Complementry subspaces

let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of V, then $\mathcal{X}$ and $\mathcal{Y}$ are said to complementary subspaces if

$$V = \mathcal{X} + \mathcal{Y} \quad \text{and} \quad \mathcal{X} \cap \mathcal{Y} = 0$$

(i.e. V is a direct sum of $\mathcal{X}$ & $\mathcal{Y}$ [write $V = \mathcal{X} \oplus \mathcal{Y}$]).

<u>Thm:</u> let $B_{\mathcal{X}}$ and $B_{\mathcal{Y}}$ be bases of $\mathcal{X}$ and $\mathcal{Y}$, then the following are equivalent:

(1) $V = \mathcal{X} \oplus \mathcal{Y}$

(2) for each $v \in V$ there are unique vectors $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ s.th. $v = x + y$.

(3) $B_{\mathcal{X}} \cap B_{\mathcal{Y}} = \phi$ and $B_{\mathcal{X}} \cup B_{\mathcal{Y}}$ is a basis for V.

<u>Prove</u> (1) $\to$ (2)

$$\dim V = \dim(\mathcal{X} + \mathcal{Y}) = \dim(\mathcal{X}) + \dim(\mathcal{Y}) - \dim(\mathcal{X} \cap \mathcal{Y})$$

if $V = \mathcal{X} \oplus \mathcal{Y}$ & $\mathcal{X} \cap \mathcal{Y} = 0 \Rightarrow$

$$\dim V = \dim \mathcal{X} + \dim \mathcal{Y}$$

let $v \in V$ and assume that there are two representation for $v \Rightarrow v = x_1 + y_1 = x_2 + y_2 \quad x_1, x_2 \in \mathcal{X}; \ y_1 \,\&\, y_2 \in \mathcal{Y}$

$$\Rightarrow x_1 - x_2 = y_2 - y_1 = \left\{ \begin{array}{l} x_1 - x_2 \in \mathcal{X} \\ \quad \& \\ x_1 - x_2 \in \mathcal{Y} \end{array} \right\} \Rightarrow x_1 - x_2 \in \mathcal{X} \cap \mathcal{Y}$$

But $\mathcal{X} \cap \mathcal{Y} = 0 \Rightarrow x_1 = x_2$ & $y_1 = y_2$.

(2) $\to$ (3)

$V = \mathcal{X} + \mathcal{Y}$ and we know that

$B_{\mathcal{X}} \cup B_{\mathcal{Y}}$ spans $\mathcal{X} + \mathcal{Y} \Rightarrow$

$B_{\mathcal{X}} \cup B_{\mathcal{Y}}$ spans V.

to prove the linearly independent, let

$B_X = \{x_1, x_2, \dots, x_r\}$ & $B_Y = \{y_1, y_2, \dots, y_s\}$

$$\Rightarrow \quad 0 = \sum_{i=1}^{r} \alpha_i x_i + \sum_{j=1}^{s} \beta_j y_j \quad (\text{But } 0 = 0 + 0)$$

$$\Rightarrow \sum_{1}^{r} \alpha_i x_i = 0 \quad \& \quad \sum_{1}^{s} \beta_j y_j = 0$$

$$\Rightarrow \alpha_1 = \dots = \alpha_r = 0 \quad \& \quad \beta_1 = \beta_2 = \dots = \beta_s = 0 .$$

because B_X & B_Y are linearly independent, and hence basis for V.

(3) $\rightarrow$ (1) If $B_X \cup B_Y$ is a basis for V, then $B_X \cup B_Y$ is a linearly independent set & $B_X \cup B_Y$ spans $X + Y$

$\Rightarrow B_X \cup B_Y$ basis for $X + Y$ as well as V.

$\Rightarrow \quad V = X + Y$

$$\dim X + \dim Y = \dim V = \dim(X + Y)$$
$$= \dim X + \dim Y - \underbrace{\dim(X \cap Y)}_{\text{zero}}$$

<u>Example</u>: let X and Y be subspaces of $\mathbb{R}^3$ that spanned by

$$B_X = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\} \quad \& \quad B_Y = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

Explain why X and Y are complementary.

<u>Solution</u>: B_X and B_Y are linearly independent so they are bases for X and Y,

$$\operatorname{rank}[X\ Y] = \operatorname{rank}\left(\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & -2 & 0 \end{array}\right) = 3$$

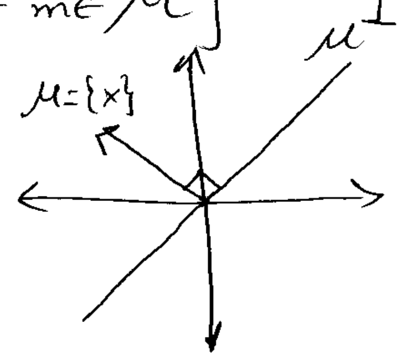
$\Rightarrow \mathcal{X} + \mathcal{Y}$ are complementary

& $\mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\mathcal{Y}}$ bases for $\mathbb{R}^3$ #.

