LU-Factorization and linear system

Consider the solution of a 3×3 lower triangular system:

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad (l_{11} l_{22} l_{33} = 0)$$

by forward substitution. From the first equation we obtain $\xi_1 = b_1/l_{11}$, the second one $\xi_2 = \dfrac{(b_2 - l_{21}\xi_1)}{l_{22}}$ and

$$\xi_3 = \frac{b_3 - l_{31}\xi_1 - l_{32}\xi_2}{l_{33}},$$

<u>Proposition</u>: (forward substitution). If $L = (l_{ii}) \in \mathbb{R}^{n\times n}$ is a lower triangular matrix $\prod_{i=1}^{n} l_{ii} \neq 0$ and $Lx = b$, then the solution is:

$$\xi_i = \Big(b_i - \sum_{k=1}^{i-1} l_{ik}\xi_k\Big)/l_{ii} \qquad i = 1, \ldots, n.$$

#

Solve the 2×2 upper triangular system

$$\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad (u_{11} u_{22} \neq 0)$$

by back substitution. From the second equation we obtain $\xi_2 = b_2/u_{22}$ and then from the first one

$$\xi_1 = (b_1 - u_{12}\xi_1)/u_{11}.$$

<u>Proposition</u> (Back substitution) If $U = (u_{ij}) \in \mathbb{R}^{n\times n}$ is an upper triangular matrix $\prod_{i=1}^{n} u_{ii} \neq 0$ and $Ux = b$, then the solution is

$$\xi_i = \Big(b_i - \sum_{k=i+1}^{n} u_{ik}\xi_k\Big)/u_{ii} \qquad i = 1, \ldots, n,$$

Proposition: (Forward substitution: row version)
If $L \in \mathbb{R}^{n\times n}$ is lower triangular $\prod_{i=1}^{n} l_{ii} \neq 0$, $Lx = b$ and ξ_1 has been found then after substitution of ξ_1 into the equations from the second to the n^{th} we obtain a new $(n-1)\times(n-1)$ lower triangular system:

$$L(2:n, 2:n)\, x(2:n) = b(2:n) - x(1)\, L(2:n, 1)$$

Proposition: (back substitution: column version)
If $U \in \mathbb{R}^{n\times n}$ is upper triangular $\prod_{i=1}^{n} u_{ii} \neq 0$, $Ux = b$, and ξ_n has been found, then after the substitution of ξ_n into the equations from the first to the $(n-1)$-th, we obtain a new $(n-1)\times(n-1)$ upper triangular system

$$U(1:(n-1), 1:(n-1))\, x(1:(n-1)) = b(1:(n-1)) - x(n)\, U(1:(n-1), n)$$

* Now we consider the simultaneous solution of several systems with a common system matrix. Let us consider the system $LX = B$, where $L \in \mathbb{R}^{n\times n}$ is a regular lower triangular matrix, $B \in \mathbb{R}^{n\times q}$ and the unknown is $X \in \mathbb{R}^{n\times q}$. We represent this system in block form

$$\begin{bmatrix} L_{11} & O & \cdots & & O \\ L_{21} & L_{22} & O & \cdots & \vdots \\ \vdots & & & & O \\ L_{N1} & L_{N2} & L_{N3} & \cdots & L_{NN} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} \qquad (1)$$

where the diagonal blocks are square. From the eqn. $L_{11} X_1 = B_1$ we can find X_1. By using (1)

$$\begin{bmatrix} L_{22} & 0 & \cdots & & 0 \\ L_{32} & L_{33} & 0 & \cdots & \vdots \\ \vdots & & & & 0 \\ L_{N2} & L_{N3} & \cdots & & L_{NN} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} B_2 - L_{21} X_1 \\ B_3 - L_{31} X_1 \\ \vdots \\ B_N - L_{N1} X_1 \end{bmatrix}$$

Continuing in this way we obtain the solution of (1).

Proposition: Triangular matrices have the following properties:

(1) the inverse of an upper (lower) triangular matrices is upper (lower) triangular.

2) the product of two upper (lower) triangular matrices is upper (lower) triangular.

LU- Factorization

Under certain conditions the system $Ax=b$ can be expressed in the form of a product of a unit lower triangular matrix L with units on the main diagonal and an upper triangular matrix U and as a result one has to solve two systems with triangular matrices.

Proposition If $A \in \mathbb{R}^{n\times n}$, $A=LU$, where L is unit lower triangular, U is regular upper triangular and $Ax=b$, then $LUx=b$ and for the solution of the system one has first to solve the system $Ly=b$ and then the system $Ux=y$.

