Proposition 2.5.1. If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n] \in \mathbf{R}^{m\times n}$ $(m \geq n)$ with linearly independent column vectors $\mathbf{a}_i$ $(i = 1\!:n)$ can be factored into $A = QR$, where $Q = [\mathbf{q}_1 \cdots \mathbf{q}_m] \in \mathbf{R}^{m\times m}$ and $R \in \mathbf{R}^{m\times n}$, then

$$span\{\mathbf{a}_1, \ldots, \mathbf{a}_k\} = span\{\mathbf{q}_1, \ldots, \mathbf{q}_k\}\ (k = 1\!:n)\,. \tag{3}$$

In particular, if

$$Q_1 = Q(1:m, 1:n),\ Q_2 = Q(1:m, n+1:m),\ R_1 = R(1:n, 1:n),$$

then

$$\mathcal{R}(A) = \mathcal{R}(Q_1) \tag{4}$$

$$\mathcal{R}(A)^{\perp} = \mathcal{R}(Q_2) \tag{5}$$

and

$$A = Q_1 R_1\,, \tag{6}$$

Proof. If $A = QR$, then

$$a_{ik} = \sum_{j=1}^{m} q_{ij} r_{jk} \overset{r_{jk}=0}{\underset{j>k}{=}} \sum_{j=1}^{k} q_{ij} r_{jk} \quad (i = 1:m,\ k = 1:n)$$

or

$$\mathbf{a}_k = \sum_{j=1}^{k} r_{jk}\mathbf{q}_j \quad (k = 1:n).$$

Thus, $\mathbf{a}_k \in span\{\mathbf{q}_1, \ldots, \mathbf{q}_k\}$ and $span\{\mathbf{a}_1, \ldots \mathbf{a}_k\} \subset span\{\mathbf{q}_1, \ldots, \mathbf{q}_k\}$. Since $rank(A) = n$, then $rank(span\{\mathbf{a}_1, \ldots \mathbf{a}_k\}) = k$, and relation (3) holds. Relation (3) for $k = n$ yields relation (4), and this yields (5). From

$$a_{ik} = \sum_{j=1}^{m} q_{ij} r_{jk} = \sum_{j=1}^{n} q_{ij} r_{jk}$$

results assertion (6). □

Singular Value Decomposition

Proposition 3.1.4 *(existence theorem of the singular value decomposition).* If $A \in \mathbf{R}^{m \times n}$, then there exist orthogonal matrices

$$U = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_m] \in \mathbf{R}^{m \times m}$$

and

$$V = [\mathbf{v}_1 \ \cdots \mathbf{v}_n] \in \mathbf{R}^{n \times n},$$

such that

$$U^T A V = \Sigma = diag(\sigma_1, \ldots, \sigma_p) \in \mathbf{R}^{m \times n} \ \ (p = \min\{m, n\}) \tag{2}$$

with

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0.$$

Definition 3.1.1. The relation in form (2) is called the *singular value decomposition* of the matrix $A \in \mathbf{R}^{m \times n}$. The elements σ_i $(i = 1 : \min\{m, n\})$ on the main diagonal of Σ are called the *singular values* of the matrix A.

Proposition 3.2.3. If $A \in \mathbf{R}^{m \times n}$ and $A = U \Sigma V^T$ is a singular value decomposition of the matrix A, then the column-vectors of $U \in \mathbf{R}^{m \times m}$ are the normed eigenvectors of AA^T and the column-vectors of $V \in \mathbf{R}^{n \times n}$ are the normed eigenvectors of $A^T A$. Singular values of the matrix A can be found as square roots of the eigenvalues of $A^T A$ or AA^T.

Proof. Proceeding from the singular value decomposition of the matrix A we will find expressions of AA^T and $A^T A$:

$$AA^T = U \Sigma V^T V \Sigma^T U^T = U(\Sigma \Sigma^T) U^T \tag{7}$$

and

$$A^T A = V \Sigma^T U^T U \Sigma V^T = V \Sigma^T \Sigma V^T. \tag{8}$$

Since the matrices $\Sigma \Sigma^T$ and $\Sigma^T \Sigma$ are diagonal matrices, the orthogonal matrices U and V in the expressions (7) and (8) must be formed by the eigenvectors of the matrices AA^T and $A^T A$ respectively. □