Orthogonal Complement

Def: For a subset $\mathcal{M}$ of an inner-product space $\mathcal{V}$, the orthogonal complement $\mathcal{M}^{\perp}$ (read "$\mathcal{M}$ perp") of $\mathcal{M}$ is defined to be the set of all vectors in $\mathcal{V}$ that are orthogonal to every vector in $\mathcal{M}$, (i.e.)

$$\mathcal{M}^{\perp} = \{ x \in \mathcal{V} \mid \langle m | x \rangle = 0 \quad \forall\, m \in \mathcal{M} \}$$



Example: (1) $\mathcal{M} = \{x\}$ in $\mathbb{R}^2$

$\mathcal{M}^{\perp}$ will be the line through the orgin that is perpendicular to x.

(2) $\mathcal{M}$ = plane through the orgin

$\mathcal{M}^{\perp}$ = line through the orgin that is perpendicular to the plane.

Note: $\mathcal{M}^{\perp}$ is a subspace of $\mathcal{V}$ even if $\mathcal{M}$ is not.
(Proof exc.)

Orthogonal Complementary subspaces :

If $\mathcal{M}$ is a subspace of a finite-dimensional inner product space V, then $V = \mathcal{M} \oplus \mathcal{M}^\perp$. Furthermore if $\mathcal{N}$ is a subspace s.th. $V = \mathcal{M} \oplus \mathcal{N}$ and $\mathcal{N} \perp \mathcal{M}$ (every vector in $\mathcal{N}$ is $\perp$ to every vector in $\mathcal{M}$), then

$$\mathcal{N} = \mathcal{M}^\perp .$$

<u>Proof</u> Observe that $\mathcal{M} \cap \mathcal{M}^\perp = 0$ because if $x \in \mathcal{M}$ & $x \in \mathcal{M}^\perp$, then $\langle x | x \rangle = 0 \Rightarrow x = 0$

to prove that $\mathcal{M} \oplus \mathcal{M}^\perp = V$?

Suppose that $B_{\mathcal{M}}$ & $B_{\mathcal{M}^\perp}$ are orthonormal bases for $\mathcal{M}$ & $\mathcal{M}^\perp$ & $\mathcal{M} \cap \mathcal{M}^\perp = 0$

$\Rightarrow$ $B_{\mathcal{M}} \cup B_{\mathcal{M}^\perp}$ is orthonormal basis for some subspace

$$S = \mathcal{M} \oplus \mathcal{M}^\perp \subseteq V$$

If $S \neq V$ then $B_{\mathcal{M}} \cup B_{\mathcal{M}^\perp} \cup \mathcal{E}$ is orthonormal bases for V

$\Rightarrow \mathcal{E} \perp B_{\mathcal{M}} \Rightarrow \mathcal{E} \perp \mathcal{M} \Rightarrow \mathcal{E} \subseteq \mathcal{M}^\perp$

$\Rightarrow \mathcal{E} \subseteq \text{span}(B_{\mathcal{M}^\perp})$ impossible.

because $B_{\mathcal{M}} \cup B_{\mathcal{M}^\perp} \cup \mathcal{E}$ is linearly independent

$\Rightarrow \mathcal{E}$ = empty set

$\Rightarrow V = \mathcal{M} \oplus \mathcal{M}^\perp$

- to prove $\mathcal{N} \perp \mathcal{M} \Rightarrow \mathcal{N} \subseteq \mathcal{M}^\perp$ & $\mathcal{M} \oplus \mathcal{M}^\perp = \mathcal{M} + \mathcal{N} = V$

$\Rightarrow \dim \mathcal{N} = \dim V - \dim \mathcal{M} = \dim \mathcal{M}^\perp$ ※.

Example: let $U_{m\times m} = (U_1 \quad U_2)$ be partitioned orthogonal matrix. Explain why $R(U_1)$ & $R(U_2)$ must be orthogonal complements of each other.

Proof: since $B_x \cap B_y = \phi$ & $B_x \cup B_y$ is a basis for V $\Rightarrow$ $\mathbb{R}^m = R(U_1) \oplus R(U_2)$ and we know that

$$R(U_1) \perp R(U_2)$$

because the columns of U are orthonormal set.

$$\Rightarrow \quad R(U_2) = R(U_1)^{\perp}.$$

Prep operation

If $\mathcal{M}$ is a subspace of n-dimensional inner-product space, then the following are true:

(1) $\dim \mathcal{M}^{\perp} = n - \dim \mathcal{M}$

(2) $(\mathcal{M}^{\perp})^{\perp} = \mathcal{M}$

Proof: (1) ~~it~~ is clear ~~that~~ since $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are complement

(2) to prove (2) we show that $(\mathcal{M}^{\perp})^{\perp} \subseteq \mathcal{M}$.

