i) Let $\dim(U) = m$ & $\dim(W) = n$.

Assume $m > n \Rightarrow$ then there exist a linearly indep. subset of W (namely basis for U) containing more than n vectors. But this is impossible since $\dim(W) = n$.

(ii) If $m = n$, but $U \neq W$, then $\exists\, x \in W$ but $x \notin U$. If B is a basis for U, then $x \notin \operatorname{span}(B)$ and $E = B \cup \{x\}$ is linearly indep. subset of W. But E contains $m+1$ $(n+1)$ vectors. Which is impossible.