2.3.3 Algorithm of Singular Value Decomposition

Algorithm 3.3.1. To find the singular value decomposition of the matrix $A \in \mathbf{R}^{m\times n}$ one has to:
I Find the eigenvalues of the matrix $A^T A$ and arrange them in descending order.
II Find the number of nonzero eigenvalues of the matrix $A^T A$.
III Find the orthogonal eigenvectors of the matrix $A^T A$ corresponding to the eigenvalues, and arrange them in the same order to form the column-vectors of the matrix $V \in \mathbf{R}^{n\times n}$.
IV Form a diagonal matrix $\Sigma \in \mathbf{R}^{m\times n}$ placing on the leading diagonal the square roots $\sigma_i = \sqrt{\lambda_i}$ of $p = \min\{m, n\}$ first eigenvalues of the matrix $A^T A$ obtained in I in descending order.
V Find the first column-vectors of the matrix $U \in \mathbf{R}^{m\times m}$:

$$\mathbf{u}_i = \sigma_i^{-1} A \mathbf{v}_i \quad (i = 1 : r). \tag{9}$$

VI Add to the matrix U the rest of $m - r$ vectors using the Gram-Schmidt orthogonalization process. □

Example 3.3.1. Let us find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \in R^{3\times 2}.$$

I Find the eigenvalues of the matrix $A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$:
$\lambda_1 = 3,\ \lambda_2 = 1.$
II Find the number of nonzero eigenvalues of the matrix $A^T A$: $r = 2$.
III Find the orthonormal eigenvectors of the matrix $A^T A$ corresponding to the eigenvalues λ_1 and λ_2:
$\mathbf{v}_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$ forming a matrix

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} \in R^{2\times 2}.$$

IV Find the singular value matrix $\Sigma \in R^{3\times 2}$:

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

on the leading diagonal of which are the square roots of the eigenvalues of the matrix $A^T A$ (in descending order) and the rest of the entries of the matrix Σ are zeros.

V Find the first two column-vectors of the matrix $U \in R^{3\times 3}$ using the formula (9)

$$\mathbf{u}_1 = \sigma_1^{-1} A \mathbf{v}_1 = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/3 \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix}$$

and

$$\mathbf{u}_2 = \sigma_2^{-1} A \mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}.$$

VI To find the vector $\mathbf{u}_3$ we shall first find, applying the Gram-Schmitd process, a vector $\grave{\mathbf{u}}_3$ perpendicular to $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\grave{\mathbf{u}}_3 = \mathbf{e}_1 - (\mathbf{u}_1^T \mathbf{e}_1)\mathbf{u}_1 - (\mathbf{u}_1^T \mathbf{e}_2)\mathbf{u}_2 = \begin{bmatrix} 1/3 & -1/3 & -1/3 \end{bmatrix}^T.$$

Norming the vector $\grave{\mathbf{u}}_3$, we get

$$\mathbf{u}_3 = \begin{bmatrix} \sqrt{3}/3 \\ -\sqrt{3}/3 \\ -\sqrt{3}/3 \end{bmatrix}.$$

Hence

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/3 & 0 & \sqrt{3}/3 \\ \sqrt{6}/6 & \sqrt{2}/2 & -\sqrt{3}/3 \\ \sqrt{6}/6 & -\sqrt{2}/2 & -\sqrt{3}/3 \end{bmatrix}$$

and the singular value decomposition of the matrix A is

$$A = \begin{bmatrix} \sqrt{6}/3 & 0 & \sqrt{3}/3 \\ \sqrt{6}/6 & -\sqrt{2}/2 & -\sqrt{3}/3 \\ \sqrt{6}/6 & \sqrt{2}/2 & -\sqrt{3}/3 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}.$$

Example 3.3.2. Let us find the singular value decomposition of the matrix $A = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix}$.
I Find the eigenvalues of the matrix $A^T A$:

$$\det(A^T A - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} 4-\lambda & 2 & -4 \\ 2 & 1-\lambda & -2 \\ -4 & -2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 9, \\ \lambda_2 = 0, \\ \lambda_3 = 0. \end{cases}$$

II Find the number of the nonzero eigenvalues of the matrix $A^T A$: $r = 1$.
III Find the eigenvector of the matrix $A^T A$:

$$\lambda_1 = 9 \ \Rightarrow \ \mathbf{v}_1 = \begin{bmatrix} -2/3 & -1/3 & 2/3 \end{bmatrix}^T,$$

$$\lambda_{2,3} = 0 \ \Rightarrow \begin{cases} \mathbf{v}_2 = \begin{bmatrix} -\sqrt{5}/5 & 2\sqrt{5}/5 & 0 \end{bmatrix}^T, \\ \mathbf{v}_3 = \begin{bmatrix} 4\sqrt{5}/15 & 2\sqrt{5}/15 & 5\sqrt{5}/15 \end{bmatrix}^T. \end{cases}$$