If $x \in (\mathcal{M}^{\perp})^{\perp} \Rightarrow x = m + n \quad m \in \mathcal{M}$ & $n \in \mathcal{M}^{\perp}$

$$\Rightarrow 0 = \langle n | x \rangle = \langle n | m+n \rangle = \langle n | m \rangle + \langle n | n \rangle$$

$$\Rightarrow \langle n | n \rangle = 0 \quad \Rightarrow n = 0 \quad \Rightarrow x \in \mathcal{M}.$$

$\Rightarrow (\mathcal{M}^{\perp})^{\perp} \subseteq \mathcal{M}$ from (1) $\dim \mathcal{M}^{\perp} = n - \dim \mathcal{M}$

& $\dim (\mathcal{M}^{\perp})^{\perp} = n - \dim \mathcal{M}^{\perp} \Rightarrow \dim (\mathcal{M}^{\perp})^{\perp} = \dim \mathcal{M}$

$$\Rightarrow (\mathcal{M}^{\perp})^{\perp} = \mathcal{M} \quad \#.$$

Note: if $\langle x|y\rangle = 0 \ \forall x \in V \Rightarrow \langle y|y\rangle = 0 \Rightarrow y = 0$

The four fundamental subspaces of $A_{m\times n}$:

~~Not~~ (1) $R(A)^{\perp} = N(A^T)$

if $x \in R(A)^{\perp}$ iff $\langle Ay|x\rangle = 0 \Rightarrow y^T A^T x = 0$

iff $\langle y|A^T x\rangle = 0$ iff $A^T x = 0$

iff $x \in N(A^T)$

$\Rightarrow \quad R(A)^{\perp} = N(A^{\perp})$

and prep both side gives $R(A^T) = N(A)^{\perp}$

Thm: for every $A \in \mathbb{R}^{m\times n}$ (orthogonal decomposition Thm).

$$R(A)^{\perp} = N(A^T) \ \ \& \ \ N(A)^{\perp} = R(A^T)$$

$$\Rightarrow \quad R^m = R(A) \oplus R(A)^{\perp} = R(A) \oplus N(A^T)$$

$$R^n = N(A) \oplus N(A)^{\perp} = N(A) \oplus R(A^T). \ \#$$

The last Thm tells us how to decompose $\mathbb{R}^m$ & $\mathbb{R}^n$ in terms of four fundamental subspaces of A, the orthogonal decomposition thm tell us how to decompose A itself into more basic components.

URV - Factorization

Suppose $\text{rank}(A) = r$, let

$\mathcal{B}_{R(A)} = \{u_1, \ldots, u_r\}$ & $\mathcal{B}_{N(A^T)} = \{u_{r+1}, \ldots, u_m\}$

be orthonormal bases for $R(A)$ & $N(A^T)$.

$\mathcal{B}_{R(A^T)} = \{v_1, \ldots, v_r\}$ & $\mathcal{B}_{N(A)} = \{v_{r+1}, \ldots, v_n\}$

be orthonormal bases for ~~A~~ $R(A^T)$ & $N(A)$.

$\Rightarrow$ $\mathcal{B}_{R(A)} \cup \mathcal{B}_{N(A^T)}$ & $\mathcal{B}_{R(A^T)} \cup \mathcal{B}_{N(A)}$ are orthonormal bases for $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Define $U_{m\times m} = [u_1\ u_2 \ldots u_m]$ & $V_{n\times n} = [v_1\ v_2 \ldots v_n]$

are orthogonal matrices.

Consider $R = U^T A V$, $\quad r_{ij} = u_i^T A v_j$

However $u_i^T A = 0$ for $i = r+1, \ldots, m$

$A v_j = 0$ for $j = r+1, \ldots, n$

$$\Rightarrow R = U^T A V = \left(\begin{array}{ccc|c} u_1^T A v_1 & \ldots & u_1^T A v_r & \\ \vdots & & & 0 \\ u_r^T A v_1 & \ldots & u_r^T A v_r & \\ \hline & 0 & & 0 \end{array}\right)$$

$$\Rightarrow R = U^T A V = U^T \left(\begin{array}{c|c} C_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right) V^T$$

where C is nonsingular matrix because it is $r \times r$, and

$\text{rank}(C) = \text{rank}\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} = \text{rank}(U^T A V) = \text{rank}(A) = r$.

this is URV - Factorization.