Example: Solve the system

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

using LU-method.

Solution: since $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

First we have to solve the system:

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$\Rightarrow \eta_1 = 1$ and $\eta_2 = 2$. Second solving the system

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \xi_2 = -1 \text{ and } \xi_1 = 3.$$

thus $x = [3 \quad -1]^T$.

Exc: By using the LU Factorization, solve the system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

The Gaussian elimination method is also applicable to LU-Factorization. Let $x \in \mathbb{R}^m$, where $\xi_k \neq 0$. If

$$\tau_i = \frac{\xi_i}{\xi_k} \quad i = k+1, \dots, m \quad , \quad t^{(k)} = [\underbrace{0 \dots 0}_{k \text{ zero's}} \ \tau_{k+1} \dots \tau_m]$$

and $M_k = I - t^{(k)} e_k^T$, then

$$M_1 = I - \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ +1 & 0 & 1 \end{bmatrix}$$

$$M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 4 & 5 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 3 & 1 \end{bmatrix}.$$

$$M_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

2.1.2 Gauss Transformation and LU-Factorization

Under certain conditions the system matrix A of the equation $A\mathbf{x} = \mathbf{b}$ can be expressed in the form of a product of a *unit lower triangular* matrix L with units on the main diagonal and an upper triangular matrix U,and as the result, one has to solve two systems with triangular matrices.

Proposition 1.2.1 (*LU−method*). If $A \in \mathbf{R}^{n\times n}$, $A = LU$, where L is unit lower triangular, U is regular upper triangular and $A\mathbf{x} = \mathbf{b}$, then $LU\mathbf{x} = \mathbf{b}$, and for the solution of the system one has first to solve the system $L\mathbf{y} = \mathbf{b}$ and then the system $U\mathbf{x} = \mathbf{y}$.

Example 1.2.1. Solve the system

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

using LU-method. Since

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix},$$

then, by Proposition 1.2.1, we have to solve first the system

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

The solution of this system is $\eta_1 = 1$ and $\eta_2 = 5 - 3 \cdot 1 = 2$. Second, solving the system

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

we find that $\xi_2 = -1$ and $\xi_1 = 1 - 2 \cdot (-1) = 3$. Thus, $\mathbf{x} = \begin{bmatrix} 3 & -1 \end{bmatrix}^T$.

The Gaussian elimination method considered in the main course of linear algebra for solution of systems of linear equations is also applicable also to the LU-factorization. Let $\mathbf{x} \in \mathbf{R}^m$, where $\xi_k \neq 0$. If

$$\tau_i = \xi_i/\xi_k \quad (i = (k+1) : m) \quad \mathbf{t}^{(k)} = [\underbrace{0 \ \cdots \ 0}_{k \text{ zeros}} \ \tau_{k+1} \ \cdots \ \tau_m]$$

and

$$M_k = I - \mathbf{t}^{(k)}\mathbf{e}_k^T \ , \tag{2}$$

then

$$M_k\mathbf{x} = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & & 1 & 0 & & 0 \\ 0 & & -\tau_{k+1} & 1 & & 0 \\ \vdots & \vdots & & \vdots & \ddots & \\ 0 & \cdots & -\tau_m & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_k \\ \xi_{k+1} \\ \vdots \\ \xi_m \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Definition 1.2.1. A matrix M_k of the form (2) is called a *Gauss matrix,* the components $\mathbf{t}((k+1):n)$ are called *Gauss multipliers,* and the vector $\mathbf{t}^{(k)}$ is called the *Gauss vector.* The transformation defined with the Gauss matrix M_k is called the *Gauss transformation.*

Definition 1.2.2. The value

$$d_k = \begin{cases} a_{11}, \text{ if } k = 1, \\ \det(A(1:k,\, 1:k))/\det(A(1:k-1,\, 1:k-1)), \text{ if } k = 2:p, \end{cases}$$

is called the $k-th$ *pivot* of the matrix $A \in \mathbf{R}^{m\times n}$, where $p = \min(m,n)$ and $\det(A(1:i,\, 1:i)) \neq 0 \ \ (i = 1:p-1)$.

If $A \in R^{n\times n}$, then for the nonzero pivots of A the Gauss matrices $M_1, \ldots, M_{n-1}$ can be found such that $M_{n-1}M_{n-2}\cdots M_2M_1A = U$ is upper triangular.