Since the eigenvalue 0 is multiple, the Gram-Schmidt orthogonalization process is used to find the vector $\mathbf{v}_3$. We compile the orthonormal matrix V :

$$V = \begin{bmatrix} -2/3 & -\sqrt{5}/5 & 4\sqrt{5}/15 \\ -1/3 & 2\sqrt{5}/5 & 2\sqrt{5}/15 \\ 2/3 & 0 & 5\sqrt{5}/15 \end{bmatrix}.$$

IV Form the singular value matrix:

$$\Sigma = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}.$$

V Calculate the unique column-vector of the matrix U applying the formula (9):

$$\mathbf{u}_1 = \frac{1}{3} A \mathbf{v}_1 = \frac{1}{3} \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2/3 & -1/3 & 2/3 \end{bmatrix}^T = \begin{bmatrix} -1 \end{bmatrix}.$$

Thus the singular value decomposition of the matrix A is

$$A = U \Sigma V^T = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ \sqrt{5}/5 & 2\sqrt{5}/5 & 0 \\ 4\sqrt{5}/15 & 2\sqrt{5}/15 & 5\sqrt{5}/15 \end{bmatrix}.$$

Example 3.3.3.* Let us find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix}.$$

The given 3×4 matrix A has three nonzero singular values. Therefore it is enough to find nonzero singular values of the matrix A using the 3×3 matrix AA^T (not the 4×4 matrix A^TA). Since

$$AA^T = \begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix} \begin{bmatrix} 2 & \frac{17}{10} & \frac{3}{5} \\ 2 & \frac{1}{10} & \frac{9}{5} \\ 2 & -\frac{17}{10} & -\frac{3}{5} \\ 2 & -\frac{1}{10} & -\frac{9}{5} \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & \frac{29}{5} & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5} \end{bmatrix},$$

then the characteristic equation of AA^T is

$$\begin{vmatrix} 16-\lambda & 0 & 0 \\ 0 & \frac{29}{5}-\lambda & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5}-\lambda \end{vmatrix} = 0$$

or

$$(16-\lambda)\left(36 - 13\lambda + \lambda^2\right) = 0,$$

and the solutions of this equation are $\lambda_1 = 16$, $\lambda_2 = 9$ and $\lambda_3 = 4$. Since $\lambda_i = \sigma_i^2$ and the matrix Σ is a 3×4 matrix, then on the leading diagonal of the matrix Σ there are the singular values of the matrix A in descending order, and all other elements of the matrix Σ are zeros:

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

The matrix U has for column-vectors the orthonormed eigenvectors of the matrix AA^T:

$$\lambda_1 = 16 \Rightarrow \mathbf{u}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T;$$

$$\lambda_2 = 9 \Rightarrow \mathbf{u}_2 = \begin{bmatrix} 0 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}^T;$$

$$\lambda_3 = 4 \Rightarrow \mathbf{u}_3 = \begin{bmatrix} 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix}^T.$$

Collecting the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, we obtain the matrix

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

According to the relation (6), we shall find the first three column-vectors of the matrix V (the matrix Σ has three nonzero entries on its leading diagonal) using the formula

$$\mathbf{v}_i = \frac{1}{\sigma_i} A^T \mathbf{u}_i.$$

Hence

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

To calculate the vector $\mathbf{v}_4$, we find first, using the Gram-Schmitd orthogonalization process, the vector $\widehat{\mathbf{v}}_4$ perpendicular to the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$:

$$\widehat{\mathbf{v}}_4 = \mathbf{e}_1 - (\mathbf{v}_1^T \mathbf{e}_1)\mathbf{v}_1 - (\mathbf{v}_2^T \mathbf{e}_1)\mathbf{v}_2 - (\mathbf{v}_3^T \mathbf{e}_1)\mathbf{v}_3 =$$

$$= \mathbf{e}_1 - \frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}^T.$$

Since $\|\widehat{\mathbf{v}}_4\|_2 = \frac{1}{2}$, then

$$\mathbf{v}_4 = 2\widehat{\mathbf{v}}_4 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T$$

and

$$V = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Let us check the result:

$$U\Sigma V^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix} = A$$

and

$$U^T AV = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} =$$

:

$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \Sigma.$$

Proposition 3.2.1. If $A \in \mathbf{R}^{m\times n}$, $A = U\Sigma V^T$, $U = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_m] \in \mathbf{R}^{m\times m}$ and
$V = [\mathbf{v}_1 \ \cdots \mathbf{v}_n] \in \mathbf{R}^{n\times n}$, then for each $i = 1 : \min\{m, n\}$ the following holds

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i \,, \tag{5}$$

$$A^T \mathbf{u}_i = \sigma_i \mathbf{v}_i \,, \tag{6}$$

$$\|A\|_F = \sigma_1^2 + \ldots + \sigma_p^2 \quad (p = \min\{m, n\}),$$

$$\|A\|_2 = \sigma_1$$

and

$$\min_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sigma_n \ \ (m \geq n).$$

Proof. Suppose $n > m$. Consider relation (3) which can be written in the form

$$A[\mathbf{v}_1 \ \cdots \mathbf{v}_n] = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_m] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_m & 0 \end{bmatrix}$$

or

$$[A\mathbf{v}_1 \cdots A\mathbf{v}_n] = \left[\begin{array}{cccc} \sigma_1\mathbf{u}_1 & \cdots & \sigma_m\mathbf{u}_m & 0 \end{array}\right].$$

The latter is (5) for the elements in the m first columns of the matrix. Consider relation (4) that can be written in the form

$$A^T[\mathbf{u}_1 \cdots \mathbf{u}_m] = [\mathbf{v}_1 \cdots \mathbf{v}_n]\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

or

$$[A^T\mathbf{u}_1 \cdots A^T\mathbf{u}_m] = \left[\begin{array}{ccc} \sigma_1\mathbf{v}_1 & \cdots & \sigma_m\mathbf{v}_m \end{array}\right],$$

which represents relation (6) by elements. We note that "0" denotes also certain blocks consisting of zeros. □

Proposition 3.2.2. If the singular values in the singular value decomposition (2) of $A \in \mathbf{R}^{m\times n}$ satisfy the inequalities

$$\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_p = 0,$$

then

1. $span\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} = \mathcal{R}(A)$;
2. $span\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} = \mathcal{R}(A^T)$;
3. $span\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m\} = \mathcal{N}(A^T)$;
4. $span\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\} = \mathcal{N}(A)$;
5. $rank(A) = r$;
6. the singular values of A are equal to the semi-axes of the hyperellipsoid $E = \{A\mathbf{x} : \ ||\mathbf{x}|| = 1\}$;
7. $A = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$.

Prove the first of these properties. Consider the relation $A = U\Sigma V^T$. Since

$$[\Sigma V^T]_{jk} = \sum_{s=1}^{n} \sigma_{js} v_{sk}^T = \begin{cases} \sigma_j v_{kj}, & \text{if } j = 1:r, \\ 0, \text{ kui } & j = r+1:m, \end{cases}$$

then

$$a_{ik} = [U\Sigma V^T]_{ik} = \sum_{j=1}^{m} u_{ij}[\Sigma V^T]_{jk} = \sum_{j=1}^{r} u_{ij}\sigma_j v_{kj}$$

or

$$\mathbf{a}_k = \sum_{j=1}^{r} \sigma_j v_{kj} \mathbf{u}_j.$$

Thus,

$$\mathbf{a}_k \in span\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} \ (k = 1:n) \Rightarrow span\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} = \mathcal{R}(A). \ \square$$

Proposition 4.2.1. If

$$\Sigma = diag(\sigma_1, \ldots, \sigma_p) \in \mathbf{R}^{m \times n} \quad (p = \min\{m, n\}) \tag{1}$$

and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_p, \tag{2}$$

then the optimum solution $\mathbf{x}^+$ of the system

$$\Sigma \mathbf{x} = \mathbf{b}$$

is given by

$$\mathbf{x}^+ = \Sigma^+ \mathbf{b},$$

where

$$\Sigma^+ = diag(1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0) \in \mathbf{R}^{n \times m}. \tag{3}$$

Definition 4.2.1. Let

$$A = U\Sigma V^T$$

be the singular value decomposition of the matrix $A \in \mathbf{R}^{m \times n}$. The *pseudoinverse matrix* of the matrix A is a matrix

$$A^+ = V\Sigma^+ U^T,$$

where Σ and Σ^+ are given by relations (1-3).

Problem 4.2.2. Let us find the pseudoinverse matrix of the matrix $A = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix}$ given in example 3.3.2. We found the singular value decomposition of the matrix A in this example

$$A = U\Sigma V^T = \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ \sqrt{5}/5 & 2\sqrt{5}/5 & 0 \\ 4\sqrt{5}/15 & 2\sqrt{5}/15 & 5\sqrt{5}/15 \end{bmatrix}.$$

Using definition 4.2.1,

$$A^+ = V\Sigma^+ U^T,$$

i.e.,

$$A^+ = \begin{bmatrix} -2/3 & \sqrt{5}/5 & 4\sqrt{5}/15 \\ -1/3 & 2\sqrt{5}/5 & 2\sqrt{5}/15 \\ 2/3 & 0 & 5\sqrt{5}/15 \end{bmatrix} \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix} = \begin{bmatrix} 2/9 \\ -1/9 \\ -2/9 \end{bmatrix}.$$