Example 1.2.2. Let us consider the finding of the Gauss matrices M_1 and M_2 and the upper triangular matrix U for

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 4 & 5 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

By relation (2), we obtain that

$$M_1 = I - \mathbf{t}^{(1)}\mathbf{e}_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 4/2 \\ (-2)/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Thus,

$$M_1A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 4 & 5 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 3 & 1 \end{bmatrix}$$

and

$$M_2 = I - \mathbf{t}^{(2)}\mathbf{e}_2^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}.$$

Therefore,

$$U = M_2M_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & -11 \end{bmatrix}.$$

Note that matrix $A^{(k-1)} = M_{k-1} \cdots M_1A$ is upper triangular in columns 1 to k-1, and for the calculation of the elements of the Gauss matrix M_k we use the matrix vector $A^{(k-1)}(k : m, k)$. The calculation of M_k is possible if $a_{kk}^{(k-1)} \neq 0$. Moreover, $M_k^{-1} = I + \mathbf{t}^{(k)}\mathbf{e}_k^T$. If to choose

$$L = M_1^{-1} \cdots M_{n-1}^{-1},$$

then

$$A = LU.$$

We stress that in our treatment the lower triangular matrix L is a unit lower triangular matrix.

Proposition 1.2.2. If $\det(A(1{:}k, 1{:}k)) \neq 0$ for (k =1:n-1),then $A \in R^{n\times n}$ has an LU factorization. If the LU factorization exists and A is regular, then the LU factorization is unique and $\det(A) = u_{11} \cdots u_{nn}$.

Proof. Suppose $k-1$ steps have been taken and the matrix $A^{(k-1)} = M_{k-1} \cdots M_1A$ has been found. The element $a_{kk}^{(k-1)}$ is the k-th pivot of A and $\det(A(1{:}k, 1{:}k)) = a_{11}^{(k-1)} \cdots a_{kk}^{(k-1)}$. Hence, if $A(1{:}k, 1{:}k)$ is regular, then $a_{kk}^{(k-1)} \neq 0$, and A has an LU factorization. Let us suppose that the regular matrix A has two LU factorizations $A = L_1U_1$ and $A = L_2U_2$. We have

$L_1U_1 = L_2U_2$ or $L_2^{-1}L_1 = U_2U_1^{-1}$. Since $L_2^{-1}L_1$ is unit lower triangular and $U_2U_1^{-1}$is upper triangular, then $L_2^{-1}L_1 = I$, $U_2U_1^{-1} = I$ and $L_2 = L_1$ and also $U_2 = U_1$. □

Example 1.2.3.* Find the LU factorization of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}.$$

Find the Gauss matrix M_1 for A :

$$M_1 = I - \mathbf{t}^{(1)}\mathbf{e}_1^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 8/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}.$$

Thus,

$$M_1A = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad M_1^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad A = LU = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

Example 1.2.4.* Find the LU factorization of

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}.$$

Find the Gauss matrix M_1 for A:

$$M_1 = I - \mathbf{t}^{(1)}\mathbf{e}_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 6/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Thus,

$$M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}, \quad M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

and since $M_1 A$ is upper triangular, then $M_2 = I$ and

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix},$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}.$$

Example 1.2.5.* By using the LU factorization, solve the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \quad \wedge \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

In example 1.2.4 we found the LU factorization for A:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}.$$

By solving the system $L\mathbf{y} = \mathbf{b}$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix},$$

we obtain

$$\mathbf{y} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

By solving the system $U\mathbf{x} = \mathbf{y}$, i.e.,

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix},$$

we obtain that

$$\mathbf{x} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Exercise 1.2.1.* Find the LU factorization of A if

$$a)\ A = \begin{bmatrix} 3 & 1 \\ 6 & 7 \end{bmatrix},\quad b)\ A = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix},\quad c)\ A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Exercise 1.2.2.* By using the LU factorization solve the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \wedge \quad \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

If the principal minors of a rectangular matrix $A \in \mathbf{R}^{m\times n}$ are nonzero, i.e.,

$$\det(A(1:k,\ 1:k) \neq 0\ (k = 1 : \min(m,n)),$$

then A has an LU factorization.

Example 1.2.6. The following equalities hold:

$$\begin{bmatrix} 2 & 1 \\ 8 & 6 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 2 & 3/2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 8 & 4 \\ 1 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 8 & 4 \\ 0 & 2 & 3 \end{bmatrix}.$$

As is known from the main course of algebra, the direct application of the Gaussian elimination, therefore also the direct realization of the LU factorization fails, if at least one of the principal minors is singular. It turns out that for a regular matrix it is possible after an appropriate interchange of matrix rows to find the LU factorization. Permutation matrices are used for interchanging the matrix rows (columns).

Definition 1.2.3. A *permutation matrix* $P \in \mathbf{R}^{n\times n}$ is the identity I with its rows reordered.